The complex inverse trigonometric and hyperbolic functions

In these notes, we examine the inverse trigonometric and hyperbolic functions, where the arguments of these functions can be complex numbers. These are all multi-valued functions. We also carefully define the corresponding single-valued principal values of the inverse trigonometric and hyperbolic functions following the conventions of Abramowitz and Stegun (see ref. 1).

1. The inverse trigonometric functions: arctan and arccot

We begin by examining the solution to the equation

$$z = \tan w = \frac{\sin w}{\cos w} = \frac{1}{i}\left(\frac{e^{iw}-e^{-iw}}{e^{iw}+e^{-iw}}\right) = \frac{1}{i}\left(\frac{e^{2iw}-1}{e^{2iw}+1}\right) .$$

We now solve for e^{2iw},

$$iz = \frac{e^{2iw}-1}{e^{2iw}+1} \qquad \Longrightarrow \qquad e^{2iw} = \frac{iz+1}{1-iz} = \frac{i-z}{i+z} .$$

Taking the complex logarithm of both sides of the equation, we can solve for w,

$$w = \frac{1}{2i}\ln\left(\frac{i-z}{i+z}\right) .$$

The solution to $z = \tan w$ is $w = \arctan z$. Hence,

$$\boxed{\arctan z = \frac{1}{2i}\ln\left(\frac{i-z}{i+z}\right)}$$

Since the complex logarithm is a multi-valued function, it follows that the arctangent function is also a multi-valued function. We can define the *principal value* of the arctangent function (denoted with a capital A) by employing the principal value of the logarithm,

$$\boxed{\text{Arctan}\, z = \frac{1}{2i}\text{Ln}\left(\frac{i-z}{i+z}\right)}$$

We now examine the principal value of the arctangent for real-valued arguments. Setting $z = x$, where x is real,

$$\text{Arctan}\, x = \frac{1}{2i}\text{Ln}\left(\frac{i-x}{i+x}\right) = \frac{1}{2i}\left[\text{Ln}\left|\frac{i-x}{i+x}\right| + i\text{Arg}\left(\frac{i-x}{i+x}\right)\right] .$$

Noting that

$$\left|\frac{i-x}{i+x}\right|^2 = \left(\frac{i-x}{i+x}\right)\left(\frac{-i-x}{-i+x}\right) = \left(\frac{1+x^2}{1+x^2}\right) = 1\,,$$

it follows that

$$\operatorname{Arctan} x = \frac{1}{2}\operatorname{Arg}\left(\frac{i-x}{i+x}\right)\,.$$

Since $-\pi < \operatorname{Arg} z \leq \pi$, the principal interval of the real-valued arctangent function is:

$$-\frac{\pi}{2} < \operatorname{Arctan} x \leq \frac{\pi}{2}\,.$$

Indeed, if we write

$$\frac{i-x}{i+x} = e^{2i\theta} \qquad \Longrightarrow \qquad 2\theta = \operatorname{Arg}\left(\frac{i-x}{i+x}\right)\,, \qquad \text{where} \quad -\pi < 2\theta \leq \pi\,,$$

and solve for x, then one obtains $x = \tan\theta$, or equivalently $\theta = \operatorname{Arctan} x$.

Starting from

$$z = \cot w = \frac{\cos w}{\sin w} = i\left(\frac{e^{iw}+e^{-iw}}{e^{iw}-e^{-iw}}\right) = i\left(\frac{e^{2iw}+1}{e^{2iw}-1}\right)\,,$$

an analogous computation yields,

$$\boxed{\operatorname{arccot} z = \frac{1}{2i}\ln\left(\frac{z+i}{z-i}\right)}$$

The principal value of the complex arccotangent function is given by

$$\boxed{\operatorname{Arccot} z = \frac{1}{2i}\operatorname{Ln}\left(\frac{z+i}{z-i}\right)}$$

Using the definitions given by the boxed equations above yield:

$$\operatorname{arccot}(z) = \arctan\left(\frac{1}{z}\right)\,, \tag{1}$$

$$\operatorname{Arccot}(z) = \operatorname{Arctan}\left(\frac{1}{z}\right)\,. \tag{2}$$

Note that eqs. (1) and (2) can be used as *definitions* of the inverse cotangent function and its principal value.

