DUE: THURSDAY FEBRUARY 11, 2010

To receive full credit, you must exhibit the intermediate steps that lead you to your final results. The nth problem in Boas from section $a.b$ is designated by $a.b$–n.

1. In class, we showed that the volume V_n of an n-dimensional hypersphere with radius $R = 1$ is given by

$$V_n = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}\,.$$

We remarked that as a function of increasing n, V_n first increases and then decreases, approaching zero as $n \to \infty$.

(a) Using Stirling's approximation for the gamma function, prove the assertion that $\lim_{n\to\infty} V_n = 0$.

(b) Using Stirling's approximation for the logarithm of the gamma function, compute the value of n at which V_n is a maximum. [*HINT:* First, estimate the location of the maximum of $\ln(V_n)$ by evaluating the derivative of $\ln(V_n)$ with respect to n and setting the derivative equal to zero. (In computing the derivative, you may neglect at first approximation any term that vanishes for large n.) Argue that your result also provides the approximate value of n for which V_n is a maximum.]

(c) Compute V_n for values of integer n near its maximum and determine which integer n corresponds to the largest value of V_n. Compare your result with part (b) and comment. For those of you who are more ambitious, use a calculator or a computer algebraic system (e.g. Mathematica or Maple) to determine the actual (non-integer) value of n for which V_n is maximal.

2. In class, we derived the following result. For any non-negative integer n,

$$\lim_{x\to -n}(x+n)\Gamma(x) = \frac{(-1)^n}{n!}\,. \tag{1}$$

In this problem, you shall determine the *behavior* of $\Gamma(x)$ as $x \to -n$.

(a) Prove that for an infinitesimal quantity, $|\epsilon| \ll 1$, eq. (1) is equivalent to the following result:

$$\Gamma(-n+\epsilon) \simeq \frac{(-1)^n}{n!}\,\frac{1}{\epsilon}\,.$$

(b) Prove that the behavior of $\Gamma(x)$ as $x \to -n$ is given by:

$$\Gamma(-n+\epsilon) \simeq \frac{(-1)^n}{n!}\left[\frac{1}{\epsilon} + \psi(n+1) + \mathcal{O}(\epsilon)\right]\,, \tag{2}$$

where $\psi(n+1)$ is the logarithmic derivative of the gamma function evaluated at the integer $n+1$, which was shown in class to be equal to:

$$\psi(n+1) = -\gamma + \sum_{k=1}^{n} \frac{1}{k},$$

and $\gamma \equiv -\Gamma'(1) \simeq 0.5772156649\cdots$ is Euler's constant.

HINT: Justify the following steps.

$$\begin{aligned}
\epsilon\Gamma(-n+\epsilon) &= \frac{\epsilon\Gamma(1-n+\epsilon)}{-n+\epsilon} = \frac{\epsilon\Gamma(2-n+\epsilon)}{(-n+\epsilon)(1-n+\epsilon)} = \cdots \\
&= \frac{\epsilon\Gamma(\epsilon)}{(-n+\epsilon)(1-n+\epsilon)(2-n+\epsilon)\cdots(-1+\epsilon)} \\
&= \frac{(-1)^n\Gamma(1+\epsilon)}{(n-\epsilon)(n-1-\epsilon)(n-2-\epsilon)\cdots(1-\epsilon)} \\
&= \frac{(-1)^n\Gamma(1+\epsilon)}{n!\left(1-\frac{\epsilon}{n}\right)\left(1-\frac{\epsilon}{n-1}\right)\left(1-\frac{\epsilon}{n-2}\right)\cdots(1-\epsilon)}\,.
\end{aligned}$$

Then, expand the right hand side above in a power series around $\epsilon = 0$, keeping only terms up to and including $\mathcal{O}(\epsilon)$. Do not forget to expand $\Gamma(1+\epsilon)$ to first order in ϵ. Show that the end result (after dividing by ϵ in the final step) is that of eq. (2).

