The complex logarithm, exponential and power functions

In these notes, we examine the logarithm, exponential and power functions, where the arguments* of these functions can be complex numbers. In particular, we are interested in how their properties differ from the properties of the corresponding real-valued functions.†

1. Review of the properties of the argument of a complex number

Before we begin, I shall review the properties of the argument of a non-zero complex number z, denoted by $\arg z$ (which is a multi-valued function), and the *principal value* of the argument, Arg z, which is single-valued and conventionally defined such that:

$$-\pi < \text{Arg } z \leq \pi \,. \tag{1}$$

Details can be found in the class handout entitled, *The argument of a complex number*. Here, we recall a number of results from that handout. One can regard $\arg z$ as a set consisting of the following elements,

$$\arg z = \text{Arg } z + 2\pi n \,, \quad n = 0\,,\, \pm 1\,,\, \pm 2\,,\, \pm 3\,,\, \ldots\,, \quad -\pi < \text{Arg } z \leq \pi \,. \tag{2}$$

One can also express Arg z in terms of $\arg z$ as follows:

$$\text{Arg } z = \arg z + 2\pi \left[\frac{1}{2} - \frac{\arg z}{2\pi} \right] \,, \tag{3}$$

where [ ] denotes the greatest integer function. That is, $[x]$ is defined to be the largest integer less than or equal to the real number x. Consequently, $[x]$ is the unique integer that satisfies the inequality

$$x - 1 < [x] \leq x \,, \qquad \text{for real } x \text{ and integer } [x] \,. \tag{4}$$

*Note that the word *argument* has two distinct meanings. In this context, given a function $w = f(z)$, we say that z is the argument of the function f. This should not be confused with the argument of a complex number, $\arg z$.

†The following three books were particularly useful in the preparation of these notes:

1. *Complex Variables and Applications*, by James Ward Brown and Ruel V. Churchill (McGraw Hill, New York, 2004).

2. *Elements of Complex Variables*, by Louis L. Pennisi, with the collaboration of Louis I. Gordon and Sim Lasher (Holt, Rinehart and Winston, New York, 1963).

3. *The Theory of Analytic Functions: A Brief Course*, by A.I. Markushevich (Mir Publishers, Moscow, 1983).

For example, $[1.5] = [1] = 1$ and $[-0.5] = -1$. One can check that Arg z as defined in eq. (3) does fall inside the principal interval specified by eq. (1).

The multi-valued function arg z satisfies the following properties,

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2 \,, \tag{5}$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2 \,. \tag{6}$$

$$\arg\left(\frac{1}{z}\right) = \arg\ \overline{z} = -\arg z \,. \tag{7}$$

Eqs. (5)–(7) should be viewed as set equalities, i.e. the elements of the sets indicated by the left-hand side and right-hand side of the above identities coincide. However, the following results are *not* set equalities:

$$\arg z + \arg z \neq 2 \arg z \,, \qquad\qquad \arg z - \arg z \neq 0 \,, \tag{8}$$

which, by virtue of eqs. (5) and (6), yield:

$$\arg z^2 = \arg z + \arg z \neq 2 \arg z \,, \qquad\qquad \arg(1) = \arg z - \arg z \neq 0 \,. \tag{9}$$

For example, $\arg(1) = 2\pi n$, for $n = 0 \pm 1, \pm 2, \ldots$. More generally,

$$\arg z^n = \underbrace{\arg z + \arg z + \cdots \arg z}_{n} \neq n \ \arg z \,. \tag{10}$$

We also note some properties of the the principal value of the argument.