We now examine the principal value of the arccotangent for real-valued arguments. Setting $z = x$, where x is real,

$$\operatorname{Arccot} x = \frac{1}{2}\operatorname{Arg}\left(\frac{x+i}{x-i}\right)\,.$$

The principal interval of the real-valued arccotangent function is:

$$-\frac{\pi}{2} < \operatorname{Arccot} x \leq \frac{\pi}{2}\,. \tag{3}$$

It is instructive to derive another relation between the arctangent and arccotangent functions. First, we first recall the property of the multi-valued complex logarithm,

$$\ln(z_1 z_2) = \ln(z_1) + \ln(z_2)\,, \tag{4}$$

as a set equality. It is convenient to define a new variable,

$$v = \frac{i-z}{i+z}\,, \qquad \Longrightarrow \qquad -\frac{1}{v} = \frac{z+i}{z-i}\,. \tag{5}$$

It follows that:

$$\arctan z + \operatorname{arccot} z = \frac{1}{2i}\left[\ln v + \ln\left(-\frac{1}{v}\right)\right] = \frac{1}{2i}\ln\left(\frac{-v}{v}\right) = \frac{1}{2i}\ln(-1)\,.$$

Since $\ln(-1) = i(\pi + 2\pi n)$ for $n = 0, \pm 1, \pm 2 \ldots$, we conclude that

$$\boxed{\arctan z + \operatorname{arccot} z = \tfrac{1}{2}\pi + \pi n\,, \qquad\qquad \text{for} \quad n = 0, \pm 1, \pm 2, \ldots}$$

To obtain the corresponding relation between the principal values, we compute

$$\begin{aligned}
\operatorname{Arctan} z + \operatorname{Arccot} z &= \frac{1}{2i}\left[\operatorname{Ln} v + \operatorname{Ln}\left(-\frac{1}{v}\right)\right] \\
&= \frac{1}{2i}\left[\operatorname{Ln}|v| + \operatorname{Ln}\left(\frac{1}{|v|}\right) + i\operatorname{Arg} v + i\operatorname{Arg}\left(-\frac{1}{v}\right)\right] \\
&= \frac{1}{2}\left[\operatorname{Arg} v + \operatorname{Arg}\left(-\frac{1}{v}\right)\right]\,.
\end{aligned} \tag{6}$$

It is straightforward to check that for any non-zero complex number v,

$$\operatorname{Arg} v + \operatorname{Arg}\left(-\frac{1}{v}\right) = \begin{cases} \pi\,, & \text{for } \operatorname{Im}\, v \geq 0\,, \\ -\pi\,, & \text{for } \operatorname{Im}\, v < 0\,. \end{cases} \tag{7}$$

Using eq. (5), we can evaluate $\operatorname{Im}\ v$ by computing

$$\frac{i-z}{i+z} = \frac{(i-z)(-i-\overline{z})}{(i+z)(-i+\overline{z})} = \frac{1-|z|^2+2i\operatorname{Re}\, z}{|z|^2+1+2\operatorname{Im}\, z}\,.$$

Writing $|z|^2 = (\operatorname{Re}\, z)^2 + (\operatorname{Im}\, z)^2$ in the denominator,

$$\frac{i-z}{i+z} = \frac{1-|z|^2+2i\operatorname{Re}\, z}{(\operatorname{Re}\, z)^2 + (\operatorname{Im}\, z+1)^2}\,.$$

Hence,

$$\mathrm{Im}\ v \equiv \mathrm{Im}\left(\frac{i-z}{i+z}\right) = \frac{2\,\mathrm{Re}\ z}{(\mathrm{Re}\ z)^2 + (\mathrm{Im}\ z + 1)^2}\,.$$

We conclude that

$$\mathrm{Im}\ v \geq 0 \quad \Longrightarrow \quad \mathrm{Re}\ z \geq 0\,, \qquad\qquad \mathrm{Im}\ v < 0 \quad \Longrightarrow \quad \mathrm{Re}\ z < 0\,.$$

Therefore, eqs. (6) and (7) yield:

$$\boxed{\mathrm{Arctan}\, z + \mathrm{Arccot}\, z = \begin{cases} \tfrac{1}{2}\pi\,, & \text{for Re } z \geq 0\,, \\ -\tfrac{1}{2}\pi\,, & \text{for Re } z < 0\,. \end{cases}} \tag{8}$$

CAUTION!!