3. The logarithmic derivative of the gamma function is defined by

$$\psi(x) \equiv \frac{d}{dx}\ln\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}\,.$$

(a) Starting from $\Gamma(x+1) = x\Gamma(x)$, take two derivatives and show that

$$\psi'(x+1) = -\frac{1}{x^2} + \psi'(x)\,, \tag{3}$$

where $\psi'(x)$ denotes the derivative of $\psi(x)$ with respect to x.

(b) Use the result of part (a) to show that for any non-negative integer n,

$$\psi'(n+1) = \psi'(1) - \sum_{k=1}^{n} \frac{1}{k^2}\,. \tag{4}$$

HINT: Use eq. (3) repeatedly for $x = 1, 2, \ldots, n$.

(c) Starting with Stirling's approximation for $\ln\Gamma(x+1)$, prove that

$$\lim_{n\to\infty} \psi'(n+1) = 0\,.$$

(d) Taking the $n \to \infty$ limit of eq. (4), compute $\psi'(1)$ and $\Gamma''(1)$, where $\Gamma''(1)$ is the second derivative of the gamma function $\Gamma(x)$ evaluated at $x = 1$.

4. In class, we derived the leading term of the asymptotic expansion of $\Gamma(x+1)$, which is valid for $x \to \infty$. In this problem, you will compute the first correction to Stirling's formula as follows. Starting with

$$\Gamma(1+x) = \int_0^\infty e^{x\ln t - t}\, dt\,,$$

one can expand the argument of the exponent in a Taylor series about $t = x$. We will need to keep four terms in this series:

$$x\ln t - t \simeq x\ln x - x - \frac{(t-x)^2}{2x} + \frac{(t-x)^3}{3x^2} - \frac{(t-x)^4}{4x^3}\,.$$

(a) Inserting this expansion into the integral above, and changing the integration variable to

$$u \equiv \frac{t-x}{\sqrt{2x}}\,,$$

show that

$$\Gamma(1+x) \simeq \sqrt{2x}e^{x\ln x - x}\int_{-\sqrt{x/2}}^\infty du\, \exp\left(-u^2 + \frac{2\sqrt{2}u^3}{3\sqrt{x}} - \frac{u^4}{x}\right)\,.$$

(b) Since x is assumed to be large, we can replace the lower limit of the integral by $-\infty$ (the resulting error in making this approximation is *exponentially* small). Moreover, the integrand can be approximated in the limit of large x to be of the form:

$$\exp\left(-u^2 + \frac{2\sqrt{2}u^3}{3\sqrt{x}} - \frac{u^4}{x}\right) = e^{-u^2}\exp\left(\frac{2\sqrt{2}u^3}{3\sqrt{x}} - \frac{u^4}{x}\right) \simeq e^{-u^2}\left[1 + \frac{A(u)}{\sqrt{x}} + \frac{B(u)}{x}\right]\,,$$

where $A(u)$ and $B(u)$ are simple u-dependent polynomials. Determine the explicit forms for $A(u)$ and $B(u)$.

(c) Using the results for $A(u)$ and $B(u)$ obtained in part (b), complete the analysis by computing the integral over u:

$$\Gamma(1+x) \simeq \sqrt{2x}e^{x\ln x - x}\int_{-\infty}^\infty du\, e^{-u^2}\left[1 + \frac{A(u)}{\sqrt{x}} + \frac{B(u)}{x}\right]\,. \tag{5}$$

Show that the final result is of the form:

$$\Gamma(x+1) \simeq \sqrt{2\pi x}e^{-x}x^x\left[1 + \frac{C}{x}\right]\,,$$

where C is determined from your computation of the integral in eq. (5).

HINT: The integrals that you need to evaluate in part (c) are very simply related to the integrals of problem 1 of homework set #4.

5. Boas, p. 558, problem 11.12–1.

6. Boas, p. 559, problem 11.12–12. To obtain the numerical value of this integral, either use the relevant mathematical software, or employ the appropriate expansion obtained in problem 5.

7. Boas, p. 88, problem 3.2–13.

8. Boas, p. 88, problem 3.2–15.

9. Boas, p. 122, problem 3.6–6.

10. Boas, p. 122, problem 3.6–7.