The complex inverse trigonometric and hyperbolic functions

In these notes, we examine the inverse trigonometric and hyperbolic functions, where the arguments of these functions can be complex numbers (see e.g. ref. 1). These are all multi-valued functions. We also carefully define the corresponding single-valued principal values of the inverse trigonometric and hyperbolic functions following the conventions employed by the computer algebra software system, Mathematica 8. These conventions are outlined in section 2.2.5 of ref. 2.

The principal value of a multi-valued complex function $f(z)$ of the complex variable z, which we denote by $F(z)$, is continuous in all regions of the complex plane, except on a specific line (or lines) called branch cuts. The function $F(z)$ has a discontinuity when z crosses a branch cut. Branch cuts end at a branch point, which is unambiguous for each function $F(z)$. But the choice of branch cuts is a matter of convention.[1] Thus, if mathematics software is employed to evaluate the function $F(z)$, you need to know the conventions of the software for the location of the branch cuts. The mathematical software needs to precisely define the principal value of $f(z)$ in order that it can produce a unique answer when the user types in $F(z)$ for a particular complex number z. There are often different possible candidates for $F(z)$ that differ only in the values assigned to them when z lies on the branch cut(s). These notes provide a careful discussion of these issues as they apply to the complex inverse trigonometric and hyperbolic functions.

1. The inverse trigonometric functions: arctan and arccot

We begin by examining the solution to the equation

$$z = \tan w = \frac{\sin w}{\cos w} = \frac{1}{i}\left(\frac{e^{iw} - e^{-iw}}{e^{iw} + e^{-iw}}\right) = \frac{1}{i}\left(\frac{e^{2iw} - 1}{e^{2iw} + 1}\right) .$$

We now solve for e^{2iw},

$$iz = \frac{e^{2iw} - 1}{e^{2iw} + 1} \qquad \Longrightarrow \qquad e^{2iw} = \frac{1 + iz}{1 - iz} .$$

[1] In these notes, the principal value of the argument of the complex number z, denoted by $\operatorname{Arg} z$, is defined to lie in the interval $-\pi < \operatorname{Arg} z \leq \pi$. That is, $\operatorname{Arg} z$ is single-valued and is continuous at all points in the complex plane excluding a branch cut along the negative real axis. The properties of $\operatorname{Arg} z$ determine the location of the branch cuts of the principal values of the logarithm the square root functions. If $f(z)$ is expressible in terms of the logarithm the square root functions, then the definition of the principal value of $F(z)$ is not unique. However given a specific definition of $F(z)$ in terms of the principal values of the logarithm the square root functions, the location of the branch cuts of $F(z)$ is inherited from that of $\operatorname{Arg} z$ and are thus uniquely determined.

Taking the complex logarithm of both sides of the equation, we can solve for w,

$$w = \frac{1}{2i} \ln \left(\frac{1+iz}{1-iz} \right) .$$

The solution to $z = \tan w$ is $w = \arctan z$. Hence,

$$\boxed{\arctan z = \frac{1}{2i} \ln \left(\frac{1+iz}{1-iz} \right)} \tag{1}$$

Since the complex logarithm is a multi-valued function, it follows that the arctangent function is also a multi-valued function. Using the definition of the multi-valued complex logarithm,

$$\arctan z = \frac{1}{2i} \operatorname{Ln} \left| \frac{1+iz}{1-iz} \right| + \tfrac{1}{2} \left[\operatorname{Arg} \left(\frac{1+iz}{1-iz} \right) + 2\pi n \right] , \qquad n = 0\,,\, \pm 1\,,\, \pm 2\,,\, \ldots\,, \tag{2}$$

where Arg is the principal value of the argument function.

Similarly,

$$z = \cot w = \frac{\cos w}{\sin w} = \left(\frac{i(e^{iw} + e^{-iw}}{e^{iw} - e^{-iw}} \right) = \left(\frac{i(e^{2iw} + 1}{e^{2iw} - 1} \right) .$$

Again, we solve for e^{2iw},

$$-iz = \frac{e^{2iw} + 1}{e^{2iw} - 1} \qquad \Longrightarrow \qquad e^{2iw} = \frac{iz-1}{iz+1} .$$

Taking the complex logarithm of both sides of the equation, we conclude that

$$w = \frac{1}{2i} \ln \left(\frac{iz-1}{iz+1} \right) = \frac{1}{2i} \ln \left(\frac{z+i}{z-i} \right) ,$$

after multiplying numerator and denominator by $-i$ to get a slightly more convenient form. The solution to $z = \cot w$ is $w = \operatorname{arccot} z$. Hence,

$$\boxed{\operatorname{arccot} z = \frac{1}{2i} \ln \left(\frac{z+i}{z-i} \right)} \tag{3}$$

Thus, the arccotangent function is a multivalued function,

$$\operatorname{arccot} z = \frac{1}{2i} \operatorname{Ln} \left| \frac{z+i}{z-i} \right| + \tfrac{1}{2} \left[\operatorname{Arg} \left(\frac{z+i}{z-i} \right) + 2\pi n \right] , \qquad n = 0\,,\, \pm 1\,,\, \pm 2\,,\, \ldots\,, \tag{4}$$

Using the definitions given by eqs. (1) and (3), the following relation is easily derived:

$$\operatorname{arccot}(z) = \arctan \left(\frac{1}{z} \right) . \tag{5}$$

Note that eq. (5) can be used as the *definition* of the arccotangent function. It is instructive to derive another relation between the arctangent and arccotangent functions. First, we first recall the property of the multi-valued complex logarithm,

$$\ln(z_1 z_2) = \ln(z_1) + \ln(z_2)\,, \tag{6}$$

as a set equality. It is convenient to define a new variable,

$$v = \frac{i-z}{i+z}\,, \qquad \Longrightarrow \qquad -\frac{1}{v} = \frac{z+i}{z-i}\,. \tag{7}$$

It follows that:

$$\arctan z + \operatorname{arccot} z = \frac{1}{2i}\left[\ln v + \ln\left(-\frac{1}{v}\right)\right] = \frac{1}{2i}\ln\left(\frac{-v}{v}\right) = \frac{1}{2i}\ln(-1)\,.$$

Since $\ln(-1) = i(\pi + 2\pi n)$ for $n = 0, \pm 1, \pm 2\ldots$, we conclude that

$$\boxed{\arctan z + \operatorname{arccot} z = \tfrac{1}{2}\pi + \pi n\,, \qquad \text{for} \quad n = 0, \pm 1, \pm 2, \ldots} \tag{8}$$

Finally, we mention two equivalent forms for the multi-valued complex arctangent and arccotangent functions. Recall that the complex logarithm satisfies

$$\ln\left(\frac{z_1}{z_2}\right) = \ln z_1 - \ln z_2\,, \tag{9}$$

where this equation is to be viewed as a set equality, where each set consists of all possible results of the multi-valued function. Thus, the multi-valued arctangent and arccotangent functions given in eqs. (1) and (5), respectively, are equivalent to

$$\arctan z = \frac{1}{2i}\Big[\ln(1+iz) - \ln(1-iz)\Big]\,, \tag{10}$$

$$\operatorname{arccot} z = \frac{1}{2i}\left[\ln\left(1+\frac{i}{z}\right) - \ln\left(1-\frac{i}{z}\right)\right]\,, \tag{11}$$

2. The principal values Arctan and Arccot

It is convenient to define principal values of the inverse trigonometric functions, which are single-valued functions, which will necessarily exhibit a discontinuity across branch cuts in the complex plane. In Mathematica 8, the principal values of the complex arctangent and arccotangent functions, denoted by Arctan and Arccot respectively (with a capital A), are defined by employing the principal values of the complex logarithms in eqs. (10) and (11),

$$\boxed{\operatorname{Arctan} z = \frac{1}{2i}\Big[\operatorname{Ln}(1+iz) - \operatorname{Ln}(1-iz)\Big]\,, \qquad z \neq \pm i} \tag{12}$$

and

$$\boxed{\operatorname{Arccot} z = \operatorname{Arctan}\left(\frac{1}{z}\right) = \frac{1}{2i}\left[\operatorname{Ln}\left(1+\frac{i}{z}\right) - \operatorname{Ln}\left(1-\frac{i}{z}\right)\right], z \neq \pm i\,,\; z \neq 0} \tag{13}$$

Note that the points $z = \pm i$ are excluded from the above definitions, as the arctangent and arccotangent are divergent at these two points. The definition of the principal value of the arccotangent given in eq. (13) is deficient in one respect since it is not well-defined at $z = 0$. We shall address this problem shortly. One useful feature of the definitions eqs. (12) and (13) is that they satisfy:

$$\operatorname{Arctan}(-z) = -\operatorname{Arctan} z\,, \qquad\qquad \operatorname{Arccot}(-z) = -\operatorname{Arccot} z\,.$$

Because the principal value of the complex logarithm Ln does not satisfy eq. (9) in all regions of the complex plane, it follows that the definitions of the complex arctangent and arccotangent functions adopted by Mathematica 8 do not coincide with alternative definitions that employ the principal value of the complex logarithms in eqs. (1) and (4) [for further details, see Appendix A].

First, we shall identify the location of the branch cuts by identifying the lines of discontinuity of the principal values of the complex arctangent and arccotangent functions in the complex plane. The principal value of the complex arctangent function is single-valued for all complex values of z, excluding the two branch points at $z \neq \pm i$. Moreover, the the principal-valued logarithms, $\operatorname{Ln}(1 \pm iz)$ are discontinuous as z crosses the lines $1 \pm iz < 0$, respectively. We conclude that $\operatorname{Arctan} z$ must be discontinuous when $z = x + iy$ crosses lines on the imaginary axis such that

$$x = 0 \quad \text{and} \quad -\infty < y < -1 \quad \text{and} \quad 1 < y < \infty\,. \tag{14}$$

These two lines that lie along the imaginary axis are the branch cuts of $\operatorname{Arctan} z$. Note that $\operatorname{Arctan} z$ is single-valued on the branch cut itself, since it inherits this property from the principal value of the complex logarithm.

