Taylor Series Expansions

In this short note, a list of well-known Taylor series expansions is provided. We focus on Taylor series about the point $x = 0$, the so-called Maclaurin series. In all cases, the interval of convergence is indicated. The variable x is real.

We begin with the infinite geometric series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \,, \qquad |x| < 1 \,.$$

If we change the sign of x, we also obtain

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \,, \qquad |x| < 1 \,. \tag{1}$$

The above expansions diverge for all $|x| \geq 1$.

Next we write down the binomial expansion,

$$(1+x)^p = 1 + \sum_{n=1}^{\infty} \frac{p(p-1)\cdots(p-n+1)}{n!}\, x^n \,, \qquad |x| < 1 \,.$$

For $x = 1$, the series converges absolutely for $p > 0$, converges conditionally for $-1 < p < 0$ and diverges for $p \leq -1$. For $x = -1$, the series converges absolutely for $p > 0$, and diverges for $p < 0$. Note that if p is a non-negative integer, then the sum above is finite and therefore converges for all x.

We now list the Taylor series for the exponential and logarithmic functions.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \,, \qquad |x| < \infty \,,$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \,, \qquad -1 < x \leq 1 \,. \tag{2}$$

Note that the Taylor expansion for $\ln(1+x)$ can be easily derived by integrating eq. (1).

$$\ln(1+x) = \int_0^x \frac{1}{1+t}\, dt = \sum_{n=0}^{\infty} (-1)^n \int_0^x t^n \, dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \,,$$

after shifting the summation index by one unit. Although eq. (1) diverges at $x = 1$, one can show that eq. (2) is conditionally convergent at $x = 1$.

Next, we examine the Taylor series of the trigonometric functions.

$$
\begin{aligned}
\sin x &= \sum_{n=0}^{\infty} (-1)^n \, \frac{x^{2n+1}}{(2n+1)!}\,, \qquad |x| < \infty\,, \\
\cos x &= \sum_{n=0}^{\infty} (-1)^n \, \frac{x^{2n}}{(2n)!}\,, \qquad |x| < \infty\,, \\
\tan x &= \sum_{n=0}^{\infty} (-1)^n T_{2n+1} \frac{x^{2n+1}}{(2n+1)!}\,, \qquad |x| < \tfrac{1}{2}\pi\,, \\
\cot x &= \frac{1}{x} - \sum_{n=1}^{\infty} (-1)^{n-1} \, 2^{2n} B_{2n} \frac{x^{2n-1}}{(2n)!}\,, \qquad 0 < |x| < \pi\,, \\
\sec x &= \sum_{n=0}^{\infty} (-1)^n \, E_{2n} \frac{x^{2n}}{(2n)!}\,, \qquad |x| < \tfrac{1}{2}\pi\,, \\
\csc x &= \frac{1}{x} + \sum_{n=1}^{\infty} (-1)^{n-1} \, 2(2^{2n-1}-1) B_{2n} \frac{x^{2n-1}}{(2n)!}\,, \qquad 0 < |x| < \pi\,,
\end{aligned}
$$

which defines the tangent numbers T_{2n+1}, the Bernoulli numbers B_{2n}, and the Euler numbers E_{2n}, for all non-negative integers n.* A table of the these numbers for $n \leq 8$ is provided below.

n	T_{2n+1}	B_{2n}	E_{2n}
0	1	1	1
1	-2	$\frac{1}{6}$	-1
2	16	$-\frac{1}{30}$	5
3	-272	$\frac{1}{42}$	-61
4	7936	$-\frac{1}{30}$	1385
5	-353792	$\frac{5}{66}$	-50521
6	22368256	$-\frac{691}{2730}$	2702765
7	-1903757312	$\frac{7}{6}$	-199360981
8	209865342976	$-\frac{3617}{510}$	19391512145

The theory of these numbers is quite interesting. Here, I shall simply summarize a number of simple facts. First, the tangent numbers can be expressed simply

*One can generalize the definitions of these numbers for all non-negative subscripts, n, by defining $B_{2n+1} = T_{2n} = E_{2n-1} = 0$ for $n = 1, 2, 3, \ldots$, and $B_1 = -\frac{1}{2}$ and $E_0 = -1$. But, we will have no need for these more general definitions here.

in terms of the Bernoulli numbers,

$$T_{2k+1} = 2^{2k+2}(2^{2k+2}-1)\frac{B_{2k+2}}{2k+2}\,. \tag{3}$$

It follows that

$$\tan x = \sum_{n=1}^{\infty}(-1)^{n-1}\,2^{2n}(2^{2n}-1)B_{2n}\frac{x^{2n-1}}{(2n)!}\,, \qquad |x| < \tfrac{1}{2}\pi\,.$$

You will also notice that whereas tangent numbers and Euler numbers are integers, the Bernoulli numbers (for $n \neq 0$) are non-integer rational numbers. In addition, T_{2n+1} and E_{2n} are positive [negative] for even [odd] n, whereas the Bernoulli numbers B_{2n} (for $n \neq 0$) are positive [negative] for odd [even] n.