Many books do not employ the definition for the principal value of the inverse cotangent function given above. Instead, they choose to define:

$$\mathrm{Arccot}\, z = \frac{\pi}{2} - \mathrm{Arctan}\, z\,. \tag{9}$$

Note that with this definition, the principal interval of the arccotangent function for real values x is

$$0 \leq \mathrm{Arccot}\, x < \pi\,,$$

instead of the interval quoted in eq. (3). One advantage of this latter definition is that the real-valued inverse cotangent function, $\mathrm{Arccot}\, x$ is continuous at $x = 0$, in contrast to eq. (8) which exhibits a discontinuity at $x = 0$.* On the other hand, one drawback of the definition of eq. (9) is that eq. (2) is no longer respected. Additional disadvantages are discussed in refs. 2 and 3, where you can find a detailed set of arguments (from the computer science community) in support of the conventions adopted by Abramowitz and Stegun (ref. 1) and employed in these notes.

2. The inverse trigonometric functions: arcsin and arccos

The arcsine function is the solution to the equation:

$$z = \sin w = \frac{e^{iw} - e^{-iw}}{2i}\,.$$

*In our conventions, the real inverse tangent function, $\mathrm{Arctan}\, x$, is a continuous single-valued function that varies smoothly from $-\frac{1}{2}\pi$ to $+\frac{1}{2}\pi$ as x varies from $-\infty$ to $+\infty$. In contrast, $\mathrm{Arccot}\, x$ varies from 0 to $-\frac{1}{2}\pi$ as x varies from $-\infty$ to zero. At $x = 0$, $\mathrm{Arccot}\, x$ jumps discontinuously up to $\frac{1}{2}\pi$. Finally, $\mathrm{Arccot}\, x$ varies from $\frac{1}{2}\pi$ to 0 as x varies from zero to $+\infty$. Following eq. (8), $\mathrm{Arccot}\, 0 = +\pi/2$, although $\mathrm{Arccot}\, x \to -\frac{1}{2}\pi$ as $x \to 0^-$ (which means as x approaches zero from the negative side of the origin).

Letting $v \equiv e^{iw}$, we solve the equation

$$v - \frac{1}{v} = 2iz\,.$$

Multiplying by v, one obtains a quadratic equation for v,

$$v^2 - 2izv - 1 = 0\,. \tag{10}$$

The solution to eq. (10) is:

$$v = iz + (1 - z^2)^{1/2}\,. \tag{11}$$

Since z is a complex variable, $(1 - z^2)^{1/2}$ is the complex square-root function. This is a multi-valued function with two possible values that differ by an overall minus sign. Hence, we do not explicitly write out the $\pm$ sign in eq. (11). To avoid ambiguity, we shall write

$$\begin{aligned} v &= iz + (1 - z^2)^{1/2} = iz + e^{\frac{1}{2}\ln(1-z^2)} = iz + e^{\frac{1}{2}\left[\mathrm{Ln}|1-z^2| + i\arg(1-z^2)\right]} \\ &= iz + |1 - z^2|^{1/2} e^{\frac{i}{2}\arg(1-z^2)}\,. \end{aligned}$$

By definition, $v \equiv e^{iw}$, from which it follows that

$$w = \frac{1}{i}\ln v = \frac{1}{i}\ln\left(iz + |1 - z^2|^{1/2} e^{\frac{i}{2}\arg(1-z^2)}\right)\,.$$

The solution to $z = \sin w$ is $w = \arcsin z$. Hence,

$$\boxed{\arcsin z = \frac{1}{i}\ln\left(iz + |1 - z^2|^{1/2} e^{\frac{i}{2}\arg(1-z^2)}\right)}$$