Likewise, the principal value of the complex arccotangent function is single-valued for all complex z, excluding the branch points $z \neq \pm i$. Moreover, the the principal-valued logarithms, $\operatorname{Ln}\left(1 \pm \frac{i}{z}\right)$ are discontinuous as z crosses the lines $1 \pm \frac{i}{z} < 0$, respectively. We conclude that $\operatorname{Arccot} z$ must be discontinuous when $z = x + iy$ crosses the branch cut located on the imaginary axis such that

$$x = 0 \quad \text{and} \quad -1 < y < 1\,. \tag{15}$$

In particular, due to the presence of the branch cut,

$$\lim_{x\to 0^-} \operatorname{Arccot}(x+iy) \neq \lim_{x\to 0^+} \operatorname{Arccot}(x+iy)\,, \qquad \text{for } -1 < y < 1\,,$$

for real values of x, where 0^+ indicates that the limit is approached from positive real axis and 0^- indicates that the limit is approached from negative real axis. If $z \neq 0$, eq. (13) provides unique values for $\operatorname{Arccot} z$ for all $z \neq \pm i$ in the complex plane, including points on the branch cut. Using eq. (12), one can easily show that if z is a non-zero complex number infinitesimally close to 0, then

$$\operatorname{Arccot} z \underset{z\to 0\,,\,z\neq 0}{=} \begin{cases} \tfrac{1}{2}\pi\,, & \text{for } \operatorname{Re} z > 0\,, \\ \tfrac{1}{2}\pi\,, & \text{for } \operatorname{Re} z = 0 \text{ and } \operatorname{Im} z < 0\,, \\ -\tfrac{1}{2}\pi\,, & \text{for } \operatorname{Re} z < 0\,, \\ -\tfrac{1}{2}\pi\,, & \text{for } \operatorname{Re} z = 0 \text{ and } \operatorname{Im} z > 0\,. \end{cases} \tag{16}$$

It is now apparent why $z = 0$ is problematical in eq. (13), since $\lim_{z\to 0} \operatorname{Arccot} z$ is not uniquely defined by eq. (16). If we wish to define a single-valued arccotangent function, then we must separately specify the value of Arccot(0). Mathematica 8 supplements the definition of the principal value of the complex arccotangent given in eq. (13) by declaring that

$$\operatorname{Arccot}(0) = \tfrac{1}{2}\pi\,. \tag{17}$$

With the definitions given in eqs. (12), (13) and (17), $\operatorname{Arctan} z$ and $\operatorname{Arccot} z$ are single-valued functions in the entire complex plane, excluding the branch points $z = \pm i$, and are continuous functions as long as the complex number z does not cross the branch cuts specified in eqs. (14) and (15), respectively.

Having defined precisely the principal values of the complex arctangent and arccotangent functions, let us check that they reduce to the conventional definitions when z is real. First consider the principal value of the real arctangent function, which satisfies

$$-\tfrac{1}{2}\pi \leq \operatorname{Arctan} x \leq \tfrac{1}{2}\pi\,, \qquad \text{for } -\infty \leq x \leq \infty\,, \tag{18}$$

where x is a real variable. The definition given by eq. (12) does reduce to the conventional definition of the principal value of the real-valued arctangent function when z is real. In particular, for real values of x,

$$\operatorname{Arctan} x = \frac{1}{2i}\Big[\operatorname{Ln}(1+ix) - \operatorname{Ln}(1-ix)\Big] = \tfrac{1}{2}\Big[\operatorname{Arg}(1+ix) - \operatorname{Arg}(1-ix)\Big]\,, \tag{19}$$

after noting that $\operatorname{Ln}|1+ix| = \operatorname{Ln}|1-ix| = \tfrac{1}{2}\operatorname{Ln}(1+x^2)$. Geometrically, the quantity $\operatorname{Arg}(1+ix) - \operatorname{Arg}(1-ix)$ is the angle between the complex numbers $1+ix$ and $1-ix$ viewed as vectors lying in the complex plane. This angle varies between $-\pi$ and π over the range $-\infty < x < \infty$. Moreover, the values $\pm\pi$ are achieved in the limit as $x \to \pm\infty$, respectively. Hence, we conclude that the principal interval of the real-valued arctangent function is indeed given by eq. (18). For all possible values of x excluding $x = -\infty$, one can check that it is permissible to subtract the two principal-valued logarithms (or equivalently the two Arg functions) using eq. (9). In the case of $x \to -\infty$, we see that $\operatorname{Arg}(1+ix) - \operatorname{Arg}(1-ix) \to -\pi$, corresponding to $N_- = -1$ in the notation of eq. (82).[2] Hence, an extra term appears when combining

[2]See eqs. (12), (13) and (55) of the class handout entitled, *The complex logarithm, exponential and power functions.*

the two logarithms that is equal to $2\pi i N_- = -2\pi i$. The end result is,

$$\mathrm{Arctan}(-\infty) = \frac{1}{2i}\left[\ln(-1) - 2\pi i\right] = -\tfrac{1}{2}\pi\,,$$

as required. As a final check, we can use the results of Tables 1 and 2 in the class handout, *The Argument of a Complex Number*, to conclude that $\mathrm{Arg}(a+bi) = \mathrm{Arctan}(b/a)$ for $a > 0$. Setting $a = 1$ and $b = x$ then yields:

$$\mathrm{Arg}\,(1+ix) = \mathrm{Arctan}\,x\,, \qquad \mathrm{Arg}\,(1-ix) = \mathrm{Arctan}(-x) = -\mathrm{Arctan}\,x\,.$$

Subtracting these two results yields eq. (19).

In contrast to the real arctangent function, there is no generally agreed definition for the principal range of the real-valued arccotangent function. However, a growing consensus among computer scientists has led to the following choice for the principal range of the real-valued arccotangent function,

$$-\tfrac{1}{2}\pi < \mathrm{Arccot}\,x \leq \tfrac{1}{2}\pi\,, \qquad \text{for } -\infty \leq x \leq \infty\,, \tag{20}$$

where x is a real variable. Note that the principal value of the arccotangent function does not include the endpoint $-\frac{1}{2}\pi$ [contrast this with eq. (18) for Arctan]. The reason for this behavior is that $\mathrm{Arccot}\,x$ is *discontinuous* at $x = 0$, with

$$\lim_{x\to 0^-} \mathrm{Arccot}\,x = -\tfrac{1}{2}\pi\,, \qquad \lim_{x\to 0^+} \mathrm{Arccot}\,x = \tfrac{1}{2}\pi\,, \tag{21}$$

as a consequence of eq. (16). In particular, eq. (20) corresponds to the convention in which $\mathrm{Arccot}(0) = \frac{1}{2}\pi$ [cf. eq. (17)]. Thus, as x increases from negative to positive values, $\mathrm{Arccot}\,x$ never reaches $-\frac{1}{2}\pi$ but jumps discontinuously to $\frac{1}{2}\pi$ at $x = 0$.

Finally, we examine the the analog of eq. (8) for the corresponding principal values. Employing the Mathematica 8 definitions for the principal values of the complex arctangent and arccotangent functions, we find that

$$\mathrm{Arctan}\,z + \mathrm{Arccot}\,z = \begin{cases} \tfrac{1}{2}\pi\,, & \text{for Re } z > 0\,, \\ \tfrac{1}{2}\pi\,, & \text{for Re } z = 0\,, \text{ and Im } z > 1 \text{ or } -1 < \text{Im } z \leq 0\,, \\ -\tfrac{1}{2}\pi\,, & \text{for Re } z < 0\,, \\ -\tfrac{1}{2}\pi\,, & \text{for Re } z = 0\,, \text{ and Im } z < -1 \text{ or } 0 < \text{Im } z < 1\,. \end{cases} \tag{22}$$

The derivation of this result will be given in Appendix B. In Mathematica, one can confirm eq. (22) with many examples.

The relations between the single-valued and multi-valued functions can be summarized by:

$$\begin{aligned} \arctan z &= \mathrm{Arctan}\,z + n\pi\,, \qquad && n = 0\,,\, \pm 1\,,\, \pm 2\,,\, \cdots\,, \\ \mathrm{arccot}\,z &= \mathrm{Arccot}\,z + n\pi\,, \qquad && n = 0\,,\, \pm 1\,,\, \pm 2\,,\, \cdots\,. \end{aligned}$$

Note that we can use these relations along with eq. (22) to confirm the result obtained in eq. (8).

3. The inverse trigonometric functions: arcsin and arccos

The arcsine function is the solution to the equation:

$$z = \sin w = \frac{e^{iw} - e^{-iw}}{2i}\,.$$

Letting $v \equiv e^{iw}$, we solve the equation

$$v - \frac{1}{v} = 2iz\,.$$

Multiplying by v, one obtains a quadratic equation for v,

$$v^2 - 2izv - 1 = 0\,. \tag{23}$$

The solution to eq. (23) is:

$$v = iz + (1-z^2)^{1/2}\,. \tag{24}$$

Since z is a complex variable, $(1-z^2)^{1/2}$ is the complex square-root function. This is a multi-valued function with two possible values that differ by an overall minus sign. Hence, we do not explicitly write out the $\pm$ sign in eq. (24). To avoid ambiguity, we shall write

$$\begin{aligned} v &= iz + (1-z^2)^{1/2} = iz + e^{\frac{1}{2}\ln(1-z^2)} = iz + e^{\frac{1}{2}\left[\mathrm{Ln}|1-z^2| + i\arg(1-z^2)\right]} \\ &= iz + |1-z^2|^{1/2} e^{\frac{i}{2}\arg(1-z^2)}\,. \end{aligned}$$

In particular, note that

$$e^{\frac{i}{2}\arg(1-z^2)} = e^{\frac{i}{2}\mathrm{Arg}(1-z^2)} e^{in\pi} = \pm e^{\frac{i}{2}\mathrm{Arg}(1-z^2)}\,, \qquad \text{for } n = 0, 1\,,$$

which exhibits the two possible sign choices.