The Taylor series for the hyperbolic functions are closely related to those of the trigonometric functions.

$$\begin{aligned}
\sinh x &= \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}\,, \qquad |x| < \infty\,,\\
\cosh x &= \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}\,, \qquad |x| < \infty\,,\\
\tanh x &= \sum_{n=0}^{\infty}T_{2n+1}\frac{x^{2n+1}}{(2n+1)!}\,, \qquad |x| < \tfrac{1}{2}\pi\,,\\
\coth x &= \frac{1}{x} + \sum_{n=1}^{\infty}2^{2n}B_{2n}\frac{x^{2n-1}}{(2n)!}\,, \qquad 0 < |x| < \pi\,,\\
\mathrm{sech} x &= \sum_{n=0}^{\infty}E_{2n}\frac{x^{2n}}{(2n)!}\,, \qquad |x| < \tfrac{1}{2}\pi\,,\\
\mathrm{csch} x &= \frac{1}{x} - \sum_{n=1}^{\infty}2(2^{2n-1}-1)B_{2n}\frac{x^{2n-1}}{(2n)!}\,, \qquad 0 < |x| < \pi\,.
\end{aligned}$$

Finally, we examine the Taylor series of the inverse trigonometric and inverse hyperbolic functions. We list only those functions that possess Taylor series about $x = 0$.

$$\begin{aligned}
\arcsin x = \tfrac{1}{2}\pi - \arccos x &= \sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}(n!)^2(2n+1)}x^{2n+1}\,, \qquad |x| \leq 1\,,\\
\arctan x &= \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}\,, \qquad |x| \leq 1\,,\\
\mathrm{arcsinh}\, x &= \sum_{n=0}^{\infty}\frac{(-1)^n\,(2n)!}{2^{2n}(n!)^2(2n+1)}x^{2n+1}\,, \qquad |x| \leq 1\,,\\
\mathrm{arctanh}\, x &= \sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1}\,, \qquad |x| < 1\,.
\end{aligned}$$

The Taylor series above for arcsin x, arccos x and arctan x correspond to the corresponding principal values of these functions, respectively. In our conventions, arccot $x \equiv \arctan(1/x)$ is not continuous at $x = 0$ and thus does not possess a Taylor series about $x = 0$. For further details, see the class handout on the inverse trigonometric and hyperbolic functions.

To end these notes, I will provide two simple algorithms for generating the Bernoulli numbers and Euler numbers. Tangent numbers can then be obtained from eq. (3). All Bernoulli numbers can be evaluated starting with B_2 by using

$$\sum_{n=1}^{N} \binom{2N}{2n-1} \frac{B_{2n}}{2n} = \frac{2N-1}{2(2N+1)}\,, \qquad N = 1, 2, 3, \ldots\,, \tag{4}$$

and all Euler numbers can be evaluated starting with E_2 by using

$$\sum_{n=1}^{N} \binom{2N}{2n} E_{2n} = -1\,, \qquad N = 1, 2, 3, \ldots\,, \tag{5}$$

where

$$\binom{n}{r} \equiv \frac{n!}{r!(n-r)!} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}\,,$$

is the binomial coefficient. Eqs. (4) and (5) can be used to build up the table of Bernoulli and Euler numbers presented earlier in this note.

You may wonder why I have not presented a formula for directly computing the Bernoulli and Euler numbers. Such formulae do exist, but they are not simple. Here is one example:

$$B_n = \sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{k} \binom{n+1}{k} \sum_{j=0}^{k-1} j^n\,.$$

The only formulae that exist for computing a particular Bernoulli or Euler number necessarily involve at least a double sum.

References

1. An excellent reference book for Taylor series of functions and many other properties of mathematical functions ca be found in Milton Abramowitz and Irene A. Stegun, *Handbook of Mathematical Functions* (Dover Publications, Inc., New York, 1965). This resource is available free on the web and can be either viewed or downloaded from: `http://www.math.ucla.edu/~cbm/aands/`.

2. The notation for tangent numbers and Bernoulli numbers varies in the mathematical literature. I have followed the conventions of Henri Cohen, *Number Theory Volume II* (Springer Science, New York, 2007). This is a rather advanced book, although it has a very nice presentation of the theory of Bernoulli and Euler numbers in Chapter 9.