Asymptotic Power Series

In this note, I will define an asymptotic power series, and contrast its definition with that of a convergent power series. All convergent power series are asymptotic series. However, an asymptotic power series may be convergent or divergent.

We first consider a power series of a function $f(x)$ expanded about the point $x = a$ (where a is finite; the point $a = 0$ is the most common example). Given a power series approximation to $f(x)$, we may write

$$f(x) = f(a) + \sum_{n=1}^{N} c_n(x-a)^n + R_N(x)\,. \tag{1}$$

where $R_N(x)$ is the remainder term. The power series in eq. (1) is *convergent* if

$$\lim_{N\to\infty}\left[f(x) - \sum_{n=0}^{N} c_n(x-a)^n\right] = 0\,, \qquad \text{for } |x-a| < r\,.$$

That is, for any value of x whose distance from a lies within the radius of convergence (which is denoted in these notes by r), $\lim_{N\to\infty} R_N(x) = 0$. A convergent power series of $f(x)$ equivalent to its Taylor series expanded about $x = a$. The coefficients of the Taylor series are uniquely determined,

$$c_n = f^{(n)}(a)/n!\,, \tag{2}$$

where $f^{(n)} \equiv (d^n f/dx^n)_{x=a}$.

It is convenient to introduce the big-oh (order) symbol. We say that

$$f(x) = \mathcal{O}(g(x))\,, \quad \text{as } x \to a\,,$$

if there exist a finite constant M such that $|f(x)| \leq M|g(x)|$ as $x \to a$. For a convergent power series,

$$f(x) = f(a) + \sum_{n=1}^{\infty} c_n(x-a)^n\,, \qquad \text{for } |x-a| < r\,,$$

it follows that when written as a finite sum plus a remainder as in eq. (1), we have

$$R_N(x) = \mathcal{O}((x-a)^{N+1})\,. \tag{3}$$

This result is a consequence of the uniqueness of the convergent power series expansion, which implies that $R_N(x) = \sum_{n=N+1}^{\infty} c_n(x-a)^n$. Eq. (3) immediately follows from the definition of the big-oh symbol.

Consider now a different limiting procedure where we fix N and study the behavior or $R_N(x)$ in the limit of $x \to a$. Since we never take N to infinity, the question of convergence or divergence does not enter. The power series of eq. (1) is *asymptotic* as $x \to a$ if

$$\lim_{x\to a} \frac{1}{(x-a)^N} \left[f(x) - \sum_{n=0}^{N} c_n (x-a)^n \right] = 0\,, \quad \text{for any fixed finite value of } N. \tag{4}$$

As a result, the remainder term of an asymptotic power series also satisfies $R_N(x) = \mathcal{O}((x-a)^{N+1})$. More explicitly, we have

$$\lim_{x\to a} \frac{1}{(x-a)^N} \left[f(x) - \sum_{n=0}^{N} c_n (x-a)^n \right] = \lim_{x\to a} [c_{N+1}(x-a) + \mathcal{O}((x-a)^2)] = 0\,. \tag{5}$$

One can check that any convergent power series is automatically an asymptotic series. However, eq. (4) alone does not imply that the asymptotic power series is convergent. Thus, an asymptotic power series may be either convergent or divergent. If the asymptotic series is divergent, then one cannot take the limit of $N \to \infty$. Nevertheless, it is common practice to write:

$$f(x) \sim \sum_{n=0}^{\infty} c_n (x-a)^n\,, \quad \text{as } x \to a\,, \tag{6}$$

for a divergent asymptotic series. This notation should be understood to mean that one only sums a finite number of terms and uses the remainder term [as defined in eq. (1)] to estimate the error.

The coefficients of an asymptotic series are determined by the following formula [which is a consequence of eq. (5)]:

$$\lim_{x\to a} \frac{1}{(x-a)^{N+1}} \left[f(x) - \sum_{n=0}^{N} c_n (x-a)^n \right] = c_{N+1} \qquad N = 0, 1, 2, \ldots\,. \tag{7}$$

Here, $c_0 = f(a)$, and the coefficients are determined in order starting with c_1.

