Here is a collection of practice problems suitable for the second midterm exam. The exam will cover material from Chapter 3 (sections 1–3 and 6–8) and Chapter 11 of Boas and homework sets 4–6.

1. Here are some quickies involving the gamma and beta functions. Do not use a calculator or a numerical table. Instead, invoke an appropriate gamma function property when needed.

(a) Evaluate $\Gamma(-\frac{5}{2})$.

(b) Evaluate $\lim_{x\to 0} \Gamma(2x)/\Gamma(x)$.

(c) Evaluate $B(\frac{1}{4}, \frac{3}{4})$ in terms of some well known numerical constants.

2. Evaluate $\psi(\frac{1}{2})$, where $\psi(x) \equiv d \ln \Gamma(x)/dx$.

HINT: The simplest way to solve this problem is to make use of the duplication formula for the gamma function (which you derived in problem 7 of homework set #4).

3. A professor posts the final grades by listing the students by the last four digits of their social security numbers. But, is this a good idea? In this problem, you will compute the probability that at least two students have social security numbers such that the last four digits are identical.

Suppose the class has n students. The probability that the last four digits of all the social security numbers of students in the class are distinct is given by:

$$P_0 = 1 \cdot \frac{N-1}{N} \cdot \frac{N-2}{N} \cdots \frac{(N-n+1)}{N},$$

where N is number of possible values for the last four digits of a social security number.

(a) Show that P_0 can be rewritten as a ratio of two factorials divided by N^n.

(b) Using Stirling's approximation, and assuming that $N \gg n \gg 1$ and $N - n \gg 1$, derive a simple formula for $\ln P_0$ as a function of n and N. In your final expression, you may drop terms of $\mathcal{O}(1/N^2)$.

(c) Given P_0, the probability that at least two social security numbers belonging to students of the class possess the same last four digits is $P = 1 - P_0$. Using the approximation obtained in part (b), find the value of n such that in a class of n students or more, the probability that two students will have social security numbers with the same last four digits is greater than 1/2.

4. Using the properties of the beta and gamma functions, evaluate the following integral in terms of *elementary* functions,

$$\int_0^\infty \frac{dx}{(1+x)x^p}\,,$$

assuming that p is a real variable that satisfies $0 < p < 1$.

5. The dilogarithm $\mathrm{Li}_2(x)$ is a special function defined via the following integral:

$$\mathrm{Li}_2(x) \equiv -\int_0^1 \frac{\ln(1-xt)}{t}\,dt\,.$$

(a) Derive the power series expansion for $\mathrm{Li}_2(x)$ about $x = 0$.

HINT: Expand the integrand in a power series and integrate term by term.

(b) Using the results of part(a), evaluate $\mathrm{Li}_2(1)$ and $\mathrm{Li}_2(-1)$ by summing the series.

6. Consider the function defined by the following integral:

$$f(x) = \int_0^\infty \frac{e^{-t}\,dt}{1+xt}\,, \qquad \text{for } \; x \geq 0\,. \tag{1}$$

(a) Evaluate the integral by the following method. First, expand $(1+xt)^{-1}$ as a power series. Then, insert the resulting series into eq. (1) and integrate term by term. Write the final result for $f(x)$ in summation notation.

(b) Prove that the sum obtained in part (a) diverges for all $x \neq 0$. This indicates that the derivation in part (a) involves an illegal mathematical step (otherwise, the resulting sum would have been convergent for some non-zero interval of x). Can you identify the illegal step?

(c) By a suitable change of integration variable, show that $f(x)$ can be expressed in terms of the incomplete gamma function. Then, using the asymptotic series for the incomplete gamma function obtained in problem 8 of homework set #4, deduce the asymptotic series for $f(x)$. How does this series compare with the result of part (a)?

7. Compute the inverse A^{-1} of the following matrix:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

8. Consider the system of equations:

$$\begin{aligned} 5x + 2y + z &= 2 \\ x + y + 2z &= 1 \\ 3x - 3z &= 0\,. \end{aligned}$$

(a) What is the augmented matrix for this system of equations?

(b) Solve this system of equations using Gaussian elimination.

(c) What is the rank of the augmented matrix of part (b)?

(d) Remove the third equation above, and solve the new system of two equations and three unknowns using Gaussian elimination. What is the rank of the corresponding augmented matrix?

9. Evaluate the following determinant by hand:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{vmatrix}\,.$$

HINT: Simplify the matrix by employing appropriate row and/or column operations.

10. In special relativity, the space-time coordinates of two inertial frames moving at relative constant velocity v are related by

$$\begin{aligned} x' &= \gamma(x - vt)\,, \\ t' &= \gamma(t - vx/c^2)\,, \end{aligned}$$

where $\gamma \equiv (1 - v^2/c^2)^{-1/2}$ and c is the velocity of light.

(a) Rewrite this system of equations in matrix form.

(b) Using Cramer's rule, solve for x and t in terms of x' and t'.

11. Consider the following matrices:

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 3 & -1 & 0 \\ 0 & 5 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 3 & -1 & 0 \end{pmatrix},$$

Compute AB, BA, $\det A$, $\det B$, $\det AB$ and $\det BA$. Verify that $AB \neq BA$ and $\det AB = (\det A)(\det B)$.

12. Consider the matrix:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} . \tag{2}$$

(a) Show that $A^n = 0$ for all integer $n \geq 3$.

(b) Evaluate $(\mathbf{I} - A)^{-1}$, where $\mathbf{I}$ is the 3×3 identity matrix.

(c) Check your answer to part (b) by the following sophisticated method. Use the geometric series to define:

$$(\mathbf{I} - A)^{-1} = \sum_{k=0}^{\infty} A^k .$$

where $A^0 = \mathbf{I}$ (assuming that the sum converges). Using the result of part (a), sum the series when A is given by eq. (2), and verify that you reproduce the result of part (b).

BONUS PROBLEM (just for fun):

13. Consider the following definition of the Bernoulli numbers B_n:

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} .$$

To simplify the notation, let us define $b_n \equiv B_n/n!$. Then it follows that:

$$x = (e^x - 1) \left(\sum_{n=0}^{\infty} b_n x^n \right) = \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \right) (1 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots) .$$

(a) If two polynomials are equal, then all their coefficients must be equal (can you justify this assertion?). Using this result, show that you get a system of linear equations for the b_n.

(b) Consider the first n linear equations thus obtained. Solve these equations using Cramer's rule and show that:

$$B_n = n! b_n = (-1)^n n! \begin{vmatrix} 1/2! & 1 & 0 & \cdots & 0 \\ 1/3! & 1/2! & 1 & \cdots & 0 \\ 1/4! & 1/3! & 1/2! & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1/(k+1)! & 1/k! & 1/(k-1)! & \cdots & 1/2! \end{vmatrix} .$$

(c) Check the formula given in part (b) by computing B_1, B_2 and B_3.