Introduction on Bernoulli's numbers

Pascal Sebah and Xavier Gourdon
numbers.computation.free.fr/Constants/constants.html

June 12, 2002

Abstract

This essay is a general and elementary overview of some of the properties of the famous numbers introduced by Bernoulli and used by Euler to express the value of the zeta function at integer even values.

1 Introduction

Bernoulli's numbers play an important and quite mysterious role in mathematics and in various places like analysis, number theory and differential topology. They first appeared in *Ars Conjectandi*, page 97, a famous (and posthumous) treatise published in 1713, by Jakob Bernoulli (1654-1705) when he studied the sums of powers of consecutive integers

$$s_p(n) = \sum_{k=1}^{n-1} k^p, \tag{1}$$

where p and n are two given positive integers.

Bernoulli's numbers also appear in the computation of the numbers

$$\zeta(2p) = \sum_{k=1}^{\infty} \frac{1}{k^{2p}}$$

and in the expansion of many usual functions as $\tan(x)$, $\tanh(x)$, $1/\sin(x)$, $\cdots$

Perhaps one of the most important result is *Euler-Maclaurin* summation formula, where Bernoulli's numbers are contained and which allows to accelerate the computation of slow converging series (see the essay on Euler's constant at [9]). They also appear in numbers theory (Fermat's theorem) and in many other domains and have caused the creation of a huge literature (see the 2700 and more entries enumerated in [6]).

According to Louis Saalschültz [17], the term *Bernoulli's numbers* was used for the first time by Abraham De Moivre (1667-1754) and also by Leonhard Euler (1707-1783) in 1755.

2 Bernoulli's approach

During the first years of the Calculus period and in its first integral computations of the function $x \mapsto x^p$, Pierre de Fermat (1601-1665) in 1636 had to evaluate the sums $s_p(n)$ defined by (1). You can see this by replacing the area under the curve $x \mapsto x^p$ by it's rectangular approximations and naturally comes the need to compute $s_p(n)$.

Also in 1631, Johann Faulhaber (1580-1635) developed explicit formulas for these sums up to $p = 17$ (read the excellent [12] for the beginnings of integration and [18] for some excerpts of Bernoulli's work). Thus, it was already known to Jakob Bernoulli that

$$
\begin{aligned}
s_0(n) &= n \\
s_1(n) &= \frac{1}{2}n^2 - \frac{1}{2}n = \frac{n(n-1)}{2} \\
s_2(n) &= \frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{1}{6}n = \frac{n(n-1)(2n-1)}{6} \\
s_3(n) &= \frac{1}{4}n^4 - \frac{1}{2}n^3 + \frac{1}{4}n^2 = \frac{n^2(n-1)^2}{4} \\
s_4(n) &= \frac{1}{5}n^5 - \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n = \frac{n(n-1)(2n-1)(3n^2-3n-1)}{30} \\
s_5(n) &= \frac{1}{6}n^6 - \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2 = \frac{n^2(2n^2-2n-1)(n-1)^2}{12} \\
&\ldots
\end{aligned}
$$

Jakob Bernoulli then, *empirically*, noticed that the polynomials $s_p(n)$ have the form

$$s_p(n) = \frac{1}{p+1}n^{p+1} - \frac{1}{2}n^p + \frac{p}{12}n^{p-1} + 0 \times n^{p-2} + \ldots$$

In this expression, the numbers $(1, -1/2, 1/12, 0, \ldots)$ are appearing and do not depend on p. More generally, the sums $s_p(n)$ can be written in the form

$$
\begin{aligned}
s_p(n) &= \sum_{k=0}^{p} \frac{B_k}{k!} \frac{p!}{(p+1-k)!} n^{p+1-k} \\
&= \frac{B_0}{0!}\frac{n^{p+1}}{p+1} + \frac{B_1}{1!}n^p + \frac{B_2}{2!}pn^{p-1} + \frac{B_3}{3!}p(p-1)n^{p-2} + \ldots + \frac{B_p}{1!}n
\end{aligned}
$$

where the B_k are numbers which are independent of p and called *Bernoulli's numbers.*

We find by identification

$$B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, \ldots$$

To illustrate the usefulness of his formula, Bernoulli computed the astonishing value of $s_{10}(1000)$ with little effort (in less than "half a quarter of an hour" he says ... [18])

914099242414242434242419242425OO

(you can check it !). To achieve this he needed to find B_0 up to B_{10}.

