The argument of a complex number

In these notes, we examine the *argument* of a non-zero complex number z, sometimes called *angle* of z or the *phase* of z. Following eq. (4.1) on p. 49 of Boas, we write:

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}\,, \tag{1}$$

where $x = \text{Re } z$ and $y = \text{Im } z$ are real numbers. The argument of z is denoted by θ, which is measured in radians. However, there is an ambiguity in definition of the argument. The problem is that

$$\sin(\theta + 2\pi) = \sin\theta\,, \qquad\qquad \cos(\theta + 2\pi) = \cos\theta\,,$$

since the sine and the cosine are periodic functions of θ with period 2π. Thus θ is defined only up to an additive integer multiple of 2π. It is common practice to establish a convention in which θ is defined to lie within an interval of length 2π. The most common convention,* which we adopt in these notes, is to take $-\pi < \theta \leq \pi$. With this definition, we identify θ as the so-called *principal value* of the argument, which we denote by Arg z (note the capital A). On the other hand, in many applications, it is convenient to define a multi-valued argument function,

$$\arg z \equiv \text{Arg } z + 2\pi n = \theta + 2\pi n\,, \qquad n = 0, \pm 1, \pm 2, \pm 3, \ldots\,.$$

This is a multi-valued function because for a given complex number z, the number arg z represents an infinite number of possible values. Although Boas does not introduce the multi-valued argument function in Chapter 2, it will become especially useful when we study the properties of the complex logarithm and complex power functions.

1. Definition of the argument function

The argument of a non-zero complex number is a multi-valued function which plays a key role in understanding the properties of the complex logarithm and power functions. Any non-zero complex number z can be written in polar form

$$z = |z|e^{i\arg z}\,, \tag{2}$$

where $\arg z$ is a multi-valued function given by:

$$\arg z = \theta + 2\pi n\,, \qquad n = 0\,, \pm 1\,, \pm 2\,, \pm 3\,, \ldots\,.$$

*Another common convention adopted in some books is to take $0 \leq \theta < 2\pi$. We shall not use this convention in these notes. I leave it to you to make the appropriate modifications if you prefer the latter choice.

Here, $\theta \equiv \text{Arg } z$ is the so-called principal value of the argument, which by convention is taken to lie in the range $-\pi < \theta \leq \pi$. That is,

$$\arg z = \text{Arg } z + 2\pi n\,, \quad n = 0\,, \pm 1\,, \pm 2\,, \pm 3\,, \cdots\,, \quad -\pi < \text{Arg } z \leq \pi\,. \tag{3}$$

It is convenient to have an explicit formula for Arg z in terms of $\arg z$. First, we introduce some notation: $[x]$ means the largest integer less than or equal to the real number x. That is, $[x]$ is the unique integer that satisfies the inequality

$$x - 1 < [x] \leq x\,, \qquad \text{for real } x \text{ and integer } [x]\,. \tag{4}$$

For example, $[1.5] = [1] = 1$ and $[-0.5] = -1$. With this notation, one can write Arg z in terms of $\arg z$ as follows:

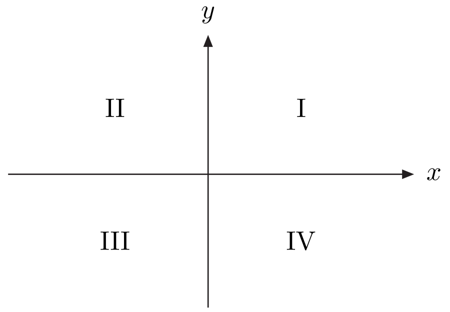
$$\text{Arg } z = \arg z + 2\pi \left[\frac{1}{2} - \frac{\arg z}{2\pi}\right]\,, \tag{5}$$

where [ ] denotes the bracket (or greatest integer) function introduced above. It is straightforward to check that Arg z as defined by eq. (5) does indeed fall inside the principal interval, $-\pi < \theta \leq \pi$.

A more useful equation for Arg z can be obtained as follows. Using the polar representation of $z = x + iy$ given in eq. (1), it follows that $x = r\cos\theta$ and $y = r\sin\theta$. From these two results, one easily derives,

$$|z| = r = \sqrt{x^2 + y^2}\,, \qquad \tan\theta = \frac{y}{x}\,. \tag{6}$$

We identify $\theta = \text{Arg } z$ in the convention where $-\pi < \theta \leq \pi$. In light of eq. (6), it is tempting to identify Arg z with $\arctan(y/x)$. However, the real function $\arctan x$ is a multi-valued function for real values of x. It is conventional to introduce a single-valued real arctangent function, called the principal value of the arctangent,[†] which is denoted by Arctan x and satisfies $-\frac{1}{2}\pi \leq \text{Arctan } x \leq \frac{1}{2}\pi$. Since $-\pi < \text{Arg } z \leq \pi$, it follows that Arg z *cannot* be identified with $\arctan(y/x)$ in all regions of the complex plane. The correct relation between these two quantities is easily ascertained by considering the four quadrants of the complex plane separately. The quadrants of the complex plane (called regions I, II, III and IV) are illustrated in the figure below:



