The Euler constant : γ

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$$\gamma = 0.57721566490153286060651209008240243104215933593992\ldots$$

1 Introduction

Euler's Constant was first introduced by Leonhard Euler (1707-1783) in 1734 as

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log(n)\right). \tag{1}$$

It is also known as the *Euler-Mascheroni constant*. According to Glaisher [4], the use of the symbol γ is probably due to the geometer Lorenzo Mascheroni (1750-1800) who used it in 1790 while Euler used the letter C.

The constant γ is deeply related to the Gamma function $\Gamma(x)$ thanks to the Weierstrass formula

$$\frac{1}{\Gamma(x)} = x\exp(\gamma x)\prod_{n>0}\left[\left(1+\frac{x}{n}\right)\exp\left(-\frac{x}{n}\right)\right].$$

This identity entails the relation

$$\Gamma'(1) = -\gamma. \tag{2}$$

It is not known if γ is an *irrational* or a *transcendental* number. The question of its irrationality has challenged mathematicians since Euler and remains a famous unresolved problem. By computing a large number of digits of γ and using continued fraction expansion, it has been shown that if γ is a rational number p/q then the denominator q must have at least 242080 digits.

Even if γ is less famous than the constants π and e, it deserves a great attention since it plays an important role in *Analysis* (Gamma function, Bessel functions, exponential-integral, ...) and occurs frequently in *Number Theory* (order of magnitude of arithmetical functions for instance [11]).

2 Computation of the Euler constant

2.1 Basic considerations

Direct use of formula (1) to compute Euler constant is of poor interest since the convergence is very slow. In fact, using the harmonic number notation

$$H_n = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n},$$

[1]This pages are from *//numbers.computation.free.fr/Constants/constants.html*

we have the estimation

$$H_n - \log(n) - \gamma \sim \frac{1}{2n}.$$

This estimation is the first term of an asymptotic expansion which can be used to compute effectively γ, as shown in next section. Nevertheless, other formulae for γ (see next sections) provide a simpler and more efficient way to compute it at a large accuracy. Better estimations are :

$$\begin{array}{rcccll}
\frac{1}{2(n+1)} & < & H_n - \log(n) - \gamma & < & \frac{1}{2n} & \text{(Young [13])} \\
0 & < & H_n - \frac{\log(n)+\log(n+1)}{2} - \gamma & < & \frac{1}{6n(n+1)} & \text{(Cesaro)} \\
\frac{-1}{48n^3} & < & H_n - \log(n+\frac{1}{2}+\frac{1}{24n}) - \gamma & < & \frac{-1}{48(n+1)^3}. & \text{(Negoi)}
\end{array}$$

Application of the third relation with $n = 100$ gives

$$-0.6127.10^{-9} < 0.577215664432730 - \gamma < 0$$

and $n = 1000$ gives

$$-0.6238.10^{-13} < 0.577215664901484 - \gamma < 0.$$

A similar estimation is given in [6].

2.2 Asymptotic expansion of the harmonic numbers

The Euler-Maclaurin summation can be used to have a complete asymptotic expansion of the harmonic numbers. We have (see the essay on Bernoulli's numbers)

$$H_n - \log(n) \approx \gamma + \frac{1}{2n} - \sum_{k\geq 1} \frac{B_{2k}}{2k}\frac{1}{n^{2k}}, \tag{3}$$

where the B_{2k} are the Bernoulli numbers. Since B_{2k} grows like $2(2k)!/(2\pi)^{2k}$, the asymptotic expansion should be stopped at a given k. For example, the first terms are given by

$$\gamma = H_n - \log(n) - \frac{1}{2n} + \frac{1}{12n^2} - \frac{1}{120n^4} + \frac{1}{252n^6} - \frac{1}{240n^8} + \frac{1}{132n^{10}} - \frac{691}{32760n^{12}} + \frac{1}{12n^{14}}.$$

This technique, *directly* inherited from the definition, can be employed to compute γ at a high precision but suffers from two major drawbacks :

- It requires the computation of the B_{2k}, which is not so easy ;
- the rate of convergence is not so good compared to other formulas with γ.

2.2.1 Euler's computation

In 1736, Euler used the asymptotic expansion 3 to compute the first 16 decimal digits of γ. He went up to $k = 7$ and $n = 10$, and wrote

$$\gamma = H_{10} - \log(10) - \frac{1}{20} + \frac{1}{1200} - \frac{1}{1,200,000} + \frac{1}{252,000,000} - \frac{1}{24,000,000,000} + ...$$

with

$$\begin{aligned} H_{10} &= 2.9289682539682539 \\ \log(10) &= 2.30258509299404568 4 \end{aligned}$$

giving the approximation

$$\gamma \approx 0.5772156649015329.$$

2.2.2 Mascheroni's mistake

During the year 1790, in "*Adnotationes ad calculum integrale Euleri*", Mascheroni made a similar calculation up to 32 decimal places. But, a few years later, in 1809, Johann von Soldner (1766-1833) found a value of the constant which was in agreement only with the first 19 decimal places of Mascheroni's calculation ... Embarrassing !