By definition, $v \equiv e^{iw}$, from which it follows that

$$w = \frac{1}{i}\ln v = \frac{1}{i}\ln\left(iz + |1-z^2|^{1/2} e^{\frac{i}{2}\arg(1-z^2)}\right)\,.$$

The solution to $z = \sin w$ is $w = \arcsin z$. Hence,

$$\boxed{\arcsin z = \frac{1}{i}\ln\left(iz + |1-z^2|^{1/2} e^{\frac{i}{2}\arg(1-z^2)}\right)}$$

The arccosine function is the solution to the equation:

$$z = \cos w = \frac{e^{iw} + e^{-iw}}{2}\,.$$

Letting $v \equiv e^{iw}$, we solve the equation

$$v + \frac{1}{v} = 2z\,.$$

Multiplying by v, one obtains a quadratic equation for v,

$$v^2 - 2zv + 1 = 0\,. \tag{25}$$

The solution to eq. (25) is:

$$v = z + (z^2-1)^{1/2}\,.$$

Following the same steps as in the analysis of arcsine, we write

$$w = \arccos z = \frac{1}{i}\ln v = \frac{1}{i}\ln\left[z + (z^2-1)^{1/2}\right]\,, \tag{26}$$

where $(z^2-1)^{1/2}$ is the multi-valued square root function. More explicitly,

$$\arccos z = \frac{1}{i}\ln\left(z + |z^2-1|^{1/2}e^{\frac{i}{2}\arg(z^2-1)}\right)\,. \tag{27}$$

It is sometimes more convenient to rewrite eq. (27) in a slightly different form. Recall that

$$\arg(z_1z_2) = \arg z + \arg z_2\,, \tag{28}$$

as a *set equality*. We now substitute $z_1 = z$ and $z_2 = -1$ into eq. (28) and note that $\arg(-1) = \pi + 2\pi n$ (for $n = 0, \pm1, \pm2, \ldots$) and $\arg z = \arg z + 2\pi n$ as a set equality. It follows that

$$\arg(-z) = \pi + \arg z\,,$$

as a set equality. Thus,

$$e^{\frac{i}{2}\arg(z^2-1)} = e^{i\pi/2}\,e^{\frac{i}{2}\arg(1-z^2)} = ie^{\frac{i}{2}\arg(1-z^2)}\,,$$

and we can rewrite eq. (26) as follows:

$$\arccos z = \frac{1}{i}\ln\left(z + i\sqrt{1-z^2}\right)\,, \tag{29}$$

which is equivalent to the more explicit form,

$$\boxed{\arccos z = \frac{1}{i}\ln\left(z + i|1-z^2|^{1/2}e^{\frac{i}{2}\arg(1-z^2)}\right)}$$

The arcsine and arccosine functions are related in a very simple way. Using eq. (24),

$$\frac{i}{v} = \frac{i}{iz+\sqrt{1-z^2}} = \frac{i(-iz+\sqrt{1-z^2})}{(iz+\sqrt{1-z^2})(-iz+\sqrt{1-z^2})} = z + i\sqrt{1-z^2}\,,$$

which we recognize as the argument of the logarithm in the definition of the arccosine [cf. eq. (29)]. Using eq. (6), it follows that

$$\arcsin z + \arccos z = \frac{1}{i}\left[\ln v + \ln\left(\frac{i}{v}\right)\right] = \frac{1}{i}\ln\left(\frac{iv}{v}\right) = \frac{1}{i}\ln i\,.$$

Since $\ln i = i(\frac{1}{2}\pi + 2\pi n)$ for $n = 0, \pm 1, \pm 2 \ldots$, we conclude that

$$\boxed{\arcsin z + \arccos z = \tfrac{1}{2}\pi + 2\pi n\,, \qquad\qquad \text{for} \quad n = 0, \pm 1, \pm 2, \ldots} \tag{30}$$

4. The principal values Arcsin and Arccos

In Mathematica 8, the principal value of the arcsine function is obtained by employing the principal value of the logarithm and the principle value of the square-root function (which corresponds to employing the principal value of the argument). Thus,

$$\operatorname{Arcsin} z = \frac{1}{i}\operatorname{Ln}\left(iz + |1 - z^2|^{1/2} e^{\frac{i}{2}\operatorname{Arg}(1-z^2)}\right)\,. \tag{31}$$

It is convenient to introduce some notation for the the principle value of the square-root function. Consider the multivalued square root function, denoted by $z^{1/2}$. Henceforth, we shall employ the symbol $\sqrt{z}$ to denote the single-valued function,

$$\sqrt{z} = \sqrt{|z|}\, e^{\frac{1}{2}\operatorname{Arg} z}\,, \tag{32}$$

where $\sqrt{|z|}$ denotes the unique positive squared root of the real number $|z|$. In this notation, eq. (31) is rewritten as:

$$\boxed{\operatorname{Arcsin} z = \frac{1}{i}\operatorname{Ln}\left(iz + \sqrt{1 - z^2}\right)} \tag{33}$$

One noteworthy property of the principal value of the arcsine function is

$$\operatorname{Arcsin}(-z) = -\operatorname{Arcsin} z\,. \tag{34}$$

To prove this result, it is convenient to define:

$$v = iz + \sqrt{1 - z^2}\,, \qquad\qquad \frac{1}{v} = \frac{1}{iz + \sqrt{1 - z^2}} = -iz + \sqrt{1 - z^2}\,. \tag{35}$$

Then,

$$\operatorname{Arcsin} z = \frac{1}{i}\operatorname{Ln} v\,, \qquad\qquad \operatorname{Arcsin}(-z) = \frac{1}{i}\operatorname{Ln}\left(\frac{1}{v}\right)\,.$$

The second logarithm above can be simplified by making use of eq. (57) of the class handout entitled, *The complex logarithm, exponential and power functions*,

$$\operatorname{Ln}(1/z) = \begin{cases} -\operatorname{Ln}(z) + 2\pi i\,, & \text{if } z \text{ is real and negative}\,, \\ -\operatorname{Ln}(z)\,, & \text{otherwise}\,. \end{cases} \tag{36}$$

In Appendix C, we prove that v can never be real and negative. Hence it follows from eq. (36) that

$$\operatorname{Arcsin}(-z) = \frac{1}{i}\operatorname{Ln}\left(\frac{1}{v}\right) = -\frac{1}{i}\operatorname{Ln} v = -\operatorname{Arcsin} z\,,$$

as asserted in eq. (34).

We now examine the principal value of the arcsine for real-valued arguments such that $-1 \leq x \leq 1$. Setting $z = x$, where x is real and $|x| \leq 1$,

$$\begin{aligned} \text{Arcsin}\, x &= \frac{1}{i}\text{Ln}\left(ix + \sqrt{1-x^2}\right) = \frac{1}{i}\left[\text{Ln}\left|ix + \sqrt{1-x^2}\right| + i\text{Arg}\left(ix + \sqrt{1-x^2}\right)\right] \\ &= \text{Arg}\left(ix + \sqrt{1-x^2}\right), \qquad \text{for } |x| \leq 1\,, \end{aligned} \tag{37}$$

since $ix + \sqrt{1-x^2}$ is a complex number with magnitude equal to 1 when x is real with $|x| \leq 1$. Moreover, $ix + \sqrt{1-x^2}$ lives either in the first or fourth quadrant of the complex plane, since $\text{Re}(ix + \sqrt{1-x^2}) \geq 0$. It follows that:

$$-\frac{\pi}{2} \leq \text{Arcsin}\, x \leq \frac{\pi}{2}\,, \qquad \text{for } \; |x| \leq 1\,.$$

In Mathematica 8, the principal value of the arccosine is defined by:

$$\text{Arccos}\, z = \tfrac{1}{2}\pi - \text{Arcsin}\, z\,. \tag{38}$$

We demonstrate below that this definition is equivalent to choosing the principal value of the complex logarithm and the principal value of the square root in eq. (29). That is,

$$\boxed{\text{Arccos}\, z = \frac{1}{i}\text{Ln}\left(z + i\sqrt{1-z^2}\right)} \tag{39}$$

To verify that eq. (38) is a consequence of eq. (39), we employ the notation of eq. (35) to obtain:

$$\begin{aligned} \text{Arcsin}\, z + \text{Arccos}\, z &= \frac{1}{i}\left[\text{Ln}|v| + \text{Ln}\left(\frac{1}{|v|}\right) + i\text{Arg}\, v + i\text{Arg}\left(\frac{i}{v}\right)\right] \\ &= \text{Arg}\; v + \text{Arg}\left(\frac{i}{v}\right)\,. \end{aligned} \tag{40}$$

It is straightforward to check that:

$$\text{Arg}\; v + \text{Arg}\left(\frac{i}{v}\right) = \tfrac{1}{2}\pi\,, \qquad \text{for Re}\; v \geq 0\,.$$

However in Appendix C, we prove that $\text{Re}\; v \equiv \text{Re}\;(iz + \sqrt{1-z^2}) \geq 0$ for all complex numbers z. Hence, eq. (40) yields:

$$\text{Arcsin}\, z + \text{Arccos}\, z = \tfrac{1}{2}\pi\,,$$

as claimed.

We now examine the principal value of the arccosine for real-valued arguments such that $-1 \leq x \leq 1$. Setting $z = x$, where x is real and $|x| \leq 1$,

$$\begin{aligned}\operatorname{Arccos} x &= \frac{1}{i}\operatorname{Ln}\left(x + i\sqrt{1-x^2}\right) = \frac{1}{i}\left[\operatorname{Ln}\left|x + i\sqrt{1-x^2}\right| + i\operatorname{Arg}\left(x + i\sqrt{1-x^2}\right)\right] \\ &= \operatorname{Arg}\left(x + i\sqrt{1-x^2}\right), \qquad \text{for } |x| \leq 1, \end{aligned} \tag{41}$$

since $x + i\sqrt{1-x^2}$ is a complex number with magnitude equal to 1 when x is real with $|x| \leq 1$. Moreover, $x + i\sqrt{1-x^2}$ lives either in the first or second quadrant of the complex plane, since $\operatorname{Im}(x + i\sqrt{1-x^2}) \geq 0$. It follows that:

$$0 \leq \operatorname{Arccos} x \leq \pi, \qquad \text{for } \; |x| \leq 1 .$$

The principal value of the complex arcsine and arccosine functions are single-valued for all complex z. The choice of branch cuts for $\operatorname{Arcsin} z$ and $\operatorname{Arccos} z$ must coincide in light of eq. (38). Moreover, due to the standard branch cut of the principal value square root function,[3] it follows that $\operatorname{Arcsin} z$ is discontinuous when $z = x + iy$ crosses lines on the real axis such that[4]

$$y = 0 \quad \text{and} \quad -\infty < x < -1 \quad \text{and} \quad 1 < x < \infty . \tag{42}$$

These two lines comprise the branch cuts of $\operatorname{Arcsin} z$ and $\operatorname{Arccos} z$; each branch cut ends at a branch point located at $x = -1$ and $x = 1$, respectively (although the square root function is not divergent at these points).[5]