There is good evidence that the famous Japanese mathematician, Seki Takakazu (1642-1708) also discovered Bernoulli's numbers at the same time. The famous Indian mathematician Srinivasa Ramanujan (1887-1920) independently studied and rediscovered those numbers in 1904. He wrote one of his first article on this subject in 1911 [15].

3 A more modern definition

An equivalent definition of the Bernoulli's numbers is obtained from the series expansion of the function $z/(e^z - 1)$:

$$G(z) = \frac{z}{e^z - 1} = \sum_{k=0}^{\infty} B_k \frac{z^k}{k!} \qquad |z| < 2\pi \tag{2}$$

In other words, the *generating function* of the Bernoulli's numbers B_k is $z/(e^z - 1)$. The first terms of the expansion of this function are

$$G(z) = \left(1 + \frac{z}{2!} + \frac{z^2}{3!} + \frac{z^3}{4!} + ...\right)^{-1} = 1 - \frac{z}{2} + \frac{z^2}{12} - \frac{z^4}{720} + \frac{z^6}{30240} - \frac{z^8}{1209600} + ...$$

which permit to obtain the first value of the Bernoulli's numbers:

$$\begin{array}{rcllll} B_0 = 1 & , & B_1 = -\frac{1}{2}, & B_2 = \frac{1}{6}, & B_3 = 0, & \\ B_4 = -\frac{1}{30} & , & B_5 = 0, & B_6 = \frac{1}{42}, & B_7 = 0, & B_8 = -\frac{1}{30}. \end{array}$$

Further, we observe that

$$G(z) + \frac{z}{2} = \frac{z}{2}\left(\frac{2}{e^z - 1} + 1\right) = \frac{z}{2}\frac{e^{z/2} + e^{-z/2}}{e^{z/2} - e^{-z/2}} = \frac{z}{2}\coth\frac{z}{2},$$

where coth is the hyperbolic tangent, hence $G(z) + z/2$ is an even function and consequently every Bernoulli's numbers of the form B_{2k+1} $(k > 0)$ is null.

3.1 Bernoulli's polynomials

With a little modification it's possible to define Bernoulli's polynomials $B_k(x)$ by

$$G(z, x) = \frac{ze^{zx}}{e^z - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{z^k}{k!}$$

and $G(z,0) = G(z)$ hence the value of the function $B_k(x)$ at $x = 0$ is B_k and because

$$\frac{\partial G}{\partial x}(z,x) = zG(z,x) = \sum_{k=0}^{\infty} \frac{dB_k}{dx}(x)\frac{z^k}{k!}$$

it follows the important relation

$$\frac{dB_k}{dx}(x) = kB_{k-1}(x).$$

Then, it's easy to deduce that $B_k(x)$ are polynomials of degree k, and the first one are

$$\begin{aligned}
B_0(x) &= 1 \\
B_1(x) &= x - \frac{1}{2} \\
B_2(x) &= x^2 - x + \frac{1}{6} \\
B_3(x) &= x^3 - \frac{3}{2}x^2 + \frac{1}{2}x \\
B_4(x) &= x^4 - 2x^3 + x^2 - \frac{1}{30} \\
B_5(x) &= x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x \\
& \ldots
\end{aligned}$$

Thanks to Bernoulli's polynomials, it's possible to rewrite the expression of the sums $s_p(n)$ as

$$s_p(n) = \sum_{k=0}^{n-1} k^p = \frac{1}{p+1}\left(B_{p+1}(n) - B_{p+1}\right).$$

There are many relations with these polynomials, for example

$$\begin{aligned}
B_k(1-x) &= (-1)^k B_k(x), \\
(-1)^k B_k(-x) &= B_k(x) + kx^{k-1}, \\
|B_{2k}(x)| &< |B_{2k}| \qquad k = 1, 2, \ldots \text{ and } 0 < x < 1, \\
B_k\left(\frac{1}{2}\right) &= -(1 - 2^{1-k})B_k \qquad k = 0, 1, \ldots \\
& \ldots
\end{aligned}$$

Consult [1] for other formulas.