It was in 1812, supervised by the famous Mathematician Gauss, that a young calculating prodigy Nicolai (1793-1846) evaluated γ up to 40 correct decimal places, in agreement with Soldner's value [4].

In order to avoid such miscalculations (see also William Shanks famous error on his determination of the value of π), digits hunters are usually doing two different calculations to check the result.

2.2.3 Stieltjes approach

In 1887, Stieltjes computed $\zeta(2), \zeta(3), ..., \zeta(70)$ to 32 decimal places and extended a previous calculation done by Legendre up to $\zeta(35)$ with 16 digits. He was then able to compute Euler's constant to 32 decimal places thanks to the fast converging series

$$\gamma = 1 - \log(\frac{3}{2}) - \sum_{k=1}^{\infty} \frac{(\zeta(2k+1) - 1)}{4^k(2k+1)}. \tag{4}$$

For large values of k we have

$$\zeta(2k+1) - 1 = \frac{1}{2^{2k+1}} + \frac{1}{3^{2k+1}} + \cdots \sim \frac{1}{2^{2k+1}}$$

hence the series converges geometrically :

$$\frac{\zeta(2k+1) - 1}{4^k} \sim \frac{1}{2.16^k},$$

This relation is issued from properties of the Gamma function and a proof is given in the Gamma function essay.

The first partial sums of series (4) are

$$\begin{aligned}
x_0 &= 0.5(9453489189183561...) = 1 - \log(\frac{3}{2}) \\
x_1 &= 0.577(69681662853609...) = \frac{13}{12} - \log(\frac{3}{2}) - \frac{\zeta(3)}{12} \\
x_5 &= 0.57721566(733782033...) \\
x_{10} &= 0.57721566490153(417...)
\end{aligned}$$

2.2.4 Knuth's computation

In 1962, Knuth used a computer to approximate γ with the Euler-Maclaurin expansion (3), with the parameters $k = 250$ and $n = 10^4$. The error is about

$$\epsilon_{k,n} = \frac{B_{(2k+2)}}{(2k+2)}\frac{1}{n^{(2k+2)}} \approx \frac{2(2k+2)!}{(2k+2)(2\pi n)^{2k+2}} \approx 10^{-1272}$$

and Knuth gave 1271 decimal places of γ [8].

Some numerical results on the error function

To appreciate the rate of convergence of this algorithm we give a table of the approximative number of digits one can find with different values for k and n. This integer in the table is the number of digits of $1/\epsilon_{k,n}$.

	$k = 10$	$k = 100$	$k = 250$	$k = 500$
$n = 10^3$	63	390	769	1235
$n = 10^4$	85	592	*1272*	2237
$n = 10^5$	107	794	1773	3239
$n = 10^6$	129	996	2275	4241

This table shows that the Euler-Maclaurin summation could not be reasonably used to compute more than a few thousands of decimal places of γ.

2.3 Exponential integral methods

An efficient way to compute decimal digits of the Euler constant is to start from the identity $\gamma = -\Gamma'(1)$ (see (2)) which entails for any positive integer n, after integrating by part the formula

$$\gamma + \log(n) = I_n - R_n, \qquad I_n = \int_0^n \frac{1-e^{-t}}{t}\,dt, \quad R_n = \int_n^\infty \frac{e^{-t}}{t}\,dt.$$

Plugging the series expansion of $(1-e^{-t})/t$ in I_n, we obtain

$$I_n = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{n^k}{k \cdot k!}.$$

The value I_n is an approximation to γ with the error bound $R_n = O(e^{-n})$. By stopping the summation at the right index, we obtain the following formula which provides an efficient way to approximate the Euler constant :

$$\gamma = \sum_{k=1}^{\alpha n} (-1)^{k-1} \frac{n^k}{k \cdot k!} - \log n + O(e^{-n}), \qquad \alpha = 3.5911\ldots \tag{5}$$

The constant α is such that $n^{\alpha n}/(\alpha n)!$ is of order e^{-n}, and satisfies $\alpha(\log(\alpha) - 1) = 1$. To obtain d decimal places of γ with (5), the formula should be used with $n \simeq d\log(10)$ and computations should be done with a precision of $2d$ decimal places to compensate cancellations in the sum I_n. This method was used by Sweeney in 1963 to compute 3566 decimal places of γ [9].