Clearly, the asymptotic series of $f(x)$ as $x \to a$ is unique, since the coefficients of the series are determined by eq. (7). However, the converse is not true. In particular, an asymptotic series does not uniquely determine the function, since there are functions whose asymptotic series are equal to zero. For example, e^{-1/x^2} is asymptotic to zero as $x \to 0$ since $\lim_{x\to 0} [e^{-1/x^2} - 0]/x^N = 0$. In the notation of eq. (6), we shall write $e^{-1/x^2} \sim 0$ as $x \to 0$. That is, exponentially small terms are *subdominant* to the power law terms that define the asymptotic expansion.

Not all functions possess asymptotic power series. But often, a function takes the form $g(x) = h(x)f(x)$, where $f(x)$ possesses an asymptotic power series. As an example, you will demonstrate on problem 9 of homework set #2 that $xe^x E_1(x)$ possesses an asymptotic power series as $x \to \infty$.

The definitions above need to be modified slightly for power series that are expanded about the point of infinity. Such a power series takes the form:

$$f(x) = f(\infty) + \sum_{n=1}^{N} \frac{c_n}{x^n} + R_N(x)\,.$$

In this case, $R_N(x) = \mathcal{O}(1/x^{N+1})$. A convergent power series would satisfy:

$$\lim_{N\to\infty} \left[f(x) - \sum_{n=0}^{N} \frac{c_n}{x^n} \right] = 0\,, \qquad \text{for } |x| > r\,.$$

The corresponding condition for an asymptotic series as $x \to \infty$ is given by:

$$\lim_{x\to\infty} x^N \left[f(x) - \sum_{n=0}^{N} \frac{c_n}{x^n} \right] = 0\,, \qquad \text{for any fixed value of } N\,.$$

The coefficients of the this asymptotic power series are then given by:

$$\lim_{x\to\infty} x^{N+1} \left[f(x) - \sum_{n=0}^{N} \frac{c_n}{x^n} \right] = c_{N+1} \qquad N = 0, 1, 2, \ldots\,,$$

where $c_0 = f(\infty)$. Again, the coefficients are determined in order starting with c_1. For a divergent asymptotic power series as $x \to \infty$, we write:

$$f(x) \sim \sum_{n=0}^{\infty} \frac{c_n}{x^n}\,, \quad \text{as } \ x \to \infty\,, \tag{8}$$

with similar caveats to the ones discussed below eq. (6).

Most asymptotic series that you will encounter are divergent. Nevertheless, they generally provide very useful approximations to the function near the point $x = a$ (or for very large x if the asymptotic series is for $x \to \infty$). Of course, when evaluating the value of the function at some point (let us call it b) that is near a, one needs to decide how large to take N. If this were a convergent series, one could choose any N and the approximation would get better and better with larger and larger N. For a divergent asymptotic series, one does not have the option of taking arbitrarily large N (after all, the series diverges). For a given b, there is always an optimal choice for N that gives the best approximation. How good this best approximation is depends on how far b is from a. The closer b is to a, the larger the N that corresponds to the optimal approximation.

For a divergent asymptotic series consisting of positive terms, the magnitudes of the coefficients c_n will initially decrease as n increases. But eventually, the magnitudes of the coefficients starting to increase again. As $n \to \infty$, one typically finds that $|c_n| \to \infty$. In this case, the optimal choice for n is often the value of n corresponding to the minimal value of c_n. That is, the optimal approximation of

$f(x)$ is obtained by truncating the series when the minimal value of c_n is reached. This rule of thumb can be rigorously justified for some divergent asymptotic series, but may not be reliable in all cases. If one has a closed form expression for $R_N(x)$, then of course this would provide the best guide for estimating the error. For divergent asymptotic series consisting of alternating positive and negative terms, the partial sums will oscillate below and above the actual value of the function. In this case, the best approximation often corresponds to the average of the two adjacent partial sums with the smallest difference.

