On Integrating and Differentiating a Series

Consider a sum of continuous and differentiable functions:

$$F(x) = \sum_{n=1}^{\infty} f_n(x) \,. \tag{1}$$

Assume that this sum is point-wise convergent on the interval $a \leq x \leq b$.

Differentiating the series

Can we compute the derivative of $F(x)$ by differentiating the series term by term? That is, does eq. (1) imply that for all $a \leq x \leq b$,

$$F'(x) = \sum_{n=1}^{\infty} f'_n(x) \,, \tag{2}$$

where $F'(x) \equiv dF/dx$. The answer is yes if the series in eq. (2) is uniformly convergent. Otherwise, it is possible that eq. (2) is false at least at one point in the interval $a \leq x \leq b$. As an example, consider

$$G(x) = \sum_{n=1}^{\infty} g_n(x) \,, \tag{3}$$

where

$$g_n(x) \equiv \frac{x^2 \sin x}{(1+nx^2)[1+(n-1)x^2]} \,. \tag{4}$$

To compute this series, use partial fractions to write:

$$\frac{x^2}{(1+nx^2)[1+(n-1)x^2]} = \frac{1}{1+(n-1)x^2} - \frac{1}{1+nx^2} \,. \tag{5}$$

Then we recognize that the sum of the first N terms of $G(x)$ is a telescoping sum, which is easily computed:

$$\begin{aligned} G_N(x) = \sum_{n=1}^{N} g_n(x) &= \sin x \left[1 - \frac{1}{1+Nx^2}\right] \,, \\ &= \frac{Nx^2 \sin x}{1+Nx^2} \,. \end{aligned} \tag{6}$$

Thus, $G(x) = \lim_{N\to\infty} G_N(x) = \sin x$. Because of the x^2 in the numerator of eq. (4), it is clear that $g_n'(0) = 0$ for all n. Thus,

$$\sum_{n=1}^{\infty} g_n'(0) = 0\,. \tag{7}$$

However, since $G(x) = \sin x$, we have $G'(x) = \cos x$, and $G'(0) = 1$. Thus,

$$G'(0) \neq \sum_{n=1}^{\infty} g_n'(0)\,. \tag{8}$$

The reason for this odd behavior is that $G(x)$ is not uniformly convergent at $x = 0$.

Integrating the series

We now turn to integration. Can we compute the integral of $F(x)$ defined in eq. (1) by integrating the series term by term? That is, does eq. (1) imply that

$$\int_a^b F(x)\,dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)\,dx\,. \tag{9}$$

The answer is yes if eq. (1) is a uniformly converging series. Otherwise, it is possible that eq. (9) is false. As an example, consider

$$F(x) = \sum_{n=1}^{\infty} f_n(x)\,, \tag{10}$$

for $|x| < 1$ [and $F(1) = 0$] where

$$f_n(x) = x^{n-1}(1-x)\left[n^2 x - (n-1)^2\right]\,. \tag{11}$$

Then we recognize that the sum of the first N terms of $F(x)$ is a telescoping sum, which is easily computed:

$$F_N(x) = \sum_{n=1}^{N} f_n(x) = N^2 x^N (1-x)\,. \tag{12}$$

Thus, $F(x) = \lim_{N\to\infty} F_N(x) = 0$ for $|x| < 1$ and $F(1) = 0$. Hence,

$$\int_0^1 F(x)\,dx = 0\,. \tag{13}$$

However, notice that

$$\begin{aligned}
\sum_{n=1}^{N} \int_0^1 f_n(x)\,dx &= \int_0^1 dx \sum_{n=1}^{N} f_n(x) = \int_0^1 F_N(x)\,dx \\
&= N^2 \int_0^1 x^N (1-x)\,dx \\
&= N^2 \left[\frac{1}{N+1} - \frac{1}{N+2}\right] = \frac{N^2}{(N+1)(N+2)}\,,
\end{aligned} \tag{14}$$

where we have used the fact that it is always permissible to interchange the order of integration and summation if the latter is a sum over a finite number of terms. Taking the limit of eq. (14) as $N \to \infty$ yields

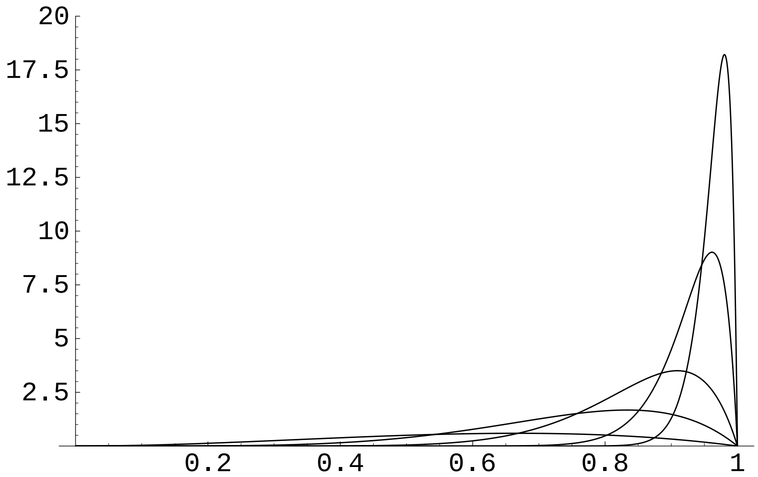
$$\sum_{n=1}^{\infty} \int_0^1 f_n(x)\, dx = 1\,. \tag{15}$$

Comparing eq. (13) with eq. (15), we conclude that

$$\int_0^1 dx \sum_{n=1}^{\infty} f_n(x) \;\neq\; \sum_{n=1}^{\infty} \int_0^1 f_n(x) dx\,. \tag{16}$$

Again, this behavior can be attributed to the fact that the sum is not uniformly convergent at $x = 1$.

To understand what is happening, consider a plot of $F_N(x)$ [eq. (12)] for various values of N. Below, I have plotted $F_N(x)$ for $N = 2$, 5, 10, 25 and 50. As N increases, the hump in the graph of $F_N(x)$ gets higher and narrower and is pushed further to the right.



As $N \to \infty$, $F_N(x) \to 0$ but in a very non-uniform way, since the area under the graph of $F_N(x)$ approaches 1 in the same limit!

Reference: David, Bressoud, *A Radical Approach to Real Analysis* (Mathematical Association of America, Washington, DC, 1994).