$$\text{Arg }(z_1 z_2) = \text{Arg } z_1 + \text{Arg } z_2 + 2\pi N_+ \,, \tag{11}$$

$$\text{Arg }(z_1/z_2) = \text{Arg } z_1 - \text{Arg } z_2 + 2\pi N_- \,, \tag{12}$$

where the integers $N_\pm$ are determined as follows:

$$N_\pm = \begin{cases} -1 \,, & \text{if} \quad \text{Arg } z_1 \pm \text{Arg } z_2 > \pi \,, \\ \ \ 0 \,, & \text{if} \quad -\pi < \text{Arg } z_1 \pm \text{Arg } z_2 \leq \pi \,, \\ \ \ 1 \,, & \text{if} \quad \text{Arg } z_1 \pm \text{Arg } z_2 \leq -\pi \,. \end{cases} \tag{13}$$

If we set $z_1 = 1$ in eq. (12), we find that

$$\text{Arg}(1/z) = \text{Arg } \overline{z} = \begin{cases} \ \ \text{Arg}\, z \,, & \text{if Im } z = 0 \ \text{ and } z \neq 0 \,, \\ -\text{Arg}\, z \,, & \text{if Im } z \neq 0 \,. \end{cases} \tag{14}$$

Note that for z real, both $1/z$ and $\overline{z}$ are also real so that in this case $z = \overline{z}$ and $\text{Arg}(1/z) = \text{Arg } \overline{z} = \text{Arg } z$. In addition, in contrast to eq. (10), we have

$$\text{Arg}(z^n) = n \,\text{Arg } z + 2\pi N_n \,, \tag{15}$$

where the integer N_n is given by:

$$N_n = \left[\frac{1}{2} - \frac{n}{2\pi}\mathrm{Arg}\; z\right]\,, \tag{16}$$

and [ ] is the greatest integer bracket function introduced in eq. (4).

2. Properties of the real-valued logarithm, exponential and power functions

Consider the logarithm of a positive real number. This function satisfies a number of properties:

$$e^{\ln x} = x\,, \tag{17}$$

$$\ln(e^a) = a\,, \tag{18}$$

$$\ln(xy) = \ln(x) + \ln(y)\,, \tag{19}$$

$$\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)\,, \tag{20}$$

$$\ln\left(\frac{1}{x}\right) = -\ln(x)\,, \tag{21}$$

$$\ln x^p = p\ln x\,, \tag{22}$$

for positive real numbers x and y and arbitrary real numbers a and p. Likewise, the power function defined over the real numbers satisfies:

$$x^a = e^{a\ln x}\,, \tag{23}$$

$$x^a x^b = x^{a+b}\,, \tag{24}$$

$$\frac{x^a}{x^b} = x^{a-b}\,, \tag{25}$$

$$\frac{1}{x^a} = x^{-a}\,, \tag{26}$$

$$(x^a)^b = x^{ab}\,, \tag{27}$$

$$(xy)^a = x^a y^a\,, \tag{28}$$

$$\left(\frac{x}{y}\right)^a = x^a y^{-a}\,, \tag{29}$$

for positive real numbers x and y and arbitrary real numbers a and b. Closely related to the power function is the generalized exponential function defined over the real numbers. This function satisfies:

$$a^x = e^{x\ln a}\,, \tag{30}$$

$$a^x a^y = a^{x+y}\,, \tag{31}$$

$$\frac{a^x}{a^y} = a^{x-y}\,, \tag{32}$$

$$\frac{1}{a^x} = a^{-x}\,, \tag{33}$$

$$(a^x)^y = a^{xy}\,, \tag{34}$$

$$(ab)^x = a^x b^x\,, \tag{35}$$

$$\left(\frac{a}{b}\right)^x = a^x b^{-x}\,. \tag{36}$$

for positive real numbers a and b and arbitrary real numbers x and y.

We would like to know which of these relations are satisfied when these functions are extended to the complex plane. It is dangerous to assume that all of the above relations are valid in the complex plane without modification, as this assumption can lead to seemingly paradoxical conclusions. Here are three examples:

1. Since $1/(-1) = (-1)/1 = -1$,

$$\sqrt{\frac{1}{-1}} = \frac{1}{i} = \sqrt{\frac{-1}{1}} = \frac{i}{1}\,. \tag{37}$$

Hence, $1/i = i$ or $i^2 = 1$. But $i^2 = -1$, so we have proven that $1 = -1$.