The principal value of the arcsine function is obtained by employing the principal value of the logarithm and the principle value of the square-root function (which corresponds to employing the principal value of the argument). Thus, we define

$$\boxed{\mathrm{Arcsin}\, z = \frac{1}{i}\mathrm{Ln}\left(iz + |1 - z^2|^{1/2} e^{\frac{i}{2}\mathrm{Arg}(1-z^2)}\right)}$$

We now examine the principal value of the arcsine for real-valued arguments. Setting $z = x$, where x is real,

$$\mathrm{Arcsin}\, x = \frac{1}{i}\mathrm{Ln}\left(ix + |1 - x^2|^{1/2} e^{\frac{i}{2}\mathrm{Arg}(1-x^2)}\right)\,.$$

For $|x| < 1$, $\mathrm{Arg}(1 - x^2) = 0$ and $|1 - x^2| = \sqrt{1 - x^2}$ defines the positive square root. Thus,

$$\begin{aligned} \mathrm{Arcsin}\, x &= \frac{1}{i}\mathrm{Ln}\left(ix + \sqrt{1 - x^2}\right) = \frac{1}{i}\left[\mathrm{Ln}\left|ix + \sqrt{1 - x^2}\right| + i\mathrm{Arg}\left(ix + \sqrt{1 - x^2}\right)\right] \\ &= \mathrm{Arg}\left(ix + \sqrt{1 - x^2}\right)\,, \end{aligned} \tag{12}$$

since $ix + \sqrt{1-x^2}$ is a complex number with magnitude equal to 1. Moreover, $ix + \sqrt{1-x^2}$ lives either in the first or fourth quadrant of the complex plane, since $\mathrm{Re}(ix + \sqrt{1-x^2}) \geq 0$. It follows that:

$$-\frac{\pi}{2} \leq \mathrm{Arcsin}\, x \leq \frac{\pi}{2}\,, \qquad \text{for } \ |x| \leq 1\,.$$

The arccosine function is the solution to the equation:

$$z = \cos w = \frac{e^{iw} + e^{-iw}}{2}\,.$$

Letting $v \equiv e^{iw}$, we solve the equation

$$v + \frac{1}{v} = 2z\,.$$

Multiplying by v, one obtains a quadratic equation for v,

$$v^2 - 2zv + 1 = 0\,. \tag{13}$$

The solution to eq. (13) is:

$$v = z + (z^2 - 1)^{1/2}\,.$$

Following the same steps as in the analysis of arcsine, we write

$$w = \arccos z = \frac{1}{i} \ln v = \frac{1}{i} \ln \left[z + (z^2 - 1)^{1/2}\right]\,, \tag{14}$$

where $(z^2 - 1)^{1/2}$ is the multi-valued square root function. More explicitly,

$$\arccos z = \frac{1}{i} \ln \left(z + |z^2 - 1|^{1/2} e^{\frac{i}{2} \arg(z^2 - 1)}\right)\,. \tag{15}$$

It is sometimes more convenient to rewrite eq. (15) in a slightly different form. Recall that

$$\arg(z_1 z_2) = \arg z + \arg z_2\,, \tag{16}$$

as a *set equality*. We now substitute $z_1 = z$ and $z_2 = -1$ into eq. (16) and note that $\arg(-1) = \pi + 2\pi n$ (for $n = 0, \pm 1, \pm 2, \ldots$) and $\arg z = \arg z + 2\pi n$ as a set equality. It follows that

$$\arg(-z) = \pi + \arg z\,,$$

as a set equality. Thus,

$$e^{\frac{i}{2} \arg(z^2 - 1)} = e^{i\pi/2}\, e^{\frac{i}{2} \arg(1 - z^2)} = i e^{\frac{i}{2} \arg(1 - z^2)}\,,$$

and we can rewrite eq. (14) as follows:

$$\arccos z = \frac{1}{i} \ln \left(z + i\sqrt{1 - z^2}\right)\,, \tag{17}$$

which is equivalent to the more explicit form,

$$\boxed{\arccos z = \frac{1}{i}\ln\left(z + i|1-z^2|^{1/2}e^{\frac{i}{2}\arg(1-z^2)}\right)}$$

