Evaluation of $\zeta(2n)$ in terms of Bernoulli numbers

In this note, I will provide an elementary method for evaluating*

$$\zeta(2n) \equiv \sum_{k=1}^{\infty} \frac{1}{k^{2n}}\,.$$

The method relies on a particular formula, whose *rigorous* proof I shall omit. However, I will provide some motivation for why this formula is true.

The formula we require is:

$$\frac{\sin x}{x} = \lim_{n\to\infty}\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{(2\pi)^2}\right)\cdots\left(1-\frac{x^2}{(n\pi)^2}\right)\,. \tag{1}$$

How might we attempt to justify such a formula? Consider a polynomial of degree n of the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0\,.$$

We shall call this a finite-order polynomial since the highest degree that appears is a finite positive integer. We know that an nth order polynomial has n roots.† If we call those roots‡ r_1, r_2, $\ldots$, r_n, then we can rewrite the polynomial in the form:

$$P(x) = a_n(x-r_1)(x-r_2)\cdots(x-r_n)\,.$$

An alternative (and equivalent) way of factoring this polynomial is:

$$P(x) = P(0)\left(1-\frac{x}{r_1}\right)\left(1-\frac{x}{r_2}\right)\cdots\left(1-\frac{x}{r_n}\right)\,, \tag{2}$$

where $P(0) = a_0$. Both forms above satisfy $P(r_i) = 0$ $[i = 1, 2, \ldots, n]$.

The Taylor series of $(\sin x)/x$ about $x = 0$ is given by:

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots \tag{3}$$

Thus, $(\sin x)/x$ is a polynomial, albeit of infinite order. Recall that $\sin n\pi = 0$ for any integer n. Since $\lim_{x\to 0}(\sin x)/x = 1$, it follows that the roots of $(\sin x)/x$

*Further details can be found in a wonderful book by George Boros and Victor H. Moll, *Irresitible Integrals*," sections 6.8 and 6.9.

†By definition, if r is a root then $P(r) = 0$.

‡Some of the r_i may not be distinct, in which case they are called multiple roots.

occur at $x = n\pi$ for $n = \pm 1, \pm 2, \pm 3, \ldots$. If eq. (2) is valid for polynomials of infinite order, then we could conclude that:

$$\frac{\sin x}{x} = \lim_{n\to\infty} \left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 + \frac{x}{2\pi}\right)\cdots\left(1 - \frac{x}{n\pi}\right)\left(1 + \frac{x}{n\pi}\right).$$

Multiplying out each pair of adjacent terms then yields eq. (1). The reason why this is not a strictly valid proof is that eq. (2) does not always hold for infinite order polynomials. Since we have a product of an infinite number of terms, there is a delicate convergence issue that must be addressed. In this particular example, our instincts turn out to be correct. But, one requires addition analysis to verify this.[§]

We shall henceforth accept the correctness of eq. (1). By multiplying out the terms in this formula, one can quickly establish a remarkable result. For example, keeping track of terms up to and including $\mathcal{O}(x^2)$,

$$\frac{\sin x}{x} = 1 - x^2 \left[\frac{1}{\pi^2} + \frac{1}{(2\pi)^2} + \frac{1}{(3\pi)^2} + \cdots\right] + \mathcal{O}(x^4). \tag{4}$$

Equating the coefficients of x^2 in eqs. (3) and (4), we conclude that:

$$-\frac{1}{6} = -\frac{1}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}, \qquad \Longrightarrow \qquad \sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}. \tag{5}$$

In fact, this was the original "proof" of eq. (5) by Euler. I placed the word proof in quotes, since Euler simply assumed (incorrectly in general) that infinite-order polynomials possess all of the properties of finite-order polynomials.

Using eq. (1), we shall now derive a formula for $\cot x$. The idea is to compute the logarithmic derivative of $\sin x$:

$$\frac{d}{dx}\ln\sin x = \frac{1}{\sin x}\frac{d}{dx}\sin x = \frac{\cos x}{\sin x} = \cot x.$$

Thus, taking the logarithm of eq. (1), we obtain

$$\ln(\sin x) = \ln x + \sum_{n=1}^{\infty}\ln\left(1 - \frac{x^2}{(n\pi)^2}\right).$$

Taking the derivative of this expression then yields:

$$\cot x = \frac{1}{x} + \sum_{n=1}^{\infty}\frac{2x}{x^2 - (n\pi)^2}, \tag{6}$$

[§]An elementary derivation of eq. (1) which proves that the resulting product is convergent and converges to the right result can be found in K. Venkatachaliengar, *American Mathematical Monthly*, **69** *(1962) 541–545.*

after a little algebraic simplification. We can resolve the denominator into partial fractions, and rewrite eq. (6) as:

$$\cot x = \frac{1}{x} + \sum_{n=1}^{\infty}\left(\frac{1}{x+n\pi} + \frac{1}{x-n\pi}\right), \tag{7}$$

as long as we keep in mind that we must keep these two terms in the sum adjacent (since the summation over the two terms taken separately each diverge).