To illustrate the last point, consider the asymptotic series:

$$f(x) \equiv \int_0^\infty \frac{e^{-t}}{1+xt}\,dt \sim \sum_{n=0}^\infty (-1)^n n!\, x^n \quad \text{as } x \to 0^+\,, \tag{9}$$

where $x \to 0^+$ means that x approaches zero from the positive side. Suppose we wish to find the numerical value of $f(0.1)$. Let us evaluate the partial sums $S_N = \sum_{n=0}^N (-1)^n n! x^n$ for $x = 0.1$ and $N = 3, 4, 5, \ldots, 26$. The results, obtained by Mathematica, are displayed in the table below (to six significant figures).

N	$S_N(0.1)$
3	0.92
4	0.914
5	0.9164
6	0.9152
7	0.91592
8	0.915416
9	0.915819
10	0.915463
11	0.915819
12	0.915420
13	0.915899
14	0.915276
15	0.916148
16	0.914840
17	0.916933
18	0.913376
19	0.919778
20	0.907614
21	0.931943
22	0.880852
23	0.993252
24	0.734732
25	1.355180
26	−0.195941

If we examine the table closely, we see that the smallest difference between two adjacent terms corresponds to $N = 10$ and $N = 11$. Averaging these two values gives us our optimal approximation: $f(0.1) \simeq 0.915641$. Numerically integrating the function with the help of Mathematica, I find $f(0.1) = 0.915633$. Thus, we have achieved four significant figure accuracy with the asymptotic expansion of $f(x)$. You can also begin to see the effects of the divergent series as N increases significantly beyond the optimal value of N. By the time we get to $N = 100$, $S_{100} = -8.47714 \times 10^{56}$ and $S_{101} = 8.48491 \times 10^{57}$. Need I say more?

The following properties of asymptotic series are noteworthy. Given two asymptotic series (in both cases as $x \to a$), then arithmetic operations such as addition, subtraction, multiplication and division can be performed term by term. If $f(x)$ is continuous and integrable near $x = a$, then one may integrate the asymptotic series given by eq. (6) term by term to get another asymptotic series:

$$\int_a^x f(t)\,dt \sim \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}\,, \quad \text{as } x \to a\,.$$

Finally, if $f(x)$ has a continuous derivative, and $f'(x)$ possesses an asymptotic power series as $x \to a$, then one may differentiate an asymptotic series given in eq. (6) term by term to produce a new asymptotic series:

$$\frac{df(x)}{dx} \sim \sum_{n=1}^{\infty} n\,c_n(x-a)^{n-1}\,, \quad \text{as } x \to a\,. \tag{10}$$

Similar results hold for asymptotic power series expanded about $x \to \infty$.

The condition for differentiating an asymptotic series is a little stronger than for integrating a series. The reason has to do with the fact that two functions differing by an exponentially subdominant term possess the same asymptotic series. However, upon differentiation these subdominant terms could end up contributing significantly to the differentiated function. The classic example is the case of $f(x) = e^{-x}\sin(e^x)$. This function has an asymptotic series $f(x) \sim 0$ as $x \to \infty$. However, $f'(x) = \cos(e^x) - e^{-x}\sin(e^x)$ oscillates as $x \to \infty$ and thus has no asymptotic expansion of the form given in eq. (8).