To obtain the relations between the single-valued and multi-valued functions, we first notice that the multi-valued nature of the logarithms imply that $\arcsin z$ can take on the values $\operatorname{Arcsin} z + 2\pi n$ and $\arccos z$ can take on the values $\operatorname{Arccos} z + 2\pi n$, where n is any integer. However, we must also take into account the fact that $(1-z^2)^{1/2}$ can take on two values, $\pm\sqrt{1-z^2}$. In particular,

$$\begin{aligned}\arcsin z &= \frac{1}{i}\ln(iz \pm \sqrt{1-z^2}) = \frac{1}{i}\ln\left(\frac{-1}{iz \mp \sqrt{1-z^2}}\right) = \frac{1}{i}\left[\ln(-1) - \ln(iz \mp \sqrt{1-z^2})\right] \\ &= -\frac{1}{i}\ln(iz \mp \sqrt{1-z^2}) + (2n+1)\pi ,\end{aligned}$$

where n is any integer. Likewise,

$$\arccos z = \frac{1}{i}\ln(z \pm i\sqrt{1-z^2}) = \frac{1}{i}\ln\left(\frac{1}{z \mp i\sqrt{1-z^2}}\right) = -\frac{1}{i}\ln(z \mp i\sqrt{1-z^2}) + 2\pi n ,$$

[3] One can check that the branch cut of the Ln function in eq. (33) is never encountered for any finite value of z. For example, in the case of $\operatorname{Arcsin} z$, the branch cut of Ln can only be reached if $iz + \sqrt{1-z^2}$ is real and negative. But this never happens since if $iz + \sqrt{1-z^2}$ is real then $z = iy$ for some real value of y, in which case $iz + \sqrt{1-z^2} = -y + \sqrt{1+y^2} > 0$.

[4] Note that for real w, we have $|\sin w| \leq 1$ and $|\cos w| \leq 1$. Hence, for both the functions $w = \operatorname{Arcsin} z$ and $w = \operatorname{Arccos} z$, it is desirable to choose the branch cuts to lie outside the interval on the real axis where $|\operatorname{Re} z| \leq 1$.

[5] The functions $\operatorname{Arcsin} z$ and $\operatorname{Arccos} z$ also possess a branch point at the point of infinity (which is defined more precisely in footnote 5). This can be verified by demonstrating that $\operatorname{Arcsin}(1/z)$ and $\operatorname{Arccos}(1/z)$ possess a branch point at $z = 0$. For further details, see e.g. Section 58 of ref 3.

where n is any integer. Hence, it follows that

$$\arcsin z = (-1)^n \text{Arcsin}\, z + n\pi\,, \qquad n = 0\,,\, \pm 1\,,\, \pm 2\,,\, \cdots\,, \tag{43}$$

$$\arccos z = \pm \text{Arccos}\, z + 2n\pi\,, \qquad n = 0\,,\, \pm 1\,,\, \pm 2\,,\, \cdots\,, \tag{44}$$

where either $\pm\text{Arccos}\, z$ can be employed to obtain a possible value of $\arccos z$. In particular, the choice of $n = 0$ in eq. (44) implies that:

$$\arccos z = -\arccos z\,, \tag{45}$$

which should be interpreted as a set equality. Note that one can use eqs. (43) and (44) along with eq. (38) to confirm the result obtained in eq. (30).

5. The inverse hyperbolic functions: arctanh and arccoth

Consider the solution to the equation

$$z = \tanh w = \frac{\sinh w}{\cosh w} = \left(\frac{e^w - e^{-w}}{e^w + e^{-w}}\right) = \left(\frac{e^{2w} - 1}{e^{2w} + 1}\right)\,.$$

We now solve for e^{2w},

$$z = \frac{e^{2w} - 1}{e^{2w} + 1} \qquad \Longrightarrow \qquad e^{2w} = \frac{1+z}{1-z}\,.$$

Taking the complex logarithm of both sides of the equation, we can solve for w,

$$w = \frac{1}{2} \ln\left(\frac{1+z}{1-z}\right)\,.$$

The solution to $z = \tanh w$ is $w = \text{arctanh} z$. Hence,

$$\boxed{\text{arctanh}\, z = \frac{1}{2} \ln\left(\frac{1+z}{1-z}\right)} \tag{46}$$

Similarly, by considering the solution to the equation

$$z = \coth w = \frac{\cosh w}{\sinh w} = \left(\frac{e^w + e^{-w}}{e^w - e^{-w}}\right) = \left(\frac{e^{2w} + 1}{e^{2w} - 1}\right)\,.$$

we end up with:

$$\boxed{\text{arccoth} z = \frac{1}{2} \ln\left(\frac{z+1}{z-1}\right)} \tag{47}$$

The above results then yield:

$$\text{arccoth}(z) = \text{arctanh}\left(\frac{1}{z}\right)\,,$$

as a set equality.

Finally, we note the relation between the inverse trigonometric and the inverse hyperbolic functions:

$$\begin{aligned} \operatorname{arctanh} z &= i \arctan(-iz)\,, \\ \operatorname{arccoth} z &= i \operatorname{arccot}(iz)\,. \end{aligned}$$

As in the discussion at the end of Section 1, one can rewrite eqs. (46) and (47) in an equivalent form:

$$\operatorname{arctanh} z = \tfrac{1}{2}\left[\ln(1+z) - \ln(1-z)\right]\,, \tag{48}$$

$$\operatorname{arccoth} z = \tfrac{1}{2}\left[\ln\left(1+\frac{1}{z}\right) - \ln\left(1-\frac{1}{z}\right)\right]\,. \tag{49}$$

6. The principal values Arctanh and Arccoth

Mathematica 8 defines the principal values of the inverse hyperbolic tangent and inverse hyperbolic cotangent, Arctanh and Arccoth, by employing the principal value of the complex logarithms in eqs. (48) and (49). We can define the principal value of the inverse hyperbolic tangent function by employing the principal value of the logarithm,

$$\boxed{\operatorname{Arctanh} z = \tfrac{1}{2}\left[\operatorname{Ln}(1+z) - \operatorname{Ln}(1-z)\right]} \tag{50}$$

and

$$\boxed{\operatorname{Arccoth} z = \operatorname{Arctanh}\left(\frac{1}{z}\right) = \tfrac{1}{2}\left[\operatorname{Ln}\left(1+\frac{1}{z}\right) - \operatorname{Ln}\left(1-\frac{1}{z}\right)\right]} \tag{51}$$

Note that the branch points at $z = \pm 1$ are excluded from the above definitions, as Arctanh z and Arccoth z are divergent at these two points. The definition of the principal value of the inverse hyperbolic cotangent given in eq. (51) is deficient in one respect since it is not well-defined at $z = 0$. For this special case, Mathematica 8 defines

$$\operatorname{Arccoth}(0) = \tfrac{1}{2} i\pi\,. \tag{52}$$

Of course, this discussion parallels that of Section 2. Moreover, alternative definitions of Arctanh z and Arccoth z analogous to those defined in Appendix A for the corresponding inverse trigonometric functions can be found in ref. 4. There is no need to repeat the analysis of Section 2 since a comparison of eqs. (12) and (13) with eqs. (50) and (51) shows that the inverse trigonometric and inverse hyperbolic tangent and cotangent functions are related by:

$$\operatorname{Arctanh} z = i\operatorname{Arctan}(-iz)\,, \tag{53}$$

$$\operatorname{Arccoth} z = i \operatorname{Arccot}(iz)\,. \tag{54}$$

Using these results, all other properties of the inverse hyperbolic tangent and cotangent functions can be easily derived from the properties of the corresponding arctangent and arccotangent functions.

For example the branch cuts of these functions are easily obtained from eqs. (14) and (15). Arctanh z is discontinuous when $z = x + iy$ crosses the branch cuts located on the real axis such that[6]

$$y = 0 \quad \text{and} \quad -\infty < x < -1 \quad \text{and} \quad 1 < x < \infty\,. \tag{55}$$

Arccoth z is discontinuous when $z = x + iy$ crosses the branch cuts located on the real axis such that

$$y = 0 \quad \text{and} \quad -1 < x < 1\,. \tag{56}$$

The relations between the single-valued and multi-valued functions can be summarized by:

$$\begin{aligned} \operatorname{arctanh} z &= \operatorname{Arctanh} z + in\pi\,, \qquad && n = 0\,, \pm 1\,, \pm 2\,, \cdots\,, \\ \operatorname{arccoth} z &= \operatorname{Arccoth} z + in\pi\,, \qquad && n = 0\,, \pm 1\,, \pm 2\,, \cdots\,. \end{aligned}$$

7. The inverse hyperbolic functions: arcsinh and arccosh

The inverse hyperbolic sine function is the solution to the equation:

$$z = \sinh w = \frac{e^w - e^{-w}}{2}\,.$$

Letting $v \equiv e^w$, we solve the equation

$$v - \frac{1}{v} = 2z\,.$$

Multiplying by v, one obtains a quadratic equation for v,

$$v^2 - 2zv - 1 = 0\,. \tag{57}$$

The solution to eq. (57) is:

$$v = z + (1 + z^2)^{1/2}\,. \tag{58}$$

Since z is a complex variable, $(1 + z^2)^{1/2}$ is the complex square-root function. This is a multi-valued function with two possible values that differ by an overall minus sign. Hence, we do not explicitly write out the $\pm$ sign in eq. (58). To avoid ambiguity, we shall write

$$\begin{aligned} v &= z + (1 + z^2)^{1/2} = z + e^{\frac{1}{2}\ln(1+z^2)} = z + e^{\frac{1}{2}\left[\operatorname{Ln}|1+z^2| + i\arg(1+z^2)\right]} \\ &= z + |1 + z^2|^{1/2} e^{\frac{i}{2}\arg(1+z^2)}\,. \end{aligned}$$

[6]Note that for real w, we have $|\tanh w| \leq 1$ and $|\coth w| \geq 1$. Hence, for $w = \operatorname{Arctanh} z$ it is desirable to choose the branch cut to lie outside the interval on the real axis where $|\mathrm{Re}\ z| \leq 1$. Likewise, for $w = \operatorname{Arccoth} z$ it is desirable to choose the branch cut to lie outside the interval on the real axis where $|\mathrm{Re}\ z| \geq 1$.