2. Since $1 = (-1)(-1)$,

$$1 = \sqrt{1} = \sqrt{(-1)(-1)} = (\sqrt{-1})(\sqrt{-1}) = i \cdot i = -1\,. \tag{38}$$

3. To prove that $\ln(-z) = \ln(z)$ for all $z \neq 0$, we proceed as follows:

$$\begin{aligned}
\ln(z^2) &= \ln[(-z)^2]\,,\\
\ln(z) + \ln(z) &= \ln(-z) + \ln(-z)\,,\\
2\ln(z) &= 2\ln(-z)\,,\\
\ln(z) &= \ln(-z)\,.
\end{aligned}$$

Of course, all these "proofs" are faulty. The fallacy in the first two proofs can be traced back to eqs. (28) and (29), which are true for real-valued functions but not true in general for complex-valued functions. The fallacy in the third proof is more subtle, and will be addressed later in these notes. A careful study of the complex logarithm, power and exponential functions will reveal how to correctly modify eqs. (17)–(36) and avoid pitfalls that can lead to false results.

3. Definition of the complex exponential function

We begin with the complex exponential function, which is defined via its power series:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \,,$$

where z is any complex number. Using this power series definition, one can verify that:

$$\boxed{e^{z_1+z_2} = e^{z_1} e^{z_2} \,, \qquad \text{for all complex } z_1 \text{ and } z_2 \,.} \tag{39}$$

In particular, if $z = x + iy$ where x and y are real, then it follows that

$$e^z = e^{x+iy} = e^x \, e^{iy} = e^x(\cos y + i \sin y) \,.$$

One can quickly verify that eqs. (30)–(33) are satisfied by the complex exponential function. In addition, eq. (34) clearly holds when the outer exponent is an integer:

$$(e^z)^n = e^{nz} \,, \qquad n = 0 \,, \pm 1 \,, \pm 2 \,, \ldots \,. \tag{40}$$

If the outer exponent is a non-integer, then the resulting expression is a multi-valued power function. We will discuss this case in more detail in section 8.

Before moving on, we record one key property of the complex exponential:

$$e^{2\pi i n} = 1 \,, \qquad n = 0 \,, \pm 1 \,, \pm 2 \,, \pm 3 \,, \ldots \,. \tag{41}$$

4. Definition of the complex logarithm

In order to define the complex logarithm, one must solve the complex equation:

$$z = e^w \,, \tag{42}$$

for w, where z is any non-zero complex number. If we write $w = u + iv$, then eq. (42) can be written as

$$e^u e^{iv} = |z| e^{i \arg z} \,. \tag{43}$$

Eq. (43) implies that:

$$|z| = e^u \,, \qquad v = \arg z \,.$$

The equation $|z| = e^u$ is a real equation, so we can write $u = \ln |z|$, where $\ln |z|$ is the ordinary logarithm evaluated with positive real number arguments. Thus,

$$w = u + iv = \ln |z| + i \arg z = \ln |z| + i(\mathrm{Arg}\ z + 2\pi n) \,, \quad n = 0 \,, \pm 1 \,, \pm 2 \,, \pm 3 \,, \ldots \tag{44}$$

We call w the complex logarithm and write $w = \ln z$. This is a somewhat awkward notation since in eq. (44) we have already used the symbol ln for the real logarithm. We shall finesse this notational quandary by denoting the real logarithm in eq. (44) by the symbol Ln. That is, $\mathrm{Ln}|z|$ shall denote the ordinary real logarithm of $|z|$. With this notational convention, we rewrite eq. (44) as:

$$\boxed{\ln z = \mathrm{Ln}|z| + i \arg z = \mathrm{Ln}|z| + i(\mathrm{Arg}\ z + 2\pi n)\,, \quad n = 0\,,\, \pm 1\,,\, \pm 2\,,\, \pm 3\,,\, \ldots} \tag{45}$$

for any non-zero complex number z.