4 Properties

4.1 Recurrence relation

In the expression

$$s_p(n) = \sum_{k=0}^{p} \frac{B_k}{k!} \frac{p!}{(p+1-k)!} n^{p+1-k}$$

we let $n = 1$, giving

$$0 = \sum_{k=0}^{p} \frac{B_k}{k!} \frac{1}{(p+1-k)!}$$

or equivalently

$$B_p = -\frac{1}{p+1} \sum_{k=0}^{p-1} \binom{p+1}{k} B_k. \tag{3}$$

The recurrence relation (3) allows an easy generation of Bernoulli's numbers and shows that the numbers B_p are all rational numbers. It's convenient to rewrite this relation as the symbolic equation

$$(B+1)^{p+1} - B^{p+1} = 0,$$

and expand the binomial $(B+1)^{p+1}$ where each power B^k must be replaced by B_k.

Example 1 *With $p = 4$ we have*

$$5B_4 + 10B_3 + 10B_2 + 5B_1 + B_0 = 0$$

thus

$$5B_4 + \left(0 + 10.\frac{1}{6} - 5.\frac{1}{2} + 1\right) = 0 \quad \textit{therefore} \quad B_4 = -\frac{1}{30}.$$

4.2 Bernoulli's numbers and the zeta function

In 1735, the solution of the *Basel problem,* expressed by Jakob Bernoulli some years before, was one of Euler's most sensational discovery. The problem was to find the limit of

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2},$$

he found it to be $\pi^2/6$. He also discovered the values of the sums

$$\zeta(2k) = \sum_{n=1}^{\infty} \frac{1}{n^{2k}}$$

for k up to 13 ([8] and for more details [7]).

It's an extraordinary result that $\zeta(2k)$ can be expressed with Bernoulli's numbers ; the values of these sums are given by

$$\zeta(2k) = \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{4^k \, |B_{2k}| \, \pi^{2k}}{2(2k)!} \qquad k > 0, \tag{4}$$

(a proof based on two different expansions of $z\,\cot(z)$ is given in [4] p. 383). No similar expression is known for the odd values of the Zeta function.

On the extension of this function to negative values, we also have

$$\zeta(1-2k) = -\frac{B_{2k}}{2k} \qquad k > 0,$$

which may also be used to compute B_{2k} (see [5]).

4.3 Asymptotic expansion of Bernoulli's numbers

From the previous relation (4) with the Zeta function, it's clear that

$$|B_{2k}| = \frac{2(2k)!}{(2\pi)^{2k}} \zeta(2k) \tag{5}$$

and because, when k becomes large, thanks to Stirling's formula

$$\begin{aligned} \zeta(2k) &\sim 1, \\ (2k)! &\sim (2k)^{2k} e^{-2k} \sqrt{4\pi k}, \end{aligned}$$

we have

$$|B_{2k}| \sim 4 \left(\frac{k}{\pi e} \right)^{2k} \sqrt{\pi k}.$$

In [16], the following results describes how the numerator N_{2k} of B_{2k} grows with k:

$$\log |N_{2k}| = 2k \log(k) + O(k).$$

4.4 Bounds

It may be useful to estimate bounds for B_{2k}, to achieve this we use the following relation between the function $\zeta(s)$ and the alternating series $\zeta_a(s)$

$$\begin{aligned} \zeta(s) &= \frac{\zeta_a(s)}{1 - 2^{1-s}} \\ \zeta_a(s) &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} \end{aligned}$$

and since

$$1 - \frac{1}{2^s} < \zeta_a(s) < 1,$$

we have the bounds for $\zeta(s)$

$$\frac{1-2^{-s}}{1-2^{1-s}} < \zeta(s) < \frac{1}{1-2^{1-s}}.$$

If we use this relation with (5), the bounds for B_{2k} are therefore

$$\frac{2(2k)!(1-4^{-k})}{(2\pi)^{2k}(1-2.4^{-k})} < |B_{2k}| < \frac{2(2k)!}{(2\pi)^{2k}(1-2.4^{-k})}.$$

5 Clausen-von Staudt's theorem

The following famous and important theorem was published in 1840 by Karl von Staudt (1798-1867) and it allows to compute easily the fractional part of Bernoulli's numbers (thus it also permits to compute the denominator of those numbers). This theorem was discovered the same year, independently, by Thomas Clausen (1801-1885).