By definition, $v \equiv e^w$, from which it follows that

$$w = \ln v = \ln\left(z + |1+z^2|^{1/2} e^{\frac{i}{2}\arg(1+z^2)}\right) .$$

The solution to $z = \sinh w$ is $w = \text{arcsinh} z$. Hence,

$$\boxed{\text{arcsinh}\, z = \ln\left(z + |1+z^2|^{1/2} e^{\frac{i}{2}\arg(1+z^2)}\right)} \tag{59}$$

The inverse hyperbolic cosine function is the solution to the equation:

$$z = \cos w = \frac{e^w + e^{-w}}{2} .$$

Letting $v \equiv e^w$, we solve the equation

$$v + \frac{1}{v} = 2z .$$

Multiplying by v, one obtains a quadratic equation for v,

$$v^2 - 2zv + 1 = 0 . \tag{60}$$

The solution to eq. (60) is:

$$v = z + (z^2 - 1)^{1/2} .$$

Following the same steps as in the analysis of inverse hyperbolic sine function, we write

$$w = \text{arccosh}\, z = \ln v = \ln\left[z + (z^2-1)^{1/2}\right] , \tag{61}$$

where $(z^2-1)^{1/2}$ is the multi-valued square root function. More explicitly,

$$\boxed{\text{arccosh}\, z = \ln\left(z + |z^2-1|^{1/2} e^{\frac{i}{2}\arg(z^2-1)}\right)}$$

The multi-valued square root function satisfies:

$$(z^2-1)^{1/2} = (z+1)^{1/2}(z-1)^{1/2} .$$

Hence, an equivalent form for the multi-valued inverse hyperbolic cosine function is:

$$\text{arccosh}\, z = \ln\left[z + (z+1)^{1/2}(z-1)^{1/2}\right] ,$$

or equivalently,

$$\text{arccosh}\, z = \ln\left(z + |z^2-1|^{1/2} e^{\frac{i}{2}\arg(z+1)} e^{\frac{i}{2}\arg(z-1)}\right) . \tag{62}$$

Finally, we note the relations between the inverse trigonometric and the inverse hyperbolic functions:

$$\text{arcsinh}\, z = i \arcsin(-iz) , \tag{63}$$

$$\text{arccosh}\, z = \pm i \ \arccos z , \tag{64}$$

where the equalities in eqs. (63) and (64) are interpreted as set equalities for the multi-valued functions. The $\pm$ in eq. (64) indicates that both signs are employed in determining the members of the set of all possible $\operatorname{arccosh} z$ values. In deriving eq. (64), we have employed eqs. (26) and (61). In particular, the origin of the two possible signs in eq. (64) is a consequence of eq. (45) [and its hyperbolic analog, eq. (73)].

8. The principal values Arcsinh and Arccosh

The principal value of the inverse hyperbolic sine function, $\operatorname{Arcsinh} z$, is defined by Mathematica 8 by replacing the complex logarithm and argument functions of eq. (59) by their principal values. That is,

$$\boxed{\operatorname{Arcsinh} z = \operatorname{Ln}\left(z + \sqrt{1+z^2}\right)} \tag{65}$$

For the principal value of the inverse hyperbolic cosine function $\operatorname{Arccosh} z$, Mathematica 8 chooses eq. (62) with the complex logarithm and argument functions replaced by their principal values. That is,

$$\boxed{\operatorname{Arccosh} z = \operatorname{Ln}\left(z + \sqrt{z+1}\sqrt{z-1}\right)} \tag{66}$$

In eqs. (65) and (66), the principal values of the square root functions are employed following the notation of eq. (32).

The relation between the principal values of the inverse trigonometric and the inverse hyperbolic sine functions is given by

$$\operatorname{Arcsinh} z = i \operatorname{Arcsin}(-iz)\,, \tag{67}$$

as one might expect in light of eq. (63). A comparison of eqs. (39) and (66) reveals that

$$\operatorname{Arccosh} z = \begin{cases} i\operatorname{Arccos} z\,, & \text{for either } \operatorname{Im} z > 0 \text{ or for } \operatorname{Im} z = 0 \text{ and } \operatorname{Re} z \leq 1\,, \\ -i\operatorname{Arccos} z\,, & \text{for either } \operatorname{Im} z < 0 \text{ or for } \operatorname{Im} z = 0 \text{ and } \operatorname{Re} z \geq 1\,. \end{cases} \tag{68}$$

The existence of two possible signs in eq. (68) is not surprising in light of the $\pm$ that appears in eq. (64). Note that either choice of sign is valid in the case of $\operatorname{Im} z = 0$ and $\operatorname{Re} z = 1$, since for this special point, $\operatorname{Arccosh}(1) = \operatorname{Arccos}(1) = 0$. For a derivation of eq. (68), see Appendix D.

The principal value of the inverse hyperbolic sine and cosine functions are single-valued for all complex z. Moreover, due to the branch cut of the principal value square root function,[7] it follows that $\operatorname{Arcsinh} z$ is discontinuous when $z = x + iy$

[7] One can check that the branch cut of the Ln function in eq. (65) is never encountered for any value of z. In particular, the branch cut of Ln can only be reached if $z + \sqrt{1+z^2}$ is real and negative. But this never happens since if $z + \sqrt{1+z^2}$ is real then z is also real. But for any real value of z, we have $z + \sqrt{1+z^2} > 0$.

crosses lines on the imaginary axis such that

$$x = 0 \quad \text{and} \quad -\infty < y < -1 \quad \text{and} \quad 1 < y < \infty\,. \tag{69}$$

These two lines comprise the branch cuts of Arcsinh z, and each branch cut ends at a branch point located at $z = -i$ and $z = i$, respectively, due to the square root function in eq. (65), although the square root function is not divergent at these points. The function Arcsinh z also possesses a branch point at the point of infinity, which can be verified by examining the behavior of Arcsinh$(1/z)$ at the point $z = 0$.[8]

The branch cut for Arccosh z derives from the standard branch cuts of the square root function and the branch cut of the complex logarithm. In particular, for real z satisfying $|z| < 1$, we have a branch cut due to $(z+1)^{1/2}(z-1)^{1/2}$, whereas for real z satisfying $-\infty < z \leq -1$, the branch cut of the complex logarithm takes over. Hence, it follows that Arccosh z is discontinuous when $z = x + iy$ crosses lines on the real axis such that[9]

$$y = 0 \quad \text{and} \quad -\infty < x < 1\,. \tag{70}$$

In particular, there are branch points at $z = \pm 1$ due to the square root functions in eq. (66) and a branch point at the point of infinity due to the logarithm [cf. footnote 5]. As a result, eq. (70) actually represents two branch cuts made up of a branch cut from $z = 1$ to $z = -1$ followed by a second branch cut from $z = -1$ to the point of infinity.[10]

The relations between the single-valued and multi-valued functions can be obtained by following the same steps used to derive eqs. (43) and (44). Alternatively, we can make use of these results along with those of eqs. (63), (64), (67) and (68). The end result is:

$$\operatorname{arcsinh} z = (-1)^n \operatorname{Arcsinh} z + in\pi\,, \qquad n = 0\,, \pm 1\,, \pm 2\,, \cdots\,, \tag{71}$$

$$\operatorname{arccosh} z = \pm \operatorname{Arccosh} z + 2in\pi\,, \qquad n = 0\,, \pm 1\,, \pm 2\,, \cdots\,, \tag{72}$$

where either $\pm$Arccosh z can be employed to obtain a possible value of arccosh z. In particular, the choice of $n = 0$ in eq. (72) implies that:

$$\operatorname{arccosh} z = -\operatorname{arccosh} z\,, \tag{73}$$

[8]In the complex plane, the behavior of the complex function $F(z)$ at the point of infinity, $z = \infty$, corresponds to the behavior of $F(1/z)$ at the origin of the complex plane, $z = 0$ [cf. footnote 3]. Since the argument of the complex number 0 is undefined, the argument of the point of infinity is likewise undefined. This means that the point of infinity (sometimes called *complex infinity*) actually corresponds to $|z| = \infty$, independently of the direction in which infinity is approached in the complex plane. Geometrically, the complex plane plus the point of infinity can be mapped onto a surface of a sphere by stereographic projection. Place the sphere on top of the complex plane such that the origin of the complex plane coincides with the south pole. Consider a straight line from any complex number in the complex plane to the north pole. Before it reaches the north pole, this line intersects the surface of the sphere at a unique point. Thus, every complex number in the complex plane is uniquely associated with a point on the surface of the sphere. In particular, the north pole itself corresponds to complex infinity. For further details, see Chapter 5 of ref. 3.

[9]Note that for real w, we have $\cosh w \geq 1$. Hence, for $w = \operatorname{Arccosh} z$ it is desirable to choose the branch cut to lie outside the interval on the real axis where Re $z \geq 1$.

[10]Given that the branch cuts of Arccosh z and iArccos z are different, it is not surprising that the relation Arccosh $z = i$Arccos z cannot be respected for all complex numbers z.

which should be interpreted as a set equality.

This completes our survey of the multi-valued complex inverse trigonometric and hyperbolic functions and their single-valued principal values.

APPENDIX A: Alternative definitions for Arctan and Arccot

The well-known reference book for mathematical functions by Abramowitz and Stegun (see ref. 1) defines the principal values of the complex arctangent and arccotangent functions by employing the principal values of the logarithms in eqs. (1) and (4). This yields,[11]

$$\operatorname{Arctan} z = \frac{1}{2i}\operatorname{Ln}\left(\frac{1+iz}{1-iz}\right)\,, \tag{74}$$

$$\operatorname{Arccot} z = \operatorname{Arctan}\left(\frac{1}{z}\right) = \frac{1}{2i}\operatorname{Ln}\left(\frac{z+i}{z-i}\right)\,. \tag{75}$$

Note that with these definitions, the branch cuts are still given by eqs. (14) and (15), respectively. Comparing the above definitions with those of eqs. (12) and (13), one can check that the two definitions differ only on the branch cuts and at certain points of infinity. In particular, there is no longer any ambiguity in how to define Arccot(0). Plugging $z=0$ into eq. (75) yields

$$\operatorname{Arccot}(0) = \frac{1}{2i}\operatorname{Ln}(-1) = \tfrac{1}{2}\pi\,.$$

It is convenient to define a new variable,

$$v = \frac{i-z}{i+z}\,, \qquad \Longrightarrow \qquad -\frac{1}{v} = \frac{z+i}{z-i}\,. \tag{76}$$

Then, we can write:

$$\begin{aligned}
\operatorname{Arctan} z + \operatorname{Arccot} z &= \frac{1}{2i}\left[\operatorname{Ln} v + \operatorname{Ln}\left(-\frac{1}{v}\right)\right] \\
&= \frac{1}{2i}\left[\operatorname{Ln}|v| + \operatorname{Ln}\left(\frac{1}{|v|}\right) + i\operatorname{Arg} v + i\operatorname{Arg}\left(-\frac{1}{v}\right)\right] \\
&= \frac{1}{2}\left[\operatorname{Arg} v + \operatorname{Arg}\left(-\frac{1}{v}\right)\right]\,.
\end{aligned} \tag{77}$$

[11] A definition of the principal value of the arccotangent function that is equivalent to eq. (75) for all complex numbers z is:

$$\operatorname{Arccot} z = \frac{1}{2i}\left[\operatorname{Ln}(iz-1) - \operatorname{Ln}(iz+1)\right]\,.$$

A proof of the equivalence of this form and that of eq. (75) can be found in Appendix C of the first reference in ref. 5.

