INSTRUCTIONS: This is a three-hour exam. During the exam, you may refer to the textbook, the class handouts, or your own personal notes. Collaboration with your neighbor is strictly forbidden. In answering the questions, it is not sufficient to simply write the final result. You must provide the intermediate steps needed to arrive at the solution in order to get full credit.

The exam consists of six problems with a total of 16 parts. Each part is worth ten points, for a total of 160 points.

1. Consider the function $F(x) = \dfrac{x}{1 - x - 2x^2}$.

(a) Use the method of partial fractions to express $F(x)$ in terms of a sum (or difference) of two simpler terms. [*HINT:* factor the denominator.]

(b) Express $F(x)$ as a power series about $x = 0$,

$$F(x) = \sum_{n=0}^{\infty} a_n x^n \,. \tag{1}$$

This is most easily done by separately expanding the two terms obtained in part (a) and then combining the two sums. Determine a closed-form expression for a_n as a function of n. Write out the first seven values of a_n (for $n = 0, 1, 2, \ldots, 6$).

(c) What is the radius of convergence of the series obtained in eq. (1)?

2. Evaluate the following integral:

$$\int_{-\infty}^{\infty} e^{-t^2} \cos(2xt)\, dt \tag{2}$$

in two different ways.

(a) Before we attempt to integrate eq. (2), consider a related integral,

$$\int_{-\infty}^{\infty} e^{-t^2} e^{2ct}\, dt = e^{c^2} \int_{-\infty}^{\infty} e^{-(t-c)^2}\, dt \,. \tag{3}$$

By a change of variables, $u = t - c$, evaluate the integral on the right-hand side of eq. (3). Use this result to evaluate the integral given eq. (2) by first writing $\cos(2xt) = \text{Re}\, e^{2ixt}$. Then, you may choose $c = ix$, assuming that your result for eq. (3) is still valid for a purely imaginary c.

(b) Expand $\cos(2xt)$ in a Taylor series about $x = 0$. Integrate term by term, and sum the resulting series. Can you reproduce the answer obtained in part (a)?

HINT: Using the duplication formula for the gamma function, given on p. 545 of Boas, show that

$$\frac{\Gamma(n+\frac{1}{2})}{\Gamma(2n+1)} = \frac{\sqrt{\pi}}{2^{2n} n!} .$$

Use this result to simplify the series obtained at the end of part (b). You should then be able to sum the series in closed form.

3. A complex number $x + iy$ can be represented by the 2×2 matrix

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix} , \tag{4}$$

where x and y are real numbers. Verify that this is a sensible representation by answering the following questions.

(a) Show that the matrix representation of $(x + iy)(a + ib)$ is equal to

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix} \begin{pmatrix} a & -b \\ b & a \end{pmatrix} .$$

To show this, you should express the product $(x+iy)(a+ib)$ in the form of $X+iY$ and show that the matrix product above, when evaluated, is consistent with the form given by eq. (4).

(b) Show that the matrix representation of the complex number $\dfrac{1}{x+iy}$ is correctly given by the inverse of eq. (4).

(c) How is the determinant of the matrix given in eq. (4) related to the corresponding complex number, $x + iy$?

4. Consider the following interesting series of numbers:

$$0, 1, 1, 3, 5, 11, 21, \ldots \tag{5}$$

This series has been generated by the following rules. First, we define

$$x_0 = 0 \quad \text{and} \quad x_1 = 1\,. \tag{6}$$

Then for all positive integers $n = 1, 2, 3, \ldots,$

$$x_{n+1} = x_n + 2x_{n-1}\,. \tag{7}$$

Starting with $x_0 = 0$ and $x_1 = 1$, we can derive the values for x_2, x_3, $x_4, \ldots$ sequentially. For example, setting $n = 1$ in eq. (7) yields $x_2 = x_1 + 2x_0 = 1$. Next we can determine $x_3 = x_2 + 2x_1 = 3$ followed by $x_4 = x_3 + 2x_2 = 5$, etc. However, this is a very inefficient way of computing x_n for some large value of n (as it would take n separate computations).

Matrix methods can help us derive a simple rule for directly determining an arbitrary term x_n in the series. Consider the matrix equation:

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix}\,. \tag{8}$$

(a) Show that this matrix equation is equivalent to the rule given in eq. (7).

(b) Defining the matrix:

$$M \equiv \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}\,,$$

which appears in eq. (8), prove that for any non-negative integer n:

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = M^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}\,. \tag{9}$$

HINT: Verify eq. (9) for $n = 0$ and $n = 1$. Then iterate the process using eq. (8).

(c) Compute M^n (for arbitrary n) by first diagonalizing the matrix M and raising the resulting diagonal matrix to the nth power. Once you have obtained an expression for M^n, use eq. (9) to write an explicit formula for x_n as a function of n. Check that your formula reproduces the series given in eq. (5).

5. We have learned two methods in this class for computing the inverse of a matrix. One method involves row reduction and the second method involves the transpose of the cofactor matrix. Consider the matrix

$$M = \begin{pmatrix} 4 & 0 & -1 \\ -2 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}. \qquad (10)$$

(a) Using one of the two methods mentioned above, compute M^{-1}. Check your result by computing MM^{-1}.

(b) Here is a third method for computing M^{-1}. Diagonalize M and take the inverse of the diagonalizing equation. Then solve for M^{-1} (your formula should involve the inverse of a diagonal matrix, which can be obtained by inspection). Apply this technique to the matrix M given by eq. (10). Verify that the result obtained for M^{-1} by this method is correct.

(c) Here is a fourth method for computing M^{-1}. By the Cayley-Hamilton theorem, M solves its own characteristic equation. Compute the characteristic equation for the matrix M given by eq. (10). Multiply this equation by M^{-1}, and show that M^{-1} can be expressed in terms of M^2, M and the identity matrix. Use this result to evaluate M^{-1}, and compare with the results of parts (a) and (b).

6. A totally antisymmetric third-rank Cartesian tensor B_{ijk} is defined by the property that B_{ijk} changes sign if any two of its indices are interchanged.

(a) If i, j, and k can assume the values 1, 2 or 3, determine the number of non-zero components of B_{ijk}. How many components of B_{ijk} vanish? You may assume that the component B_{123} is nonzero.

(b) Show that B_{ijk} is proportional to the Levi-Civita tensor ϵ_{ijk}. What is the constant of proportionality?