We also know the Taylor series expansion for $\cot x$. In the class handout on Taylor series, you will find:

$$\cot x = \frac{1}{x} - \sum_{n=1}^{\infty}(-1)^{n-1}\, 2^{2n} B_{2n} \frac{x^{2n-1}}{(2n)!}, \qquad 0 < |x| < \pi\,. \tag{8}$$

That is, we now possess two series expansions [eqs. (7) and (8)] for $\cot x$. By comparing the two series, we will discover the remarkable connection between the Bernoulli numbers B_{2n} and the Riemann zeta function $\zeta(2n)$.

In order to make this comparison, we need to convert eq. (7) into the form of a power series. To accomplish this, we expand out the two denominators of eq. (7) in a power series about $x = 0$.

$$\begin{aligned}
\cot x &= \frac{1}{x} + \sum_{n=1}^{\infty}\frac{1}{n\pi}\left(\frac{1}{1+\dfrac{x}{n\pi}} - \frac{1}{1-\dfrac{x}{n\pi}}\right), \\
&= \frac{1}{x} + \sum_{n=1}^{\infty}\frac{1}{n\pi}\left[\sum_{k=0}^{\infty}(-1)^k\left(\frac{x}{n\pi}\right)^k - \sum_{k=0}^{\infty}\left(\frac{x}{n\pi}\right)^k\right].
\end{aligned}$$

Note that in the sum over k, all the even k terms cancel exactly. Thus, all we need to do is sum over odd values of k. Setting $k = 2j-1$ and interchanging the order of summation,

$$\begin{aligned}
\cot x &= \frac{1}{x} - 2\sum_{n=1}^{\infty}\frac{1}{n\pi}\sum_{j=1}^{\infty}\left(\frac{x}{n\pi}\right)^{2j-1} \\
&= \frac{1}{x} - 2\sum_{j=1}^{\infty}\frac{x^{2j-1}}{\pi^{2j}}\sum_{n=1}^{\infty}\frac{1}{n^{2j}} \\
&= \frac{1}{x} - 2\sum_{j=1}^{\infty}\frac{x^{2j-1}}{\pi^{2j}}\,\zeta(2j)\,.
\end{aligned}$$

We have therefore proven that

$$\cot x = \frac{1}{x} - \sum_{n=1}^{\infty}(-1)^{n-1}B_{2n}\frac{2^{2n}x^{2n-1}}{(2n)!} = \frac{1}{x} - 2\sum_{n=1}^{\infty}\frac{x^{2n-1}}{\pi^{2n}}\,\zeta(2n)\,,$$

after relabeling the j index and calling it n. By comparing the coefficients of x^{2n-1}, it follows that:

$$\boxed{\zeta(2n) = (-1)^{n-1}\,\frac{(2\pi)^{2n}}{2(2n)!}\,B_{2n}\,, \qquad \text{for } n = 1, 2, 3, \ldots\,.} \tag{9}$$

This is the desired result. Finally, by noting that the Bernoulli number B_{2n} is positve (negative) whn n is odd (even), it follows that $B_{2n} = (-1)^{n+1}|B_{2n}|$. Hence, we can rewrite eq. (9) as:

$$\zeta(2n) = \frac{(2\pi)^{2n}}{2(2n)!}\,|B_{2n}|\,, \qquad \text{for } n = 1, 2, 3, \ldots\,.$$

For completeness, we can also evaluate $\zeta(1-2n)$ using the functional relation:

$$\zeta(x) = 2^{\,x}\,\pi^{x-1}\,\sin\left(\tfrac{1}{2}\pi x\right)\Gamma(1-x)\,\zeta(1-x)\,.$$

Let $x = 1-2n$ (assuming n is a positive integer), and note that $\Gamma(2n) = (2n-1)!$ and

$$\sin(\tfrac{1}{2}\pi(1-2n)) = \sin(\tfrac{1}{2}\pi - \pi n) = \sin(\pi/2)\cos(\pi n) - \cos(\pi/2)\sin(\pi n) = (-1)^n\,,$$

since $\cos(\pi n) = (-1)^n$ and $\sin(\pi n) = 0$ for integer values of n. Hence, it follows that:

$$\boxed{\zeta(1-2n) = -\frac{B_{2n}}{2n}\,, \qquad n = 1, 2, 3, \ldots\,.}$$