It is straightforward to check that for any non-zero complex number v,

$$\operatorname{Arg} v + \operatorname{Arg}\left(-\frac{1}{v}\right) = \begin{cases} \pi\,, & \text{for } \operatorname{Im} v \geq 0\,, \\ -\pi\,, & \text{for } \operatorname{Im} v < 0\,. \end{cases} \tag{78}$$

Using eq. (76), we can evaluate Im v by computing

$$\frac{i-z}{i+z} = \frac{(i-z)(-i-\overline{z})}{(i+z)(-i+\overline{z})} = \frac{1-|z|^2+2i\,\mathrm{Re}\,z}{|z|^2+1+2\,\mathrm{Im}\,z}\,.$$

Writing $|z|^2 = (\mathrm{Re}\,z)^2 + (\mathrm{Im}\,z)^2$ in the denominator,

$$\frac{i-z}{i+z} = \frac{1-|z|^2+2i\,\mathrm{Re}\,z}{(\mathrm{Re}\,z)^2+(\mathrm{Im}\,z+1)^2}\,.$$

Hence,

$$\mathrm{Im}\,v \equiv \mathrm{Im}\left(\frac{i-z}{i+z}\right) = \frac{2\,\mathrm{Re}\,z}{(\mathrm{Re}\,z)^2+(\mathrm{Im}\,z+1)^2}\,.$$

We conclude that

$$\mathrm{Im}\,v \geq 0 \quad\Longrightarrow\quad \mathrm{Re}\,z \geq 0\,, \qquad\qquad \mathrm{Im}\,v < 0 \quad\Longrightarrow\quad \mathrm{Re}\,z < 0\,.$$

Therefore, eqs. (77) and (78) yield:

$$\operatorname{Arctan} z + \operatorname{Arccot} z = \begin{cases} \tfrac{1}{2}\pi\,, & \text{for } \mathrm{Re}\,z \geq 0\,, \\ -\tfrac{1}{2}\pi\,, & \text{for } \mathrm{Re}\,z < 0\,. \end{cases} \tag{79}$$

This relation differs from eq. (22) when z lives on one of the branch cuts, i.e. for Re $z = 0$ and $z \neq \pm i$.

One disadvantage of the definition of the principal value of the arctangent given by eq. (74) concerns the value of Arctan$(-\infty)$. In particular, if $z = x$ is real,

$$\left|\frac{1+ix}{1-ix}\right| = 1\,, \tag{80}$$

Since $\operatorname{Ln} 1 = 0$, it would follow from eq. (74) that for all real x,

$$\operatorname{Arctan} x = \tfrac{1}{2}\operatorname{Arg}\left(\frac{1+ix}{1-ix}\right)\,. \tag{81}$$

Indeed, eq. (81) is correct for all finite real values of x. It also correctly yields Arctan $(\infty) = \frac{1}{2}\mathrm{Arg}(-1) = \frac{1}{2}\pi$, as expected. However, if we take $x \to -\infty$ in eq. (81), we would also get Arctan$\,(-\infty) = \frac{1}{2}\mathrm{Arg}(-1) = \frac{1}{2}\pi$, in contradiction with the conventional definition of the principal value of the real-valued arctangent function, where Arctan $(-\infty) = -\frac{1}{2}\pi$. This slight inconsistency is not surprising, since the principal value of the argument of any complex number z must lie in the range

$-\pi < \operatorname{Arg} z \leq \pi$. Consequently, eq. (81) implies that $-\frac{1}{2}\pi < \operatorname{Arctan} x \leq \frac{1}{2}\pi$, which is not quite consistent with eq. (18) as the endpoint at $-\frac{1}{2}\pi$ is missing.

Some authors finesse this defect by defining the value of Arctan $(-\infty)$ as the limit of Arctan (x) as $x \to -\infty$. Note that

$$\lim_{x\to-\infty} \operatorname{Arg}\left(\frac{1+ix}{1-ix}\right) = -\pi\,,$$

since for any finite real value of $x < -1$, the complex number $(1 + ix)/(1 - ix)$ lies in Quadrant III[12] and approaches the negative real axis as $x \to -\infty$. Hence, eq. (81) yields

$$\lim_{x\to-\infty} \operatorname{Arctan}(x) = -\tfrac{1}{2}\pi\,.$$

With this interpretation, eq. (74) is a perfectly good definition for the principal value of the arctangent function.

It is instructive to consider the difference of the two definitions of Arctan z given by eqs. (12) and (74). Using eqs. (13) and (55) of the class handout entitled, *The complex logarithm, exponential and power functions*, it follows that

$$\operatorname{Ln}\left(\frac{1+iz}{1-iz}\right) - [\operatorname{Ln}(1+iz) - \operatorname{Ln}(1-iz)] = 2\pi i N_-\,,$$

where

$$N_- = \begin{cases} -1\,, & \text{if } \operatorname{Arg}(1+iz) - \operatorname{Arg}(1-iz) > \pi\,, \\ 0\,, & \text{if } -\pi < \operatorname{Arg}(1+iz) - \operatorname{Arg}(1-iz) \leq \pi\,, \\ 1\,, & \text{if } \operatorname{Arg}(1+iz) \pm \operatorname{Arg}(1-iz) \leq -\pi\,. \end{cases} \tag{82}$$

To evaluate N_- explicitly, we must examine the quantity $\operatorname{Arg}(1+iz) - \operatorname{Arg}(1-iz)$ as a function of the complex number $z = x + iy$. Hence, we shall focus on the quantity $\operatorname{Arg}(1 - y + ix) - \operatorname{Arg}(1 + y - ix)$ as a function of x and y. If we plot the numbers $1 - y + ix$ and $1 + y - ix$ in the complex plane, it is evident that for finite values of x and y and $x \neq 0$ then

$$-\pi < \operatorname{Arg}(1 - y + ix) - \operatorname{Arg}(1 + y - ix) < \pi\,.$$

The case of $x = 0$ is easily treated separately, and we find that

$$\operatorname{Arg}(1-y) - \operatorname{Arg}(1+y) = \begin{cases} \pi\,, & \text{if } y > 1\,, \\ 0\,, & \text{if } -1 < y < 1\,, \\ -\pi\,, & \text{if } y < -1\,. \end{cases}$$

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[12]This is easily verified. We write:

$$z \equiv \frac{1+ix}{1-ix} = \frac{1+ix}{1-ix} \cdot \frac{1+ix}{1+ix} = \frac{1-x^2+2ix}{1+x^2}\,.$$

Thus, for real values of $x < -1$, it follows that $\operatorname{Re} z < 0$ and $\operatorname{Im} z < 0$, i.e. the complex number z lies in Quadrant III. Moreover, as $x \to -\infty$, we see that $\operatorname{Re} z \to -1$ and $\operatorname{Im} z \to 0^-$, where 0^- indicates that one is approaching 0 from the negative side. Some authors write $\lim_{x\to\infty}(1+ix)/(1-ix) = -1 - i0$ to indicate this behavior, and then define $\operatorname{Arg}(-1 - i0) = -\pi$.

Note that we have excluded the points $x = 0$, $y = \pm 1$, which correspond to the branch points where the arctangent function diverges.

Therefore, it follows that in the finite complex plane excluding the branch points at $z = \pm i$,

$$N_- = \begin{cases} 1\,, & \text{if } \operatorname{Re} z = 0 \text{ and } \operatorname{Im} z < -1\,, \\ 0\,, & \text{otherwise.} \end{cases}$$

This means that in the finite complex plane, the two possible definitions for the principal value of the arctangent function given by eqs. (12) and (74) differ only on the branch cut along the negative imaginary axis below $z = -i$. That is, for *finite* values of $z \neq \pm i$,

$$\frac{1}{2i}\operatorname{Ln}\left(\frac{1+iz}{1-iz}\right) = \begin{cases} \pi + \dfrac{1}{2i}\left[\operatorname{Ln}(1+iz) - \operatorname{Ln}(1-iz)\right]\,, & \text{if } \operatorname{Re} z = 0 \text{ and } \operatorname{Im} z < -1\,, \\ \dfrac{1}{2i}\left[\operatorname{Ln}(1+iz) - \operatorname{Ln}(1-iz)\right]\,, & \text{otherwise}\,. \end{cases} \tag{83}$$

In order to compare the two definitions of $\operatorname{Arctan} z$ in the limit of $|z| \to \infty$, we can employ the relation $\operatorname{Arccot} z = \operatorname{Arctan}(1/z)$ [which holds for both sets of definitions], and examine the behavior of $\operatorname{Arccot} z$ in the limit of $z \to 0$.