Theorem 2 *The value B_{2k}, added to the sum of the inverse of prime numbers p such that $(p-1)$ divides $2k$, is an integer. In other words,*

$$-B_{2k} \equiv \sum_{(p-1)|2k} \frac{1}{p} \pmod 1$$

Proof. *A complete proof is given in [11], p. 91.* ■

When $k>1$, we observe that the primes $p=2,3$ are such as $(p-1)$ divides $2k$. Let's illustrate this theorem with a few examples. For $k=1$, it becomes

$$\begin{aligned} -B_2 &\equiv \sum_{(p-1)|2} \frac{1}{p} = \frac{1}{2}+\frac{1}{3} = \frac{5}{6} \pmod 1 \\ B_2 &\equiv \frac{1}{6} \pmod 1 \end{aligned}$$

for $k=5$

$$\begin{aligned} -B_{10} &\equiv \sum_{(p-1)|10} \frac{1}{p} = \frac{1}{2}+\frac{1}{3}+\frac{1}{11} = \frac{61}{66} \pmod 1 \\ B_{10} &\equiv \frac{5}{66} \pmod 1 \end{aligned}$$

and for $k=8$

$$\begin{aligned} -B_{16} &\equiv \sum_{(p-1)|16} \frac{1}{p} = \frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{17} = \frac{47}{510} \pmod 1 \\ B_{16} &\equiv \frac{463}{510} \pmod 1 \end{aligned}$$

Corollary 3 *(due to Rado 1934) For every prime numbers k of the form $3n+1$*

$$B_{2k} \equiv \frac{1}{6} \pmod 1.$$

Proof. *It's an easy consequence of the Staudt's theorem since $p-1$ divides $2k = 2(3n+1)$ only if $p-1$ is one of $1, 2, 3n+1, 6n+2$, that is p is one of $2, 3, 3n+2, 6n+3$. But $6n+3$ is divisible by 3 and $3n+2$ is divisible by 2 because $3n+1$ is prime so the only primes p candidates are 2 and 3.* ■

Example 4 *The first primes of the form $3n+1$ are $7, 13, 19, 31, 37, 43, 61, 67, \ldots$ hence we have*

$$B_{14} \equiv B_{26} \equiv B_{38} \equiv B_{62} \equiv B_{74} \equiv B_{86} \equiv B_{122} \equiv B_{134} \equiv \frac{1}{6} \pmod 1.$$

Clausen-von Staudt's theorem also permits to compute exactly a Bernoulli's number as soon as a sufficiently good approximation of it is known.

6 Expansion of usual functions

In a previous section we gave the definition

$$\frac{z}{e^z - 1} = \sum_{k=0}^{\infty} B_k \frac{z^k}{k!} \qquad |z| < 2\pi$$

and the consequence

$$\frac{z}{2} \coth \frac{z}{2} = \sum_{k=0}^{\infty} B_{2k} \frac{z^{2k}}{(2k)!}.$$

It obviously leads to the two following expansions

$$\begin{aligned} z\coth(z) &= \sum_{k=0}^{\infty} 4^k B_{2k} \frac{z^{2k}}{(2k)!} = 1 + \frac{z^2}{3} - \frac{z^4}{45} + \frac{2z^6}{945} - \ldots \qquad |z| < \pi \\ z\cot(z) &= \sum_{k=0}^{\infty} (-4)^k B_{2k} \frac{z^{2k}}{(2k)!} = 1 - \frac{z^2}{3} - \frac{z^4}{45} - \frac{2z^6}{945} - \ldots \qquad |z| < \pi \end{aligned}$$

where $\cot(z) = \cos(z)/\sin(z) = i\coth(iz)$ is the cotangent function.