The principal value of the arccosine is then defined as

$$\boxed{\text{Arccos}z = \frac{1}{i}\text{Ln}\left(z + i|1-z^2|^{1/2}e^{\frac{i}{2}\text{Arg}(1-z^2)}\right)} \tag{18}$$

We now examine the principal value of the arccosine for real-valued arguments. Setting $z = x$, where x is real,

$$\text{Arccos}\, x = \frac{1}{i}\text{Ln}\left(x + i|1-x^2|^{1/2}e^{\frac{i}{2}\text{Arg}(1-x^2)}\right).$$

As previously noted, for $|x| < 1$ we have $\text{Arg}(1-x^2) = 0$, and $|1-x^2| = \sqrt{1-x^2}$ defines the positive square root. Thus,

$$\begin{aligned}\text{Arccos}\, x &= \frac{1}{i}\text{Ln}\left(x + i\sqrt{1-x^2}\right) = \frac{1}{i}\left[\text{Ln}\left|x + i\sqrt{1-x^2}\right| + i\text{Arg}\left(x + i\sqrt{1-x^2}\right)\right] \\ &= \text{Arg}\left(x + i\sqrt{1-x^2}\right),\end{aligned} \tag{19}$$

since $x + i\sqrt{1-x^2}$ is a complex number with magnitude equal to 1. Moreover, $x + i\sqrt{1-x^2}$ lives either in the first or second quadrant of the complex plane, since $\text{Im}(x + i\sqrt{1-x^2}) \geq 0$. It follows that:

$$0 \leq \text{Arccos}\, x \leq \pi, \qquad \text{for} \quad |x| \leq 1.$$

The arcsine and arccosine functions are related in a very simple way. Using eq. (11),

$$\frac{i}{v} = \frac{i}{iz + \sqrt{1-z^2}} = \frac{i(-iz + \sqrt{1-z^2})}{(iz + \sqrt{1-z^2})(-iz + \sqrt{1-z^2})} = z + i\sqrt{1-z^2},$$

which we recognize as the argument of the logarithm in the definition of the arccosine [cf. eq. (17)]. Using eq. (4), it follows that

$$\arcsin z + \arccos z = \frac{1}{i}\left[\ln v + \ln\left(\frac{i}{v}\right)\right] = \frac{1}{i}\ln\left(\frac{iv}{v}\right) = \frac{1}{i}\ln i.$$

Since $\ln i = i(\frac{1}{2}\pi + 2\pi n)$ for $n = 0, \pm 1, \pm 2 \ldots$, we conclude that

$$\boxed{\arcsin z + \arccos z = \tfrac{1}{2}\pi + 2\pi n, \qquad \text{for} \quad n = 0, \pm 1, \pm 2, \ldots}$$

This result is even simpler for the corresponding principal values.

$$\begin{aligned}\mathrm{Arcsin}\, z + \mathrm{Arccos}\, z &= \frac{1}{i}\left[\mathrm{Ln}|v| + \mathrm{Ln}\left(\frac{1}{|v|}\right) + i\mathrm{Arg}\, v + i\mathrm{Arg}\left(\frac{i}{v}\right)\right] \\ &= \mathrm{Arg}\ v + \mathrm{Arg}\left(\frac{i}{v}\right)\,. \end{aligned} \tag{20}$$

Here v and i/v are defined using the principal value of the square root:

$$v = iz + |1-z^2|^{1/2} e^{\frac{i}{2}\mathrm{Arg}(1-z^2)}\,, \qquad\qquad \frac{i}{v} = z + i|1-z^2|^{1/2} e^{\frac{i}{2}\mathrm{Arg}(1-z^2)}\,.$$

In particular, since Re $(\pm iz) = \mp$Im z for any complex number z,

$$\mathrm{Re}\ v = -\mathrm{Im}\ z + |1-z^2|^{1/2}\cos\left[\tfrac{1}{2}\mathrm{Arg}(1-z^2)\right]\,, \tag{21}$$

$$\mathrm{Re}\left(\frac{1}{v}\right) = \mathrm{Im}\ z + |1-z^2|^{1/2}\cos\left[\tfrac{1}{2}\mathrm{Arg}(1-z^2)\right]\,. \tag{22}$$