The difference of the two definitions of $\operatorname{Arccot} z$ given by eqs. (13) and (75) follows immediately from eq. (83). For $z \neq \pm i$ and $z \neq 0$,[13]

$$\frac{1}{2i}\operatorname{Ln}\left(\frac{z+i}{z-i}\right) = \begin{cases} \pi + \dfrac{1}{2i}\left[\operatorname{Ln}\left(1+\dfrac{i}{z}\right) - \operatorname{Ln}\left(1-\dfrac{i}{z}\right)\right]\,, & \text{if } \operatorname{Re} z = 0 \text{ and } 0 < \operatorname{Im} z < 1\,, \\ \dfrac{1}{2i}\left[\operatorname{Ln}\left(1+\dfrac{i}{z}\right) - \operatorname{Ln}\left(1-\dfrac{i}{z}\right)\right]\,, & \text{otherwise}\,. \end{cases} \tag{84}$$

We can now derive the behavior of $\operatorname{Arccot} z$ when z is a non-zero complex number infinitesimally close to $z = 0$. Using the results of eq. (84), it follows that eq. (16) is modified to:

$$\operatorname{Arccot} z \underset{z\to 0\,,\,z\neq 0}{=} \begin{cases} \tfrac{1}{2}\pi\,, & \text{for } \operatorname{Re} z \geq 0\,, \\ -\tfrac{1}{2}\pi\,, & \text{for } \operatorname{Re} z < 0\,. \end{cases}$$

As expected, $\operatorname{Arccot} z$ is discontinuous across the branch cut, which corresponds to the line in the complex plane corresponding to $\operatorname{Re} z = 0$ and $|\operatorname{Im} z| < 1$. However, $\operatorname{Arccot} z$ as defined by eq. (75) is a continuous function of z along the branch cut, with

$$\lim_{y\to 0^+} \operatorname{Arccot}(iy) = \lim_{y\to 0^-} \operatorname{Arccot}(iy) = \tfrac{1}{2}\pi\,. \tag{85}$$

[13]Eq. (84) is also valid for $|z| \to \infty$, in which case both definitions of the arccotangent yield $\operatorname{Arccot}(\infty) = 0$, independently of the direction in the complex plane in which z approaches complex infinity. This behavior is equivalent to the statement that both definitions of the principal value of the arctangent in eq. (83) yield $\operatorname{Arctan}(0) = 0$.

This is in contrast to the behavior of Arccot as defined in Mathematica 8 [cf. eq. (13)], where the value of Arccot is discontinuous at $z = 0$ on the branch cut, in which case one must separately define the value of Arccot(0).

So which set of conventions is best? Of course, there is no one right or wrong answer to this question. The authors of refs. 4–6 argue for choosing eq. (12) to define the principal value of the arctangent and eq. (75) to define the principal value of the arccotangent. This has the benefit of ensuring that eq. (85) is satisfied so that Arccot(0) is unambiguously defined. But, it will lead to corrections to the relation $\operatorname{Arccot} z = \operatorname{Arctan}(1/z)$ for a certain range of complex numbers that lie on the branch cuts. In particular, with the definitions of $\operatorname{Arctan} z$ and $\operatorname{Arccot} z$ given by eqs. (12) and (85), we immediately find from eq. (84) that

$$\operatorname{Arccot} z = \begin{cases} \pi + \operatorname{Arctan}\left(\dfrac{1}{z}\right), & \text{if } \operatorname{Re} z = 0 \text{ and } 0 < \operatorname{Im} z < 1\,, \\ \operatorname{Arctan}\left(\dfrac{1}{z}\right), & \text{otherwise}\,, \end{cases}$$

excluding the branch points $z = \pm i$ where $\operatorname{Arctan} z$ and $\operatorname{Arccot} z$ both diverge. Likewise, with the definitions of $\operatorname{Arctan} z$ and $\operatorname{Arccot} z$ given by eqs. (12) and (85), the expression for $\operatorname{Arctan} z + \operatorname{Arccot} z$ [given in eqs. (22) and (79)] will also be modified on the branch cuts,

$$\operatorname{Arctan} z + \operatorname{Arccot} z = \begin{cases} \tfrac{1}{2}\pi\,, & \text{for } \operatorname{Re} z > 0\,, \\ \tfrac{1}{2}\pi\,, & \text{for } \operatorname{Re} z = 0\,, \text{ and } \operatorname{Im} z > -1\,, \\ -\tfrac{1}{2}\pi\,, & \text{for } \operatorname{Re} z < 0\,, \\ -\tfrac{1}{2}\pi\,, & \text{for } \operatorname{Re} z = 0\,, \text{ and } \operatorname{Im} z < -1\,. \end{cases} \tag{86}$$

A similar set of issues arise in the definitions of the principal values of the inverse hyperbolic tangent and cotangent functions. It is most convenient to define these functions in terms of the corresponding principal values of the arctangent and arccotangent functions following eqs. (53) and (54),

$$\operatorname{Arctanh} z = i\operatorname{Arctan}(-iz)\,, \qquad \operatorname{Arccoth} z = i\operatorname{Arccot}(-iz)\,.$$

As a practical matter, I usually employ the Mathematica 8 definitions, as this is a program that I use most often in my research.

<u>CAUTION!!</u>

The principal value of the arccotangent is given in terms the principal value of the arctangent,

$$\operatorname{Arccot} z = \operatorname{Arctan}\left(\frac{1}{z}\right), \tag{87}$$

for both the Mathematica 8 definition [eq. (13)] or the alternative definition presented in eq. (75). However, many books define the principal value of the arccotangent differently via the relation,

$$\text{Arccot}\, z = \tfrac{1}{2}\pi - \text{Arctan}\, z\,. \tag{88}$$

This relation should be compared with the corresponding relations, eqs. (22) and (79), which are satisfied with the definitions of the principal value of the arccotangent introduced in eqs. (13) and (75), respectively. Eq. (88) is adopted by the Maple 14 computer algebra system, which is one of the main competitors to Mathematica.

The main motivation for eq. (88) is that the principal interval for real values x is

$$0 \leq \text{Arccot}\, x \leq \pi\,,$$

instead of the interval quoted in eq. (20). One advantage of this latter definition is that the real-valued arccotangent function, Arccot x, is continuous at $x = 0$, in contrast to eq. (87) which exhibits a discontinuity at $x = 0$. Note that if one adopts eq. (88) as the the definition of the principal value of the arccotangent, then the branch cuts of Arccot z are the same as those of Arctan z, namely eq. (14). The disadvantages of the definition given in eq. (88) are discussed in refs. 4 and 5.

Which convention does your calculator and/or your favorite mathematics software use? Try evaluating Arccot(-1). In the convention of eq. (13) or eq. (75), we have Arccot$(-1) = -\frac{1}{4}\pi$, whereas in the convention of eq. (88), we have Arccot$(-1) = \frac{3}{4}\pi$.

APPENDIX B: Derivation of eq. (22)

To derive eq. (22), we will make use of the computations provided in Appendix A. Start from eq. (79), which is based on the definitions of the principal values of the arctangent and arccotangent given in eqs. (74) and (75), respectively. We then use eqs. (83) and (84) which allow us to translate between the definitions of eqs. (74) and (75) and the Mathematica 8 definitions of the principal values of the arctangent and arccotangent given in eqs. (12) and (13), respectively. Eqs. (83) and (84) imply that the result for Arctan z + Arccot z does not change if Re $z \neq 0$. For the case of Re $z = 0$, Arctan z + Arccot z changes from $\frac{1}{2}\pi$ to $-\frac{1}{2}\pi$ if $0 < \text{Im}\ z < 1$ or $\text{Im}\ z < -1$. This is precisely what is exhibited in eq. (22).

APPENDIX C: Proof that $\text{Re}\,(\pm iz + \sqrt{z^2 - 1}) > 0$

It is convenient to define:

$$v = iz + \sqrt{1 - z^2}\,, \qquad\qquad \frac{1}{v} = \frac{1}{iz + \sqrt{1 - z^2}} = -iz + \sqrt{1 - z^2}\,.$$

In this Appendix, we shall prove that:

$$\text{Re}\ v \geq 0\,, \qquad \text{and} \qquad \text{Re}\ \left(\frac{1}{v}\right) \geq 0\,. \tag{89}$$

Using the fact that Re $(\pm iz) = \mp$Im z for any complex number z,

$$\text{Re } v = -\text{Im } z + |1-z^2|^{1/2}\cos\left[\tfrac{1}{2}\text{Arg}(1-z^2)\right] , \tag{90}$$

$$\text{Re}\left(\frac{1}{v}\right) = \text{Im } z + |1-z^2|^{1/2}\cos\left[\tfrac{1}{2}\text{Arg}(1-z^2)\right] . \tag{91}$$

One can now prove that

$$\text{Re } v \geq 0 , \qquad \text{and} \qquad \text{Re}\left(\frac{1}{v}\right) \geq 0 , \tag{92}$$

for any finite complex number z by considering separately the cases of Im $z < 0$, Im $z = 0$ and Im $z > 0$. The case of Im $z = 0$ is the simplest, since in this case Re $v = 0$ for $|z| \leq 1$ and Re $v > 0$ for $|z| > 1$ (since the principal value of the square root of a positive number is always positive). In the case of Im $z \neq 0$, we first note that that $-\pi < \text{Arg}(1-z^2) \leq \pi$ implies that $\cos\left[\frac{1}{2}\text{Arg}(1-z^2)\right] \geq 0$. Thus if Im $z < 0$, then it immediately follows from eq. (90) that Re $v > 0$. Likewise, if Im $z > 0$, then it immediately follows from eq. (91) that Re $(1/v) > 0$. However, the sign of the real part of any complex number z is the *same* as the sign of the real part of $1/z$, since

$$\frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} .$$

Hence, it follows that both Re $v \geq 0$ and Re $(1/v) \geq 0$, and eq. (89) is proven.