Clearly, $\ln z$ is a multi-valued function (as its value depends on the integer n). It is useful to define a single-valued *complex* function, $\mathrm{Ln}\, z$, called the principal value of $\ln z$ as follows:

$$\boxed{\mathrm{Ln}\, z = \mathrm{Ln}\,|z| + i\mathrm{Arg}\ z\,, \qquad -\pi < \mathrm{Arg}\ z \leq \pi\,,} \tag{46}$$

which extends the definition of Ln z to the entire complex plane (excluding the origin, $z = 0$, where the logarithmic function is singular). In particular, eq. (46) implies that $\mathrm{Ln}(-1) = i\pi$. Note that for real positive z, we have $\mathrm{Arg}\ z = 0$, so that eq. (46) simply reduces to the usual real logarithmic function in this limit.

The relation between $\ln z$ and its principal value is simple:

$$\ln z = \mathrm{Ln}\ z + 2\pi i n\,, \quad n = 0\,,\, \pm 1\,,\, \pm 2\,,\, \pm 3\,,\, \ldots\,.$$

5. Properties of the complex logarithm

We now consider which of the properties given in eqs. (17)–(22) apply to the complex logarithm. Since we have defined the multi-value function $\ln z$ and the single-valued function Ln z, we should examine the properties of both these functions. We begin with the multi-valued function $\ln z$. First, we examine eq. (17). Using eq. (45), it follows that:

$$e^{\ln z} = e^{\mathrm{Ln}|z|} e^{i\mathrm{Arg}\ z} e^{2\pi i n} = |z| e^{i\mathrm{Arg}\ z} = z\,. \tag{47}$$

Thus, eq. (17) is satisfied. Next, we examine eq. (18) for $z = x + iy$:

$$\ln(e^z) = \mathrm{Ln}|e^z| + i(\arg\ e^z) = \mathrm{Ln}(e^x) + i(y + 2\pi k) = x + iy + 2\pi i k = z + 2\pi i k\,,$$

where k is an arbitrary integer. In deriving this result, we used the fact that $e^z = e^x e^{iy}$, which implies that $\arg(e^z) = y + 2\pi k$.[‡] Thus,

$$\ln(e^z) = z + 2\pi i k \neq z\,, \qquad \text{unless } k = 0\,. \tag{48}$$

[‡]Note that $\mathrm{Arg}\ e^z = y + 2\pi N$, where N is chosen such that $-\pi < y + 2\pi N \leq \pi$. Moreover, eq. (2) implies that $\arg e^z = \mathrm{Arg}\ e^z + 2\pi n$, where $n = 0, \pm 1, \pm 2, \ldots$. Hence, $\arg(e^z) = y + 2\pi k$, where $k = n + N$ is still some integer.

This is not surprising, since $\ln(e^z)$ is a multi-valued function, which cannot be equal to the single-valued function z. Indeed eq. (18) is false for the multi-valued complex logarithm.

As a check, let us compute $\ln(e^{\ln z})$ in two different ways. First, using eq. (47), it follows that $\ln(e^{\ln z}) = \ln z$. Second, using eq. (48), $\ln(e^{\ln z}) = \ln z + 2\pi ik$. This seems to imply that $\ln z = \ln z + 2\pi ik$. In fact, the latter is completely valid as a *set equality* in light of eq. (45).