[†]In defining the principal value of the arctangent, we follow the conventions of Keith B. Oldham, Jan Myland and Jerome Spanier, *An Atlas of Functions* (Springer Science, New York, 2009), Chapter 35.

Table 1: Formulae for the argument of a complex number $z = x + iy$.

Quadrant	Sign of x and y	Arg z
I	$x > 0\,,\ y > 0$	$\mathrm{Arctan}(y/x)$
II	$x < 0\,,\ y > 0$	$\pi + \mathrm{Arctan}(y/x)$
III	$x < 0\,,\ y < 0$	$-\pi + \mathrm{Arctan}(y/x)$
IV	$x > 0\,,\ y < 0$	$\mathrm{Arctan}(y/x)$

Table 2: Formulae for the argument of a complex number $z = x+iy$ when z is real or pure imaginary. By convention, the principal value of the argument satisfies $-\pi < \mathrm{Arg}\ z \leq \pi$.

Quadrant border	type of complex number z	Conditions on x and y	Arg z
IV/I	real and positive	$x > 0\,,\ y = 0$	0
I/II	pure imaginary with Im $z > 0$	$x = 0\,,\ y > 0$	$\frac{1}{2}\pi$
II/III	real and negative	$x < 0\,,\ y = 0$	π
III/IV	pure imaginary with Im $z < 0$	$x = 0\,,\ y < 0$	$-\frac{1}{2}\pi$
origin	zero	$x = y = 0$	undefined

The principal value of the argument of $z = x + iy$ in terms of its real part x and imaginary part y is given in Table 1, assuming that z lies within one of the four quadrants of the complex plane. Note that $\mathrm{Arg}\ z = \mathrm{Arctan}(y/x)$ is valid only in quadrants I and IV. If z lies within quadrants II or III, one must add or subtract π in order to ensure that $\frac{1}{2}\pi < \mathrm{Arg}\ z < \pi$ or $-\pi < \mathrm{Arg}\ z < -\frac{1}{2}\pi$, respectively.‡ Cases where z lies on the border between two adjacent quadrants are considered separately in Table 2. In particular, note that the argument of zero is undefined.

Since $z = 0$ if and only if $|z| = 0$, eq. (2) remains valid despite the fact that $\arg 0$ is not defined. When studying the properties of $\arg z$ and $\mathrm{Arg}\, z$ below, we shall always assume implicitly that $z \neq 0$.

‡For finite non-zero values of y/x, the principal value of the arctangent function lies inside the interval $0 < \mathrm{Arctan}(y/x) < \frac{1}{2}\pi$ if $y/x > 0$ and $-\frac{1}{2}\pi < \mathrm{Arctan}(y/x) < 0$ if $y/x < 0$. For completeness, we note that

$$\mathrm{Arctan}(y/x) = \begin{cases} 0\,, & \text{if } y = 0 \text{ and } x \neq 0\,, \\ \frac{1}{2}\pi\,, & \text{if } x = 0 \text{ and } y > 0\,, \\ -\frac{1}{2}\pi\,, & \text{if } x = 0 \text{ and } y < 0\,, \\ \text{undefined}\,, & \text{if } x = y = 0\,. \end{cases}$$

2. Properties of the multi-valued argument function

We can view a multi-valued function $f(z)$ evaluated at z as a set of values, where each element of the set corresponds to a different choice of some integer n. For example, given the multi-valued function $\arg z$ whose principal value is $\mathrm{Arg}\ z \equiv \theta$, then $\arg z$ consists of the set of values:

$$\arg z = \{\theta\,,\ \theta+2\pi\,,\ \theta-2\pi\,,\ \theta+4\pi\,,\ \theta-4\pi\,,\ \cdots\}\,. \tag{7}$$

Given two multi-valued functions, *e.g.*, $f(z) = F(z)+2\pi n$ and $g(z) = G(z)+2\pi n$, where $F(z)$ and $G(z)$ are the principal values of $f(z)$ and $g(z)$ respectively, then $f(z) = g(z)$ if and only if for each point z, the corresponding set of values of $f(z)$ and $g(z)$ precisely coincide:

$$\{F(z)\,,\ F(z)+2\pi\,,\ F(z)-2\pi\,,\ \cdots\} \;=\; \{G(z)\,,\ G(z)+2\pi\,,\ G(z)-2\pi\,,\ \cdots\}\,. \tag{8}$$

Sometimes, one refers to the equation $f(z) = g(z)$ as a *set equality* since all the elements of the two sets in eq. (8) must coincide. We add two additional rules to the concept of set equality. First, the ordering of terms within the set is unimportant. Second, we only care about the distinct elements of each set. That is, if our list of set elements has repeated entries, we omit all duplicate elements.

