Here is a collection of practice problems covering material from Chapters 3 and 10 of Boas and homework sets 7–9. Along with the first two practice problem sets, this may help you in preparing for the final exam.

1. One of the eigenvalues of the matrix A is $\lambda = 0$. Prove that A^{-1} does not exist.

2. If $A^{\mathsf{T}} = -A$, then we say that A is a skew-symmetric (or antisymmetric) matrix. Prove that if A is antisymmetric and B is symmetric, then $\mathrm{Tr}(AB) = 0$.

HINT: Note that for any matrix M, $\mathrm{Tr}\, M^{\mathsf{T}} = \mathrm{Tr}\, M$.

3. Let A be a complex $n \times n$ matrix. Prove that the eigenvalues of $AA^\dagger$ are real and non-negative.

HINT: Let $\vec{\boldsymbol{w}} = A^\dagger \vec{\boldsymbol{v}}$, where $\vec{\boldsymbol{v}}$ is an eigenvector of $AA^\dagger$. Investigate the consequence of the fact that the inner product $\langle \vec{\boldsymbol{w}} | \vec{\boldsymbol{w}} \rangle$ is non-negative in a complex Euclidean space.

4. Suppose that the $n \times n$ matrix A is diagonalizable. Then, we can find an invertible matrix S such that $S^{-1}AS = D$ where D is diagonal. Show that $A^n = S^{-1}D^nS$, where n is a positive integer. This provides a simple way to compute A^n since raising a diagonal matrix to a power is easy. Using these considerations, compute M^{10} where

$$M = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} .$$

5. Determine whether the following matrices are diagonalizable. If diagonalizable, indicate whether it is possible to diagonalize the matrix with a unitary similarity transformation.

$$\text{(a) } A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} , \qquad \text{(b) } A = \begin{pmatrix} 1 & -4 & 2 \\ -4 & 1 & -2 \\ 2 & -2 & -2 \end{pmatrix} .$$

6. A linear transformation T that is represented by a matrix M with respect to the standard basis $\mathcal{B}$ is represented by the matrix $P^{-1}MP$ with respect to a different basis $\mathcal{B}'$.

(a) Show that the characteristic equation for determining the eigenvalues of T does not depend on the choice of basis.

HINT: Note that $P^{-1}MP - \lambda\mathbf{I} = P^{-1}(M - \lambda\mathbf{I})P$.

(b) Show that the eigenvalues of T do not depend on the choice of basis.

7. Four equal mass balls lying along the x-axis are attached by three springs. The two outermost balls are fixed, while the two innermost balls are free to oscillate in the x direction. Denote the displacement from equilibrium of the two balls by x_1 and x_2. The total potential energy of the system is:

$$V = \tfrac{1}{2}kx_1^2 + \tfrac{1}{2}k(x_2 - x_1)^2 .$$

(a) Show that the equations of motion for the two displacements can be cast into the matrix equation:

$$\frac{d^2\vec{\boldsymbol{x}}}{dt^2} = A\vec{\boldsymbol{x}} ,$$

where $\vec{\boldsymbol{x}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and A is a 2×2 matrix.

(b) Diagonalize the matrix A and determine the two possible frequencies of vibrations. These are the *normal modes* of the system.

8. If A is a diagonalizable matrix, prove that

$$e^{\mathrm{Tr}A} = \det e^A .$$

HINT: First prove this result for a diagonal matrix. Then, try to prove the more general result by diagonalizing A. This result is true for *any* matrix A, but if A is not diagonalizable, a more sophisticated technique is required.

9. A linear transformation A is represented by the matrix:

$$A = \begin{pmatrix} 1 & -4 & 2 \\ -4 & 1 & -2 \\ 2 & -2 & -2 \end{pmatrix} ,$$

with respect to the standard basis $\mathcal{B} = \{(1,0,0), (0,1,0), (0,0,1)\}$. Consider a new basis, $\mathcal{B}' = \{(2,-2,1), (1,1,0), (-1,1,4)\}$, where the components of the basis vectors of $\mathcal{B}'$ are given with respect to the standard basis $\mathcal{B}$.

(a) What are the components of the basis vectors of $\mathcal{B}$ when expressed relative to the basis $\mathcal{B}'$?

(b) Determine the matrix representation of A relative to the basis $\mathcal{B}'$.

10. Consider tensors that live in n-dimensional Euclidean space.

(a) How many independent components does a symmetric second-rank tensor, S_{ij}, possess?

(b) How many independent components does an antisymmetric second-rank tensor, A_{ij}, possess?

11. In three dimensional space, the components of the vector cross product (in rectangular coordinates) is defined as

$$(\vec{\boldsymbol{b}} \times \vec{\boldsymbol{c}})_i = \sum_{j=1}^{3} \sum_{k=1}^{3} \epsilon_{ijk} b_j c_k \,,$$

where ϵ_{ijk} is the Levi-Civita symbol.

(a) Using the formula for the determinant given on p. 509 of Boas, prove that:

$$\vec{\boldsymbol{a}} \cdot (\vec{\boldsymbol{b}} \times \vec{\boldsymbol{c}}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} .$$

(b) Using the properties of the determinant, prove that

$$\begin{aligned} &\text{(i)} \quad \vec{\boldsymbol{a}} \cdot (\vec{\boldsymbol{b}} \times \vec{\boldsymbol{c}}) = \vec{\boldsymbol{b}} \cdot (\vec{\boldsymbol{c}} \times \vec{\boldsymbol{a}}) = \vec{\boldsymbol{c}} \cdot (\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}}) \,, \\ &\text{(ii)} \quad \vec{\boldsymbol{a}} \cdot (\vec{\boldsymbol{b}} \times \vec{\boldsymbol{c}}) = (\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}}) \cdot \vec{\boldsymbol{c}} \,, \qquad \text{[interchange of the dot and cross]} \,, \\ &\text{(iii)} \quad \vec{\boldsymbol{a}} \cdot (\vec{\boldsymbol{b}} \times \vec{\boldsymbol{c}}) = -\vec{\boldsymbol{c}} \cdot (\vec{\boldsymbol{b}} \times \vec{\boldsymbol{a}}) \,. \end{aligned}$$

12. A product of Levi-Civita ϵ symbols can be expressed in terms of products of Kronecker deltas.

(a) Show that the following determinantal identity is satisfied:

$$\epsilon_{ijk}\epsilon_{\ell mn} = \begin{vmatrix} \delta_{i\ell} & \delta_{im} & \delta_{in} \\ \delta_{j\ell} & \delta_{jm} & \delta_{jn} \\ \delta_{k\ell} & \delta_{km} & \delta_{kn} \end{vmatrix} .$$

HINT: You may find eq. (5.5) on p. 509 of Boas useful.

(b) Set $k = \ell$ in part (a) and sum the resulting expression from $k = 1$ to 3. Show that the result coincides with eq. (5.8) on p. 510 of Boas.