We now consider the properties exhibited in eqs. (19)–(22). Using the definition of the multi-valued complex logarithms and the properties of $\arg\ z$ given in eqs. (5)–(7), it follows that eqs. (19)–(21) are satisfied as set equalities:

$$\ln(z_1 z_2) = \ln z_1 + \ln z_2 \,, \tag{49}$$

$$\ln\left(\frac{z_1}{z_2}\right) = \ln z_1 - \ln z_2 \,. \tag{50}$$

$$\ln\left(\frac{1}{z}\right) = -\ln z \,. \tag{51}$$

However, one must be careful in employing these results. One should not make the mistake of writing, for example, $\ln z + \ln z \stackrel{?}{=} 2\ln z$ or $\ln z - \ln z \stackrel{?}{=} 0$. Both these latter statements are false for the same reasons that eqs. (8) and (9) are *not* identities under set equality. In particular, the multi-valued complex logarithm does *not* satisfy eq. (22) when p is an integer n:

$$\ln z^n = \underbrace{\ln z + \ln z + \cdots + \ln z}_{n} \neq n \ln z \,, \tag{52}$$

which follows from eq. (10). If p is not an integer, then z^p is a complex multi-valued function, and one needs further analysis to determine whether eq. (22) is valid. In section 6, we will prove [see eq. (62)] that eq. (22) is satisfied by the complex logarithm only if $p = 1/n$ where n is an integer. In this case,

$$\ln(z^{1/n}) = \frac{1}{n}\ln z \,, \qquad n = 1\,,\,2\,,\,3\,,\,\ldots\,. \tag{53}$$

We next examine the properties of the single-valued function $\text{Ln}\, z$. Again, we examine the six properties given by eqs. (17)–(22). First, eq. (17) is trivially satisfied since

$$e^{\text{Ln}\, z} = e^{\text{Ln}|z|} e^{i\text{Arg}\ z} = |z| e^{i\text{Arg}\ z} = z \,. \tag{54}$$

However, eq. (18) is generally false. In particular, for $z = x + iy$

$$\begin{aligned} \text{Ln}(e^z) &= \text{Ln}\ |e^z| + i(\text{Arg}\ e^z) = \text{Ln}(e^x) + i(\text{Arg}\ e^{iy}) = x + i\text{Arg}\ (e^{iy}) \\ &= x + i\arg(e^{iy}) + 2\pi i\left[\frac{1}{2} - \frac{\arg(e^{iy})}{2\pi}\right] = x + iy + 2\pi i\left[\frac{1}{2} - \frac{y}{2\pi}\right] \\ &= z + 2\pi i\left[\frac{1}{2} - \frac{\text{Im}\ z}{2\pi}\right] \,, \end{aligned} \tag{55}$$

after using eq. (3), where [ ] is the greatest integer bracket function defined in eq. (4). Thus, eq. (18) is satisfied only when $-\pi < y \leq \pi$. For values of y outside the principal interval, eq. (18) contains an additive correction term as shown in eq. (55).

As a check, let us compute $\mathrm{Ln}(e^{\mathrm{Ln} z})$ in two different ways. First, using eq. (54), it follows that $\mathrm{Ln}(e^{\mathrm{Ln}\, z}) = \mathrm{Ln}\, z$. Second, using eq. (55),

$$\mathrm{Ln}(e^{\mathrm{Ln}\, z}) = \mathrm{Ln}\, z + 2\pi i \left[\frac{1}{2} - \frac{\mathrm{Im}\,\mathrm{Ln}\, z}{2\pi}\right] = \mathrm{Ln}\, z + 2\pi i \left[\frac{1}{2} - \frac{\mathrm{Arg}\, z}{2\pi}\right] = \mathrm{Ln}\, z\,,$$

where we have used $\mathrm{Im}\,\mathrm{Ln}\, z = \mathrm{Arg}\, z$ [see eq. (46)]. In the last step, we noted that

$$0 \leq \frac{1}{2} - \frac{\mathrm{Arg}\, z}{2\pi} < 1\,,$$

due to eq. (1), which implies that the integer part of $\frac{1}{2} - \frac{1}{2\pi}\mathrm{Arg}\, z$ is zero. Thus, the two computations agree.