To see how the set equality of two multi-valued functions works, let us consider the multi-valued function $\arg z$. One can prove that:

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2\,, \tag{9}$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2\,. \tag{10}$$

$$\arg\left(\frac{1}{z}\right) = \arg\overline{z} = -\arg z\,. \tag{11}$$

To prove eq. (9), consider $z_1 = |z_1|e^{i\arg\ z_1}$ and $z_2 = |z_2|e^{i\arg\ z_2}$. The arguments of these two complex numbers are: $\arg z_1 = \mathrm{Arg}\ z_1 + 2\pi n_1$ and $\arg z_2 = \mathrm{Arg}\ z_2 + 2\pi n_2$, where n_1 and n_2 are arbitrary integers. [One can also write $\arg z_1$ and $\arg z_2$ in set notation as in eq. (7).] Thus, one can also write $z_1 = |z_1|e^{i\mathrm{Arg}\ z_1}$ and $z_2 = |z_2|e^{i\mathrm{Arg}\ z_2}$, since $e^{2\pi i n} = 1$ for any integer n. It then follows that

$$z_1 z_2 = |z_1 z_2| e^{i(\mathrm{Arg}\ z_1 + \mathrm{Arg}\ z_2)}\,,$$

where we have used $|z_1||z_2| = |z_1 z_2|$. Thus, $\arg(z_1 z_2) = \mathrm{Arg}\ z_1 + \mathrm{Arg}\ z_2 + 2\pi n_{12}$, where n_{12} is also an arbitrary integer. Therefore, we have established that:

$$\arg z_1 + \arg z_2 = \mathrm{Arg}\ z_1 + \mathrm{Arg}\ z_2 + 2\pi(n_1+n_2)\,,$$

$$\arg(z_1 z_2) = \mathrm{Arg}\ z_1 + \mathrm{Arg}\ z_2 + 2\pi n_{12}\,,$$

where n_1, n_2 and n_{12} are arbitrary integers. Thus, $\arg z_1 + \arg z_2$ and $\arg(z_1 z_2)$ coincide as sets, and so eq. (9) is confirmed. One can easily prove eqs. (10) and (11) by a similar method. In particular, if one writes $z = |z|e^{i \arg z}$ and employs the definition of the complex conjugate (which yields $|\overline{z}| = |z|$ and $\overline{z} = |z|e^{-i \arg z}$), then it follows that $\arg(1/z) = \arg \overline{z} = -\arg z$. As an instructive example, consider the last relation in the case of $z = -1$. It then follows that

$$\arg(-1) = -\arg(-1)\,,$$

as a set equality. This is not paradoxical, since the sets,

$$\arg(-1) = \{\pm\pi\,,\, \pm 3\pi\,,\, \pm 5\pi\,,\, \ldots\} \qquad \text{and} \qquad -\arg(-1) = \{\mp\pi\,,\, \mp 3\pi\,,\, \mp 5\pi\,,\, \ldots\}\,,$$

coincide, as they possess precisely the same list of elements.

Now, for a little surprise:

$$\arg z^2 \neq 2 \arg z\,. \tag{12}$$

To see why this statement is surprising, consider the following false proof. Use eq. (9) with $z_1 = z_2 = z$ to derive:

$$\arg z^2 = \arg z + \arg z \stackrel{?}{=} 2 \arg z\,, \qquad \text{[FALSE!!]}. \tag{13}$$