A refinement is obtained by approximating R_n by its asymptotic expansion, leading to the formula

$$\gamma = \sum_{k=1}^{\beta n} (-1)^{k-1} \frac{n^k}{k \cdot k!} - \log(n) - \frac{e^{-n}}{n} \sum_{k=0}^{n-2} \frac{k!}{(-n)^k} + O(e^{-2n}), \quad \beta = 4.32\ldots \tag{6}$$

The constant β is such that $\beta(\log(\beta) - 1) = 2$. This improvement, also due to Sweeney [9], permits to take $n \simeq d/2\log(10)$ and to work with a precision of $3d/2$ decimal places to obtain d decimal places of γ.

Notice that R_n can be approximated as accurately as desired by using Euler's continued fraction

$$e^n R_n = 1/n + 1/1 + 1/n + 2/1 + 2/n + 3/1 + 3/n + \cdots$$

This can be used to improve the efficiency of the technique, but leads to a much more complicated algorithm.

More information about this technique can be found in [12].

2.4 Bessel function method

A better method (see also [12]) is based on the modified Bessel functions and leads to the formula

$$\gamma = \frac{A_n}{B_n} - \log(n) + O(e^{-4n}),$$

with

$$A_n = \sum_{k=0}^{\alpha n} \left(\frac{n^k}{k!}\right)^2 H_k, \qquad B_n = \sum_{k=0}^{\alpha n} \left(\frac{n^k}{k!}\right)^2,$$

where $\alpha = 3.5911\ldots$ satisfies $\alpha(\log(\alpha) - 1) = 1$.

This technique is quite easy, fast and it has a great advantage compared to Exponential integral techniques : to obtain d decimal places of γ, the intermediate computations can be done with d decimal places.

A refinement can be obtained from an asymptotic series of the error term. It consists in computing

$$C_n = \frac{1}{4n} \sum_{k=0}^{2n} \frac{[(2k)!]^3}{(k!)^4 (16n)^{2k}}.$$

Brent and McMillan in [12] suggest that

$$\gamma = \frac{A_n}{B_n} - \frac{C_n}{B_n^2} - \log(n) + O(e^{-8n}). \tag{7}$$

This time, the summations in A_n and B_n should go up to βn where $\beta = 4.970625759\ldots$ satisfies $\beta(\log(\beta) - 1) = 3$. The error $O(e^{-8n})$ followed an empirical evidence but the result had not been proved by Brent and McMillan. Formula (7) has been used by Xavier Gourdon with a *binary splitting process* to obtain more than 100 millions decimal digits of γ in 1999.

Unlike the constant π with the *AGM iteration* for instance, no quadratically (or more) convergent algorithm is known for γ.

3 Collection of formulae for the Euler constant

Integral and series formulae for the Euler constant can be found in *Collection of formulae for the Euler constant*.

4 Records of computation

Number of digits	When	Who	Notes
5	1734	L. Euler	He found $\gamma = 0.577218$.
15	1736	L. Euler	The Euler-Maclaurin summation was used [1].
19	1790	L. Mascheroni	Mascheroni computed 32 decimal places, but only 19 were correct.
24	1809	J. von Soldner	In a work on the logarithm-integral function.
40	1812	F.B.G. Nicolai	In agreement with Soldner's calculation.
19	1825	A.M. Legendre	Euler-Maclaurin summation was used with $n = 10$ [2].
34	1857	Lindman	Euler-Maclaurin summation was used with $n = 100$.
41	1861	Oettinger	Euler-Maclaurin summation was used with $n = 100$.
59	1869	W. Shanks	Euler-Maclaurin summation was used with $n = 1000$.
110	1871	W. Shanks	
263	1878	J.C. Adams	Adams also computed the first 62 Bernoullian numbers [5].
32	1887	T. J. Stieltjes	He used a series based on the zeta function.
???	1952	J.W. Wrench	Euler-Maclaurin summation [7].
1271	1962	D.E. Knuth	Euler-Maclaurin summation [8].
3566	1962	D.W. Sweeney	The exponential integral method was used [9].
20,700	1977	R.P. Brent	Brent used Sweeney's approach [10].
30,100	1980	R.P. Brent and E.M. McMillan	The Bessel function method [12] was used
172,000	1993	J. Borwein	A variant of Brent's method was used.
1,000,000	1997	T. Papanikolaou	This is the first gamma computation based on a binary splitting approach. He used a Sun SPARC Ultra, and the computation took 160 hours. He also proved that if γ is rational, its denominator has at least 242080 decimal digits.
7,286,255	1998 Dec.	X. Gourdon	Sweeney's method (with $N = 2^{23}$) with binary splitting was used. The computation took 47 hours on a SGI R10000 (256 Mo). The verification was done with the value $N = 2^{23} + 1$.
108,000,000	1999 Oct.	X. Gourdon and P. Demichel	Formula (7) was used with a *binary splitting process.* The program was from X. Gourdon and Launched by P. Demichel on a HP J5000, 2 processors PA 8500 (440 Mhz) with 2 Go of memory.

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