We now consider the properties exhibited in eqs. (19)–(22). $\mathrm{Ln}\, z$ may not satisfy any of these properties due to the fact that the principal value of the complex logarithm must lie in the interval $-\pi < \mathrm{Im}\,\mathrm{Ln}\, z \leq \pi$. Using the results of eqs. (11)–(16), it follows that

$$\mathrm{Ln}\,(z_1 z_2) = \mathrm{Ln}\, z_1 + \mathrm{Ln}\, z_2 + 2\pi i N_+\,, \tag{56}$$

$$\mathrm{Ln}\,(z_1/z_2) = \mathrm{Ln}\, z_1 - \mathrm{Ln}\, z_2 + 2\pi i N_-\,, \tag{57}$$

$$\mathrm{Ln}(z^n) = n\,\mathrm{Ln}\, z + 2\pi i N_n \qquad (\text{integer } n)\,, \tag{58}$$

where the integers $N_\pm = -1$, 0 or $+1$ and N_n are determined by eqs. (13) and (16), respectively, and

$$\mathrm{Ln}(1/z) = \begin{cases} -\mathrm{Ln}(z) + 2\pi i\,, & \text{if } z \text{ is real and negative}\,, \\ -\mathrm{Ln}(z)\,, & \text{otherwise}\,. \end{cases} \tag{59}$$

Note that eq. (19) is satisfied if $\mathrm{Re}\, z_1 > 0$ and $\mathrm{Re}\, z_2 > 0$ (in which case $N_+ = 0$). In other cases, $N_+ \neq 0$ and eq. (19) fails. Similar considerations also apply to eqs. (20)–(22). For example, eq. (21) is satisfied by $\mathrm{Ln}\, z$ unless $\mathrm{Arg}\, z = \pi$ (equivalently for negative real values of z), as indicated by eq. (59). In particular, one may use eq. (58) to verify that:

$$\mathrm{Ln}[(-1)^{-1}] = -\mathrm{Ln}(-1) + 2\pi i = -\pi i + 2\pi i = \pi i = \mathrm{Ln}(-1)\,,$$

as expected, since $(-1)^{-1} = -1$.

We cannot yet check whether eq. (22) is satisfied if p is a non-integer, since in this case z^p is a multi-valued function. Thus, we now turn our attention to the complex power functions (and the related generalized exponential functions).

6. Definition of the generalized power and exponential functions

The generalized complex power function is *defined* via the following equation:

$$w = z^c = e^{c \ln z}\,, \qquad z \neq 0\,. \tag{60}$$

To motivate this definition, we first note that if $c = k$ is an integer, then for $z = |z|e^{i \arg z}$,

$$z^k = |z|^k e^{ki \arg z} = e^{k \operatorname{Ln}|z|} e^{ki \arg z} = e^{k(\operatorname{Ln}|z| + i \arg z)} = e^{k \ln z}\,.$$

In this case, $w = z^k$ is a single-valued function, since

$$z^k = e^{k \ln z} = e^{k(\operatorname{Ln} z + 2\pi i n)} = e^{k \operatorname{Ln} z}\,.$$

If $c = 1/k$ (where k is an integer), then we have:

$$z^{1/k} = |z|^{1/k} e^{i \arg(z)/k} = e^{\operatorname{Ln}|z|/k} e^{i \arg(z)/k} = e^{(\operatorname{Ln}|z| + i \arg z)/k} = e^{\ln(z)/k}\,,$$

where $|z|^{1/k}$ refers to the positive real kth root of $|z|$. Combining the two results just obtained, we can easily prove that eq. (60) holds for any rational real number c. Since any irrational real number can be approximated (to any desired accuracy) by a rational number, it follows by continuity that eq. (60) must hold for any real number c. These arguments provide the motivation for defining the generalized complex power function as in eq. (60) for an arbitrary complex power c.