Now it's possible to find the expansion for $\tanh(z)$ and $\tan(z)$, if we observe that

$$2\coth(2z) - \coth(z) = 2\frac{\cosh(2z)}{\sinh(2z)} - \frac{\cosh(z)}{\sinh(z)} = \frac{\cosh^2(z) + \sinh^2(z)}{\sinh(z)\cosh(z)} - \frac{\cosh(z)}{\sinh(z)} = \tanh(z)$$

so that

$$\tanh(z) = \sum_{k=1}^{\infty} 4^k (4^k - 1) B_{2k} \frac{z^{2k-1}}{(2k)!} = z - \frac{z^3}{3} + \frac{2z^5}{15} - \frac{17z^7}{315} + \ldots \qquad |z| < \frac{\pi}{2}$$

$$\tan(z) = \sum_{k=1}^{\infty}(-4)^k(1-4^k)B_{2k}\frac{z^{2k-1}}{(2k)!} = z + \frac{z^3}{3} + \frac{2z^5}{15} + \frac{17z^7}{315} + ... \qquad |z| < \frac{\pi}{2}.$$

Bernoulli's numbers also occur in the expansions of other classical functions

$$\frac{z}{\sin(z)}, \frac{z}{\sinh(z)}, \log\left(\frac{\sin(z)}{z}\right), \log(\cos(z)), \log\left(\frac{\tan(z)}{z}\right), ...$$

6.1 Series

Setting $z = 1$ or $z = -1$ in the expansion (2) of $z/(e^z - 1)$ leads to the fast converging series

$$\frac{1}{e-1} = \sum_{k=0}^{\infty}\frac{B_k}{k!} = \frac{1}{2} + \sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}$$

and

$$\frac{B_{2k}}{(2k)!} \sim \frac{2}{(4\pi^2)^k} \approx \frac{2}{39.48^k}.$$

In one of his famous *notebook*, Ramanujan stated without proof the following result:

Theorem 5 *Let (a, b) two positive real numbers such as $ab = \pi^2$, let $n > 1$ an integer, then*

$$a^n\sum_{k=1}^{\infty}\frac{k^{2n-1}}{e^{2ak}-1} - (-b)^n\sum_{k=1}^{\infty}\frac{k^{2n-1}}{e^{2bk}-1} = (a^n - (-b)^n)\frac{B_{2n}}{4n}.$$

Proof. *See [3].* ■

Corollary 6 *Let $n \geq 1$, then*

$$\sum_{k=1}^{\infty}\frac{k^{4n+1}}{e^{2\pi k}-1} = \frac{B_{4n+2}}{8n+4}.$$

Proof. *Just apply the theorem with $a = b = \pi$ and replace n by $2n + 1$.* ■

It's interesting to compare this result with the classical integral representation valid for $n \geq 1$

$$\int_0^{\infty}\frac{x^{2n-1}}{e^{2\pi x}-1}dx = (-1)^{n-1}\frac{B_{2n}}{4n}$$

which implies that for $n \geq 0$

$$\int_0^{\infty}\frac{x^{4n+1}}{e^{2\pi x}-1}dx = \frac{B_{4n+2}}{8n+4}.$$

7 Euler-Maclaurin formula

Let $f(x)$ be a function of class C^{2p+2} on an interval $[a,b]$ and let $h=(b-a)/m$ a subdivision of this interval into m equal parts then we have the important result:

Theorem 7 *There exist* $0<\vartheta<1$ *and*

$$\begin{aligned}\sum_{k=0}^{m} f(a+kh) &= \frac{1}{h}\int_a^b f(x)dx+\frac{1}{2}\left(f(a)+f(b)\right)+\sum_{k=1}^{p}\frac{h^{2k-1}}{(2k)!}B_{2k}\left(f^{(2k-1)}(b)-f^{(2k-1)}(a)\right)\\ &\quad+\frac{h^{2p+2}}{(2p+2)!}B_{2p+2}\sum_{k=0}^{m-1}f^{(2p+2)}(a+kh+\vartheta h).\end{aligned}$$

Proof. *A proof is given in [10].* ■

This formula was first studied by Euler in 1732 and independently by Colin Maclaurin (1698-1746) in 1742 [13]. Euler used it to compute sums of slow converging series and Maclaurin used it as a numerical quadrature formula.