One can now prove that

$$\mathrm{Re}\ v \geq 0\,, \tag{23}$$

for any complex number z by considering separately the cases of Im $z \leq 0$ and Im $z > 0$. Note that $-\pi < \mathrm{Arg}(1-z^2) \leq \pi$ implies that $\cos\left[\frac{1}{2}\mathrm{Arg}(1-z^2)\right] \geq 0$. Thus if Im $z \leq 0$, then it immediately follows from eq. (21) that Re $v \geq 0$. Likewise, if Im $z > 0$, then it immediately follows from eq. (22) that Re $(1/v) > 0$. However, the sign of the real part of any complex number z is the *same* as the sign of the real part of $1/z$, since

$$\frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}\,.$$

Hence, we again conclude that Re $v > 0$, and eq. (23) is proven.

It is straightforward to check that:

$$\mathrm{Arg}\ v + \mathrm{Arg}\left(\frac{i}{v}\right) = \tfrac{1}{2}\pi\,, \qquad \text{for Re } v \geq 0\,.$$

Hence, eq. (20) yields:

$$\boxed{\mathrm{Arcsin}\, z + \mathrm{Arccos}\, z = \tfrac{1}{2}\pi}$$

3. The inverse hyperbolic functions: arctanh and arccoth

Consider the solution to the equation

$$z = \tanh w = \frac{\sinh w}{\cosh w} = \left(\frac{e^w - e^{-w}}{e^w + e^{-w}}\right) = \left(\frac{e^{2w} - 1}{e^{2w} + 1}\right)\,.$$

We now solve for e^{2w},

$$z=\frac{e^{2w}-1}{e^{2w}+1} \qquad \Longrightarrow \qquad e^{2w}=\frac{1+z}{1-z}\,.$$

Taking the complex logarithm of both sides of the equation, we can solve for w,

$$w=\frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)\,.$$

The solution to $z=\tanh w$ is $w=\text{arctanh}z$. Hence,

$$\boxed{\text{arctanh}\, z=\frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)}$$

We can define the principal value of the inverse hyperbolic tangent function by employing the principal value of the logarithm,

$$\boxed{\text{Arctanh}\, z=\frac{1}{2}\text{Ln}\left(\frac{1+z}{1-z}\right)}$$

Similarly, by considering the solution to the equation

$$z=\coth w=\frac{\cosh w}{\sinh w}=\left(\frac{e^{w}+e^{-w}}{e^{w}-e^{-w}}\right)=\left(\frac{e^{2w}+1}{e^{2w}-1}\right)\,.$$

we end up with:

$$\boxed{\text{arccoth} z=\frac{1}{2}\ln\left(\frac{z+1}{z-1}\right)}$$

We can define the principal value of the inverse hyperbolic cotangent function by employing the principal value of the logarithm,

$$\boxed{\text{Arccoth}\, z=\frac{1}{2}\text{Ln}\left(\frac{z+1}{z-1}\right)}$$

The above results then yield:

$$\text{arccoth}(z)=\text{arctanh}\left(\frac{1}{z}\right)\,, \qquad\qquad \text{Arccoth}(z)=\text{Arctanh}\left(\frac{1}{z}\right)\,.$$

Finally, we note the relation between the inverse trigonometric and the inverse hyperbolic functions:

$$\begin{aligned} \text{arctanh}\, z &= i\,\text{arctan}(-iz)\,, & \qquad \text{arccoth}\, z &= i\,\text{arccot}(iz)\,,\\ \text{Arctanh}\, z &= i\text{Arctan}(-iz)\,, & \qquad \text{Arccoth}\, z &= i\,\text{Arccot}(iz)\,. \end{aligned}$$