The false step is the one indicated by the symbol $\stackrel{?}{=}$ above. Given $z = |z|e^{i \arg z}$, one finds that $z^2 = |z|^2 e^{2i(\mathrm{Arg}\ z + 2\pi n)} = |z|^2 e^{2i \mathrm{Arg}\ z}$, and so the possible values of $\arg(z^2)$ are:

$$\arg(z^2) = \{2\mathrm{Arg}\ z\,,\, 2\mathrm{Arg}\ z + 2\pi\,,\, 2\mathrm{Arg}\ z - 2\pi\,,\, 2\mathrm{Arg}\ z + 4\pi\,,\, 2\mathrm{Arg}\ z - 4\pi\,,\, \cdots\}\,,$$

whereas the possible values of $2 \arg z$ are:

$$\begin{aligned} 2 \arg z &= \{2\mathrm{Arg}\ z\,,\, 2(\mathrm{Arg}\ z + 2\pi)\,,\, 2(\mathrm{Arg}\ z - 2\pi)\,,\, 2(\mathrm{Arg}\ z + 4\pi)\,,\, \cdots\} \\ &= \{2\mathrm{Arg}\ z\,,\, 2\mathrm{Arg}\ z + 4\pi\,,\, 2\mathrm{Arg}\ z - 4\pi\,,\, 2\mathrm{Arg}\ z + 8\pi\,,\, 2\mathrm{Arg}\ z - 8\pi\,,\, \cdots\}\,. \end{aligned}$$

Thus, $2 \arg z$ is a *subset* of $\arg(z^2)$, but half the elements of $\arg(z^2)$ are missing from $2 \arg z$. These are therefore unequal sets, as indicated by eq. (12). Now, you should be able to see what is wrong with the statement:

$$\arg z + \arg z \stackrel{?}{=} 2 \arg z\,. \tag{14}$$

When you add $\arg\ z$ as a set to itself, the element you choose from the first $\arg z$ need not be the same as the element you choose from the second $\arg z$. In contrast, $2 \arg z$ means take the set $\arg z$ and multiply each element by two. The end result is that $2 \arg z$ contains only half the elements of $\arg z + \arg z$ as shown above.

Here is one more example of an incorrect proof. Consider eq. (10) with $z_1 = z_2 \equiv z$. Then, you might be tempted to write:

$$\arg\left(\frac{z}{z}\right) = \arg(1) = \arg z - \arg z \stackrel{?}{=} 0\,.$$

This is clearly wrong since $\arg(1) = 2\pi n$, where n is the set of integers. Again, the error occurs with the step:

$$\arg z - \arg z \stackrel{?}{=} 0\,. \tag{15}$$

The fallacy of this statement is the same as above. When you subtract $\arg z$ as a set from itself, the element you choose from the first $\arg z$ need not be the same as the element you choose from the second $\arg z$.

3. Properties of the principal value of the argument

The properties of the principal value Arg z are not as simple as those given in eqs. (9)–(11), since the range of Arg z is restricted to lie within the principal range $-\pi < \text{Arg } z \leq \pi$. Instead, the following relations are satisfied:

$$\text{Arg }(z_1 z_2) = \text{Arg } z_1 + \text{Arg } z_2 + 2\pi N_+\,, \tag{16}$$

$$\text{Arg }(z_1/z_2) = \text{Arg } z_1 - \text{Arg } z_2 + 2\pi N_-\,, \tag{17}$$

where the integers $N_\pm$ are determined as follows:

$$N_\pm = \begin{cases} -1\,, & \text{if } \text{Arg } z_1 \pm \text{Arg } z_2 > \pi\,, \\ 0\,, & \text{if } -\pi < \text{Arg } z_1 \pm \text{Arg } z_2 \leq \pi\,, \\ 1\,, & \text{if } \text{Arg } z_1 \pm \text{Arg } z_2 \leq -\pi\,. \end{cases} \tag{18}$$

If we set $z_1 = 1$ in eq. (17), we find that

$$\text{Arg}(1/z) = \text{Arg } \overline{z} = \begin{cases} \text{Arg}\, z\,, & \text{if Im } z = 0 \text{ and } z \neq 0\,, \\ -\text{Arg}\, z\,, & \text{if Im } z \neq 0\,. \end{cases} \tag{19}$$

Note that for z real, both $1/z$ and $\overline{z}$ are also real so that in this case $z = \overline{z}$ and $\text{Arg}(1/z) = \text{Arg } \overline{z} = \text{Arg } z$.

If n is an integer, then

$$\arg z^n = \arg z + \arg z + \cdots + \arg z \neq n \arg z\,, \tag{20}$$

where the final inequality above was noted in the case of $n = 2$ in eq. (12). The corresponding property of Arg z is much simpler:

$$\text{Arg}(z^n) = n\text{Arg } z + 2\pi N_n\,, \tag{21}$$

where the integer N_n is given by:

$$N_n = \left[\frac{1}{2} - \frac{n}{2\pi}\text{Arg } z\right]\,, \tag{22}$$

and [ ] is the greatest integer bracket function introduced in eq. (4). It is straightforward to verify eqs. (16)–(19) and eq. (21). These formulae follow from the corresponding properties of $\arg z$, taking into account the requirement that Arg z must lie within the principal interval, $-\pi < \theta \leq \pi$.