With the same conditions, setting $n=m+1, a=1, b=n, h=1$ the theorem becomes:

Theorem 8

$$\sum_{k=1}^{n} f(k)=\int_1^n f(x)dx+\frac{1}{2}\left(f(1)+f(n)\right)+\sum_{k=1}^{p}\frac{B_{2k}}{(2k)!}\left(f^{(2k-1)}(n)-f^{(2k-1)}(1)\right)+R_n(f,p),$$

where $R_n(f,p)$ *is the remainder bounded by*

$$R_n(f,p)\le\frac{2}{(2\pi)^{2p}}\int_1^n\left|f^{(2p+1)}(x)\right|dx$$

7.1 Applications

1. $f(x)=x^2$, the remainder is null since $f^{(p)}(x)=0$ for $p>2$:

$$\begin{aligned}\sum_{k=1}^{n}k^2 &= \int_1^n x^2dx+\frac{1}{2}\left(1+n^2\right)+\frac{B_2}{2}(2n-2)+0\\ &= \frac{n^3-1}{3}+\frac{1+n^2}{2}+\frac{n-1}{6}=\frac{n(n+1)(2n+1)}{6}.\end{aligned}$$

2. $f(x)=1/x$, $f^{(2k-1)}(x)=-(2k-1)!/x^{2k}$, Euler-Maclaurin formula yields for a given p:

$$\sum_{k=1}^{n}\frac{1}{k}-\log(n)=\frac{1}{2}+\frac{1}{2n}+\sum_{k=1}^{p}\frac{B_{2k}}{2k}\left(1-\frac{1}{n^{2k}}\right)+R_n(f,p)$$

when $n \to \infty$, the left hand side of the equality tends to γ (Euler's constant) and the equality gives

$$\gamma = \frac{1}{2} + \sum_{k=1}^{p} \frac{B_{2k}}{2k} + R_\infty(f,p),$$

finally

$$\sum_{k=1}^{n} \frac{1}{k} - \log(n) = \gamma + \frac{1}{2n} - \sum_{k=1}^{p} \frac{B_{2k}}{2k} \frac{1}{n^{2k}} + \left(R_n(f,p) - R_\infty(f,p)\right).$$

(check that $R_n(f,p) - R_\infty(f,p) = O(1/n^{2p+2})$)

3. $f(x) = \log(x)$, with the same method (left as exercise) Euler-Maclaurin formula becomes

$$\sum_{k=1}^{n} \log(k) = n\log(n) - n + \frac{\log(n)}{2} + \log(\sqrt{2\pi}) + \sum_{k=1}^{p} \frac{B_{2k}}{2k(2k-1)} \frac{1}{n^{2k-1}} + O\left(\frac{1}{n^{2p+1}}\right)$$

so, for example, with $p = 3$

$$\log(n!) = n\log(n) - n + \log(\sqrt{2\pi n}) + \frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5} + O\left(\frac{1}{n^7}\right)$$

and taking the exponential

$$n! = n^n e^{-n} \sqrt{2\pi n} \exp\left(\frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5} + O\left(\frac{1}{n^7}\right)\right).$$

This is the *asymptotic Stirling formula*. Using the series expansion of the exponential function near the origin, it's more convenient to write it as

$$n! = n^n e^{-n} \sqrt{2\pi n} \left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} - \frac{571}{2488320n^4} + \frac{163879}{209018880n^5} + O\left(\frac{1}{n^6}\right)\right)$$

8 Bernoulli's numbers and Fermat's last theorem

The famous *Fermat's last theorem* states that the equation

$$x^n + y^n = z^n$$

never has non-zero integer solutions for $n > 2$. Since Fermat expressed this result around 1630, the pursuit of a proof occupied generations of mathematicians.

A big step was made in 1850 by Ernst Kummer (1810-1893) when he proved Fermat's theorem for $n = p$, whenever p is, what is called today, a *regular prime.* Kummer gave the beautiful regularity criterion:

p is a regular prime if and only if p does not divide the numerator of $B_2, B_4, ..., B_{p-3}$.

He showed that all primes before 37 where regular, hence Fermat's theorem was proved for those primes. 37 is the first non regular prime because it divides the numerator of

$$B_{32} = -\frac{7709321041217}{510} = -\frac{208360028141 \times 37}{510}.$$

The next irregular primes (less than 300) are

$$59, 67, 101, 103, 131, 149, 157, 233, 257, 263, 271, 283, 293, ...$$

For example, 157 divides the numerators of B_{62} and B_{110}.

Thanks to arithmetical properties of Bernoulli's numbers, Johann Ludwig Jensen (1859-1925) proved in 1915 that the number of irregular primes is infinite. Even if it's probable that the number of regular primes is infinite, a proof remains unknown [16].