4. The inverse hyperbolic functions: arcsinh and arccosh

The inverse hyperbolic sine function is the solution to the equation:

$$z = \sinh w = \frac{e^w - e^{-w}}{2}\,.$$

Letting $v \equiv e^w$, we solve the equation

$$v - \frac{1}{v} = 2z\,.$$

Multiplying by v, one obtains a quadratic equation for v,

$$v^2 - 2zv - 1 = 0\,. \tag{24}$$

The solution to eq. (24) is:

$$v = z + (1+z^2)^{1/2}\,. \tag{25}$$

Since z is a complex variable, $(1+z^2)^{1/2}$ is the complex square-root function. This is a multi-valued function with two possible values that differ by an overall minus sign. Hence, we do not explicitly write out the $\pm$ sign in eq. (25). To avoid ambiguity, we shall write

$$\begin{aligned} v &= z + (1+z^2)^{1/2} = z + e^{\frac{1}{2}\ln(1+z^2)} = z + e^{\frac{1}{2}\left[\mathrm{Ln}|1+z^2| + i\arg(1+z^2)\right]} \\ &= z + |1+z^2|^{1/2} e^{\frac{i}{2}\arg(1+z^2)}\,. \end{aligned}$$

By definition, $v \equiv e^w$, from which it follows that

$$w = \ln v = \ln\left(z + |1+z^2|^{1/2} e^{\frac{i}{2}\arg(1+z^2)}\right)\,.$$

The solution to $z = \sinh w$ is $w = \mathrm{arcsinh} z$. Hence,

$$\boxed{\mathrm{arcsinh} z = \ln\left(z + |1+z^2|^{1/2} e^{\frac{i}{2}\arg(1+z^2)}\right)}$$

The principal value of the inverse hyperbolic sine function is obtained by employing the principal value of the logarithm and the principle value of the square-root function (which corresponds to employing the principal value of the argument). Thus, we define

$$\boxed{\mathrm{Arcsinh}\, z = \mathrm{Ln}\left(z + |1+z^2|^{1/2} e^{\frac{i}{2}\mathrm{Arg}(1+z^2)}\right)}$$

The inverse hyperbolic cosine function is the solution to the equation:

$$z = \cos w = \frac{e^w + e^{-w}}{2}\,.$$

Letting $v \equiv e^w$, we solve the equation

$$v + \frac{1}{v} = 2z\,.$$

Multiplying by v, one obtains a quadratic equation for v,

$$v^2 - 2zv + 1 = 0\,. \tag{26}$$

The solution to eq. (26) is:

$$v = z + (z^2 - 1)^{1/2}\,.$$

Following the same steps as in the analysis of inverse hyperbolic sine function, we write

$$w = \operatorname{arccosh} z = \ln v = \ln\left[z + (z^2 - 1)^{1/2}\right]\,,$$

where $(z^2 - 1)^{1/2}$ is the multi-valued square root function. More explicitly,

$$\boxed{\operatorname{arccosh} z = \ln\left(z + |z^2 - 1|^{1/2} e^{\frac{i}{2}\arg(z^2-1)}\right)}$$

The corresponding principal value is:

$$\boxed{\operatorname{Arccosh} z = \operatorname{Ln}\left(z + |z^2 - 1|^{1/2} e^{\frac{i}{2}\operatorname{Arg}(z^2-1)}\right)} \tag{27}$$

Finally, we note the relation between the inverse trigonometric and the inverse hyperbolic functions:

$$\operatorname{arcsinh} z = i \arcsin(-iz)\,, \qquad \operatorname{arccosh} z = i \, \arccos z\,,$$

$$\operatorname{Arcsinh} z = i\operatorname{Arcsin}(-iz)\,.$$

Unfortunately, a comparison of eqs. (18) and (27) reveals that

$$\operatorname{Arccosh} z = \pm i \operatorname{Arccos} z\,,$$

where the sign depends on the location of z in the complex plane. I leave the determination of this condition as an exercise for the reader.

References

1. A comprehensive treatment of the properties of the inverse trigonometric and inverse hyperbolic functions can be found in Milton Abramowitz and Irene A. Stegun, *Handbook of Mathematical Functions* (Dover Publications, Inc., New York, 1972).

2. W. Kahan, *Branch Cuts for Complex Elementary Functions*, in *The State of Art in Numerical Analysis*, edited by A. Iserles and M.J.D. Powell (Clarendon Press, Oxford, UK, 1987) pp. 165–211.

3. R.M. Corless, D.J. Jeffrey, S.M. Watt and J.H. Davenport, *"According to Abramowitz and Stegun" or arccoth needn't be uncouth*, ACM SIGSAM Bulletin **34**, 58–65 (2000).