APPENDIX D: Derivation of eq. (68)

We begin with the definitions given in eqs. (39) and (66),[14]

$$i\text{Arccos}\, z = \text{Ln}\left(z + i\sqrt{1-z^2}\right) , \tag{94}$$

$$\text{Arccosh}\, z = \text{Ln}\left(z + \sqrt{z+1}\sqrt{z-1}\right) , \tag{95}$$

where the principal values of the square root functions are employed following the notation of eq. (32). Our first task is to relate $\sqrt{z+1}\sqrt{z-1}$ to $\sqrt{z^2-1}$. Of course,

[14]We caution the reader that some authors employ different choices for the definitions of the principal values of $\arccos z$ and $\text{arccosh}\, z$ and their branch cuts. The most common alternative definitions are:

$$\text{Arccosh}\, z = i\,\text{Arccos}\, z = \text{Ln}(z + \sqrt{z^2-1}) , \tag{93}$$

which differ from the definitions, eqs. (94) and (95), employed by Mathematica 8 and these notes. In particular, with the alternative definitions given by eq. (93), Arccos z now possesses the same set of branch cuts as Arccosh z given by eq. (70), in contrast to eq. (42). Moreover, Arccos z no longer satisfies eq. (38) if either (Re z)(Im z) < 0 or if $|\text{Re } z| > 1$ and Im $z = 0$ [cf. eq. (99)]. Other disadvantages of the alternative definitions of Arccos z and Arccosh z are discussed in ref. 4.

these two quantities are equal for all real numbers $z \geq 1$. But, as these quantities are principal values of the square roots of complex numbers, one must be more careful in the general case. We shall make use of eqs. (13) and (77) of the class handout entitled, *The complex logarithm, exponential and power functions*, in which the following formula is obtained:

$$\sqrt{z_1 z_2} = e^{\frac{1}{2}\mathrm{Ln}(z_1 z_2)} = e^{\frac{1}{2}(\mathrm{Ln}\ z_1 + \mathrm{Ln}\ z_2 + 2\pi i N_+)} = \sqrt{z_1}\sqrt{z_2}\, e^{\pi i N_+}\,,$$

where

$$N_+ = \begin{cases} -1\,, & \text{if} \quad \mathrm{Arg}\ z_1 + \mathrm{Arg}\ z_2 > \pi\,, \\ \ \ 0\,, & \text{if} \quad -\pi < \mathrm{Arg}\ z_1 + \mathrm{Arg}\ z_2 \leq \pi\,, \\ \ \ 1\,, & \text{if} \quad \mathrm{Arg}\ z_1 + \mathrm{Arg}\ z_2 \leq -\pi\,. \end{cases}$$

That is,

$$\sqrt{z_1 z_2} = \varepsilon\sqrt{z_1}\sqrt{z_2}\,, \qquad \varepsilon = \pm 1\,, \tag{96}$$

where the choice of sign is determined by:

$$\varepsilon = \begin{cases} +1\,, & \text{if } -\pi < \mathrm{Arg}\ z_1 + \mathrm{Arg}\ z_2 \leq \pi\,, \\ -1\,, & \text{otherwise.} \end{cases}$$

Thus, we must determine in which interval the quantity $\mathrm{Arg}(z+1) + \mathrm{Arg}(z-1)$ lies as a function of z. The special cases of $z = \pm 1$ must be treated separately, since $\mathrm{Arg}\,0$ is not defined. By plotting the complex points $z+1$ and $z-1$ in the complex plane, one can easily show that for $z \neq \pm 1$,

$$-\pi < \mathrm{Arg}\ (z+1) + \mathrm{Arg}\ (z-1) \leq \pi\,, \qquad \text{if} \begin{cases} \mathrm{Im}\ z > 0 \quad \text{and} \quad \mathrm{Re}\ z \geq 0\,, \\ \qquad\qquad \text{or} \\ \mathrm{Im}\ z = 0 \quad \text{and} \quad \mathrm{Re}\ z > -1\,, \\ \qquad\qquad \text{or} \\ \mathrm{Im}\ z < 0 \quad \text{and} \quad \mathrm{Re}\ z > 0\,. \end{cases}$$

If the above conditions do not hold, then $\mathrm{Arg}(z+1) + \mathrm{Arg}(z-1)$ lies outside the range of the principal value of the argument function. Hence, we conclude that if $z_1 = z+1$ and $z_2 = z-1$ then if $\mathrm{Im}\ z \neq 0$ then ε in eq. (96) is given by:

$$\varepsilon = \begin{cases} +1\,, & \text{if } \mathrm{Re}\ z > 0\,,\ \mathrm{Im}\ z \neq 0 \quad \text{or} \quad \mathrm{Re}\ z = 0\,,\ \mathrm{Im}\ z > 0\,, \\ -1\,, & \text{if } \mathrm{Re}\ z < 0\,,\ \mathrm{Im}\ z \neq 0 \quad \text{or} \quad \mathrm{Re}\ z = 0\,,\ \mathrm{Im}\ z < 0\,. \end{cases}$$

In the case of $\mathrm{Im}\ z = 0$, we must exclude the points $z = \pm 1$, in which case we also have

$$\varepsilon = \begin{cases} +1\,, & \text{if } \mathrm{Im}\ z = 0 \quad \text{and} \quad \mathrm{Re}\ z > -1 \quad \text{with} \quad \mathrm{Re}\ z \neq 1\,, \\ -1\,, & \text{if } \mathrm{Im}\ z = 0 \quad \text{and} \quad \mathrm{Re}\ z < -1\,. \end{cases}$$

It follow that $\text{Arccosh}\, z = \text{Ln}(z \pm \sqrt{z^2-1})$, where the sign is identified with ε above. Noting that $z - \sqrt{z^2-1} = [z + \sqrt{z^2-1}]^{-1}$, where $z + \sqrt{z^2-1}$ is real and negative if and only if $\text{Im}\ z = 0$ and $\text{Re}\ z \leq -1$,[15] one finds after applying eq. (36) that:

$$\text{Ln}(z - \sqrt{z^2-1}) = \begin{cases} 2\pi i - \text{Ln}(z + \sqrt{z^2-1})\,, & \text{for Im } z = 0 \text{ and Re } z \leq -1\,, \\ -\text{Ln}(z + \sqrt{z^2-1})\,, & \text{otherwise}\,. \end{cases}$$

To complete this part of the analysis, we must consider separately the points $z = \pm 1$. At these two points, eq. (95) yields $\text{Arccosh}(1) = 0$ and $\text{Arccosh}(-1) = \text{Ln}(-1) = \pi i$. Collecting all of the above results then yields:

$$\text{Arccosh}\, z = \begin{cases} \text{Ln}(z + \sqrt{z^2-1})\,, & \text{if Im } z > 0\,,\ \text{Re } z \geq 0 \ \text{ or } \ \text{Im } z = 0\,,\ \text{Re } z \geq -1 \\ & \qquad \text{or } \ \text{Im } z < 0, \text{Re } z > 0\,, \\ -\text{Ln}(z + \sqrt{z^2-1})\,, & \text{if Im } z > 0\,,\ \text{Re } z < 0 \ \text{ or } \ \text{Im } z < 0\,,\ \text{Re } z \leq 0\,, \\ 2\pi i - \text{Ln}(z + \sqrt{z^2-1})\,, & \text{if Im } z = 0, \text{Re } z \leq -1\,. \end{cases} \tag{97}$$

Note that the cases of $z = \pm 1$ are each covered twice in eq. (97) but in both respective cases the two results are consistent.

Our second task is to relate $i\sqrt{1-z^2}$ to $\sqrt{z^2-1}$. To accomplish this, we first note that for any non-zero complex number z, the principal value of the argument of $-z$ is given by:

$$\text{Arg}(-z) = \begin{cases} \text{Arg}\, z - \pi\,, & \text{if Arg}\, z > 0\,, \\ \text{Arg}\, z + \pi\,, & \text{if Arg}\, z \leq 0\,. \end{cases} \tag{98}$$

This result is easily checked by considering the locations of the complex numbers z and $-z$ in the complex plane. Hence, by making use of eqs. (32) and (98) along with $i = e^{i\pi/2}$, it follows that:

$$i\sqrt{1-z^2} = \sqrt{|z^2-1|}\, e^{\frac{1}{2}\left[\pi + \text{Arg}(1-z^2)\right]} = \eta\sqrt{z^2-1}\,, \qquad \eta = \pm 1\,,$$

where the sign η is determined by:

$$\eta = \begin{cases} +1\,, & \text{if Arg}(1-z^2) \leq 0\,, \\ -1\,, & \text{if Arg}(1-z^2) > 0\,, \end{cases}$$

assuming that $z \neq \pm 1$. If we put $z = x + iy$, then $1 - z^2 = 1 - x^2 + y^2 - 2ixy$, and

[15]Let $w = z + \sqrt{z^2-1}$, and assume that $\text{Im}\ w = 0$ and $\text{Re}\ w \neq 0$. That is, w is real and nonzero, in which case $\text{Im}\ w^2 = 0$. But

$$0 = \text{Im}\ w^2 = \text{Im}\ \left[2z^2 - 1 + 2z\sqrt{z^2-1}\right] = \text{Im}\ (2zw - 1) = 2w\text{Im}\ z\,,$$

which confirms that $\text{Im}\ z = 0$, i.e. z must be real. If we require in addition that that $\text{Re}\ w < 0$, then we also must have $\text{Re}\ z \leq -1$.

we deduce that

$$\text{Arg}(1-z^2) \text{ is } \begin{cases} \text{positive}\,, & \text{either if } xy<0 \quad \text{or} \quad \text{if } y=0 \text{ and } |x|>1\,, \\ \text{zero}\,, & \text{either if } x=0 \quad \text{or} \quad \text{if } y=0 \text{ and } |x|<1\,, \\ \text{negative}\,, & \text{if } xy>0\,. \end{cases}$$

We exclude the points $z = \pm 1$ (corresponding to $y = 0$ and $x = \pm 1$) where $\text{Arg}(1-z^2)$ is undefined. Treating these two points separately, eq. (94) yields $\text{Arccos}(1) = 0$ and $i\text{Arccos}(-1) = \text{Ln}(-1) = \pi i$. Collecting all of the above results then yields:

$$i\text{Arccos}z = \begin{cases} \text{Ln}(z+\sqrt{z^2-1})\,, & \text{if Im } z>0\,,\ \text{Re } z\geq 0 \ \text{ or } \ \text{Im } z<0\,,\ \text{Re } z\leq 0 \\ & \qquad \text{or } \ \text{Im } z=0\,,\ |\text{Re } z|\leq 1\,, \\ -\text{Ln}(z+\sqrt{z^2-1})\,, & \text{if Im } z>0\,,\ \text{Re } z<0 \ \text{ or } \ \text{Im } z<0\,,\ \text{Re } z>0\,, \\ & \qquad \text{or } \ \text{Im } z=0\,,\ \text{Re } z\geq 1\,, \\ 2\pi i-\text{Ln}(z+\sqrt{z^2-1})\,, & \text{if Im } z=0\,,\ \text{Re } z\leq -1\,. \end{cases} \tag{99}$$

Note that the cases of $z = \pm 1$ are each covered twice in eq. (99) but in both respective cases the two results are consistent.

Comparing eqs. (97) and (99) established eq. (68) and our proof is complete.

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