9 The first Bernoulli's numbers

9.1 First numbers

Here is the list of the first Bernoulli's numbers. Except for B_1 numbers of the form B_{2k+1} are null.

$$\begin{aligned}
B_0 &= 1\\
B_1 &= -1/2\\
B_2 &= 1/6\\
B_4 &= -1/30\\
B_6 &= 1/42\\
B_8 &= -1/30\\
B_{10} &= 5/66,\\
B_{12} &= -691/2730\\
B_{14} &= 7/6\\
B_{16} &= -3617/510\\
B_{18} &= 43867/798\\
B_{20} &= -174611/330\\
B_{22} &= 854513/138\\
B_{24} &= -236364091/2730
\end{aligned}$$

$$
\begin{aligned}
B_{26} &= 8553103/6 \\
B_{28} &= -23749461029/870 \\
B_{30} &= 8615841276005/14322 \\
B_{32} &= -7709321041217/510 \\
B_{34} &= 2577687858367/6 \\
B_{36} &= -26315271553053477373/1919190 \\
B_{38} &= 2929993913841559/6 \\
B_{40} &= -261082718496449122051/13530 \\
&\ldots
\end{aligned}
$$

More numbers are given in [1] and in [17].

9.2 Some computations

Bernoulli himself computed the numbers that now bear his name up to B_{10}. Later, Euler computed these numbers up to B_{30}, then Martin Ohm extended the calculation up to B_{62} in 1840 [14]. A few years later, in 1877, Adams made the impressive computation of all Bernoulli's numbers up to B_{124} (or B_{62}^* according to his convention) [2]. For instance, the numerator of B_{124} has 110 digits and the denominator is the number 30.

In 1996, Simon Plouffe and Greg J. Fee computed B_{200000} a huge number of about 800000 digits, the computation took about 2 hours on a work station. In 2002, the same authors improved the record to B_{600000} which has 2727474 digits by a 12 hours computation on a personal computer. The method is based on the formula (5) which allow a direct computation of the required number without the need to compute the previous ones.

References

[1] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions*, Dover, New York, (1964)

[2] J.C. Adams, *On the calculation of Bernoulli's numbers up to B_{62} by means of Staudt's theorem*, Rep. Brit. Ass., (1877)

[3] B.C. Berndt, *Ramanujan's Notebooks, Part II*, Springer-Verlag, New York, (1989)

[4] J.M. Borwein and P.B. Borwein, *Pi and the AGM - A study in Analytic Number Theory and Computational Complexity*, A Wiley-Interscience Publication, New York, (1987)

[5] P. Borwein, *An efficient algorithm for the Riemann Zeta function*, (1995)

[6] K. Dilcher, *A Bibliography of Bernoulli Numbers*, World Wide Web site at the address: `http://www.mscs.dal.ca/~dilcher/bernoulli.html`

[7] W. Dunham, *Euler The Master of Us All*, The Mathematical Association of America, (1999)

[8] L. Euler, *Introduction à l'analyse infinitésimale* (french traduction by Labey), Barrois, ainé, Librairie, (original 1748, traduction 1796), vol. 1

[9] X. Gourdon and P. Sebah, *Numbers, Constants and Computation*, World Wide Web site at the address: `http://numbers.computation.free.fr/Constants/constants.html`, (1999)

[10] R.L. Graham, D.E. Knuth and O. Patashnik, *Concrete Mathematics*, Addison-Wesley, (1994)

[11] G.H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Oxford Science Publications, (1979)

[12] V.J. Katz, *A History of Mathematics-An Introduction*, Addison-Wesley, (1998)

[13] C. Maclaurin, *A Treatise of fluxions*, Edinburgh, (1742)

[14] M. Ohm, *Etwas über die Bernoulli'schen Zahlen*, J. Reine Angew. Math., (1840), vol. 20, p. 11-12

[15] S. Ramanujan, *Some properties of Bernoulli's numbers*, J. Indian Math. Soc., (1911), vol. 3, p. 219-234

[16] P. Ribenboim, *The new Book of Prime Number Records*, Springer, (1996)

[17] L. Saalschütz, *Vorlesungen über die Bernoullischen Zahlen*, Berlin, Verlag von Julius Springer, (1893)

[18] D.E. Smith, *A Source Book in Mathematics*, Dover Publications, New York, (1959, first edition 1929)