1. The reflection formula for the gamma function is: $\Gamma(x)\Gamma(1-x) = \pi/\sin(\pi x)$.

   (a) Multiplying this equation by x, deduce an expression for $\Gamma(1+x)\Gamma(1-x)$.

   (b) The integral definition of the gamma function,

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt\,,$$

converges for all complex z such that Re $z > 0$. Show that:

$$\overline{\Gamma(z)} = \Gamma(\overline{z})\,,$$

where $\overline{z}$ is the complex conjugate of z and $\overline{\Gamma(z)}$ is the complex conjugate of $\Gamma(z)$.

   (c) Using the results of parts (a) and (b), prove that:

$$|\Gamma(1+iy)|^2 = \frac{\pi y}{\sinh(\pi y)}\,,$$

where y is a real number.

2. Use Stirling's formula to evaluate the following two limits:

$$\text{(a)}\ \lim_{n\to\infty} \frac{\Gamma(n+\frac{3}{2})}{\sqrt{n}\,\Gamma(n+1)}\,, \qquad\qquad \text{(b)}\ \lim_{n\to\infty} \frac{(n!)^{1/n}}{n}\,.$$

3. Let A be a 3×3 matrix. Assume that $A \neq 0$. The determinant of A is denoted by $\det A$.

   (a) Is the equation $\det(3A) = 3\det A$ true or false? Explain.

   (b) Suppose that $\det A = 1$. Let B be a matrix obtained from A by permuting the order of the rows so that the first row of A is the second row of B, the second row of A is the third row of B and the third row of A is the first row of B. (This is called a *cyclic permutation*.) What is the value of $\det B$?

   (c) Suppose that the 3×3 matrix $A \neq 0$ but $\det A = 0$. What can you say about the rank of A?

4. Consider the system of equations:

$$
\begin{aligned}
x_1 + 3x_2 - x_3 &= 4\,, \\
x_1 + 2x_2 + x_3 &= 2\,, \\
3x_1 + 7x_2 + x_3 &= c\,,
\end{aligned}
$$

where c is some unspecified real number.

(a) Is there any value of c for which there is a unique solution to the system of equations above? Explain your answer.

(b) There exists one value of c for which there are an infinite number of solutions to the above system of equations. Find that value of c and determine the allowed solutions.

HINT: Solve the system of equations with c arbitrary by constructing the augmented matrix and reducing it to reduced row echelon form. At the end of your computation, you can read off the required value of c.