Verification of the asymptotic series given in eq. (9)

We end this short introduction with a practical example. Let us derive the asymptotic power series for the function $f(x)$ given in eq. (9):

$$f(x) \equiv \int_0^\infty \frac{e^{-t}}{1+xt}\, dt \sim \sum_{n=0}^{\infty} (-1)^n n!\, x^n \quad \text{as } x \to 0^+\,, \tag{11}$$

where $x \to 0^+$ means that x approaches zero from the positive side. Recall that for a finite geometric series:

$$\sum_{n=0}^{N} (-1)^n (xt)^n = \frac{1-(-xt)^{N+1}}{1+xt}\,.$$

Thus, we can insert

$$\frac{1}{1+xt} = \sum_{n=0}^{N} (-1)^n (xt)^n + \frac{(-xt)^{N+1}}{1+xt}$$

into the expression for $f(x)$ given in eq. (11) to obtain:*

$$\begin{aligned} f(x) &= \int_0^\infty e^{-t}\, dt \sum_{n=0}^{N} (-1)^n (xt)^n + R_N(x) \\ &= \sum_{n=0}^{N} (-1)^n\, x^n \int_0^\infty e^{-t} t^n\, dt + R_N(x) \\ &= \sum_{n=0}^{N} (-1)^n\, n!\, x^n + R_N(x)\,, \end{aligned}$$

where

$$R_N(x) \equiv \int_0^\infty \frac{e^{-t}(-xt)^{N+1}}{1+xt}\, dt\,.$$

To conclude that eq. (11) is the correct asymptotic power series expansion for $f(x)$, all we need to prove is that $R_N(x) = \mathcal{O}(x^{N+1})$ as $x \to 0^+$. This is most easily accomplished by noting that $(1+xt)^{-1} \leq 1$ for $x > 0$ and $t \geq 0$, Hence,

$$|R_N(x)| \leq x^{N+1} \int_0^\infty e^{-t}\, t^{N+1}\, dt = (N+1)!\, x^{N+1}\,.$$

Thus, we have proved that $\lim_{x\to 0} x^{-(N+1)} |R_{N+1}| \leq (N+1)!$, which means that $R_N(x) = \mathcal{O}(x^{N+1})$. Equivalently, we have verified that $\lim_{x\to 0} x^{-N}\, R_{N+1} = 0$ which coincides with the definition of an asymptotic expansion given in eq. (4).

*The interchange of the order of the sum and integral is always possible when the sum involves a finite number of terms.

It is tempting to use a shortcut in deriving the asymptotic series of eq. (11) directly by inserting the expansion

$$(1+xt)^{-1} = \sum_{N=0}^{\infty} (-1)^n \, (xt)^n \tag{12}$$

directly into the integral and then integrating the sum term by term. Indeed, this does yield the expansion given in eq. (11) as follows

$$\begin{aligned} f(x) &\stackrel{?}{=} \int_0^\infty e^{-t}\, dt \sum_{n=0}^{\infty} (-1)^n (xt)^n \\ &\stackrel{?}{=} \sum_{n=0}^{\infty} (-1)^n \, x^n \int_0^\infty e^{-t}\, t^n \, dt \\ &\stackrel{?}{=} \sum_{n=0}^{\infty} (-1)^n \, n! \, x^n \, . \end{aligned}$$

However, this procedure is not mathematically valid, since eq. (12) converges only when $|xt| < 1$, which is equivalent to $|t| < 1/|x|$. However, the range of integration is $0 \leq t < \infty$, so some range of values of t in the integration region lies outside the validity of eq. (12). This observation provides the explanation for why the resulting asymptotic series is divergent. If the use of eq. (12) had been mathematically correct throughout the entire range of integration, the result of integrating term by term would have been convergent.[†] Thus, it is not surprising that the resulting asymptotic series is divergent. Nevertheless, if one employs an infinite series in the evaluation of an integral whose integration range is larger than the radius of convergence of the infinite series, there is no guarantee in general that the end result will correspond to the desired asymptotic series. To be completely confident of the final result, one must check that the remainder term of any finite sum satisfies the requirement of eq. (3).

[†] This last statement assumes that the interchange of the order of integration and the infinite summation is valid. Such an interchange is permitted for a uniformly convergent sum and integral.