Note that due to the multi-valued nature of $\ln z$, it follows that $w = z^c = e^{c \ln z}$ is also multi-valued for any non-integer value of c, with a branch point at $z = 0$:

$$w = z^c = e^{c \ln z} = e^{c \operatorname{Ln} z} e^{2\pi i n c}\,, \qquad n = 0\,, \pm 1\,, \pm 2\,, \pm 3\,, \cdots\,. \tag{61}$$

If c is a rational number, then it can always be expressed in the form $c = m/k$, where m is an integer and k is a positive integer such that m and k possess no common divisor. One can then assume that $n = 0, 1, 2, \ldots, k-1$ in eq. (61), since other values of n will not produce any new values of $z^{m/k}$. It follows that the multi-valued function $w = z^{m/k}$ has precisely k distinct branches. If c is irrational or complex, then the number of branches is infinite (with one branch for each possible choice of integer n).

Having defined the multi-valued complex power function, we are now able to compute $\ln(z^c)$. In light of eq. (48),

$$\ln(z^c) = \ln(e^{c \ln z}) = c \ln z + 2\pi i k = c\left(\ln z + \frac{2\pi i k}{c}\right)\,, \tag{62}$$

where k is an arbitrary integer. Thus, $\ln(z^c) = c \ln z$ in the sense of *set equality* (in which case the sets corresponding to $\ln z$ and $\ln z + 2\pi i k/c$ coincide) if and only if k/c is an integer for all values of k. The only way to satisfy this latter requirement is to take $c = 1/n$, where n is an integer. Thus, eq. (53) is now verified.

We can define a single-valued power function by selecting the principal value of $\ln z$ in eq. (60). Consequently, the *principal value* of z^c is defined by

$$Z^c = e^{c \operatorname{Ln} z}, \qquad z \neq 0. \tag{63}$$

For a lack of a better notation, I will indicate the principal value by capitalizing the variable Z as above. The principal value definition of z^c can lead to some unexpected results. For example, consider the principal value of the cube root function $w = Z^{1/3} = e^{\operatorname{Ln}(z)/3}$. Then, for $z = -1$, the principal value of

$$\sqrt[3]{-1} = e^{\operatorname{Ln}(-1)/3} = e^{\pi i/3} = \tfrac{1}{2}\left(1 + i\sqrt{3}\right).$$

This may have surprised you, if you were expecting that $\sqrt[3]{-1} = -1$. To obtain the latter result would require a different choice of the principal interval in the definition of the principal value of $z^{1/3}$.

We are now in the position to check eq. (22) in the case that both the complex logarithm and complex power function are defined by their principal values. That is, we compute:

$$\operatorname{Ln}(Z^c) = \operatorname{Ln}(e^{c \operatorname{Ln} z}) = c \operatorname{Ln} z + 2\pi i N_c, \tag{64}$$

after using eq. (55), where N_c is an integer determined by

$$N_c \equiv \left[\frac{1}{2} - \frac{\operatorname{Im}\,(c \operatorname{Ln} z)}{2\pi}\right], \tag{65}$$

and [ ] is the greatest integer bracket function defined in eq. (4). N_c can be evaluated by noting that:

$$\operatorname{Im}\,(c \operatorname{Ln} z) = \operatorname{Im}\,\{c\,(\operatorname{Ln}|z| + i\operatorname{Arg}\,z)\} = \operatorname{Arg}\,z \operatorname{Re}\,c + \operatorname{Ln}|z| \operatorname{Im}\,c.$$

Note that if $c = n$ where n is an integer, then eq. (64) simply reduces to eq. (58), as expected. We conclude that eq. (22) is generally false both for the multi-valued complex logarithm and its principal value.

A function that in some respects is similar to the complex power function is the generalized exponential function. A possible definition of the generalized exponential function for $c \neq 0$ is:

$$w = c^z = e^{z \ln c} = e^{z(\operatorname{Ln}\,c + 2\pi i n)}, \qquad n = 0, \pm 1, \pm 2, \pm 3, \cdots. \tag{66}$$

However, the multi-valued nature of this function differs somewhat from the multi-valued power function. In contrast to the latter, the generalized exponential function possesses no branch point (or any other type of singularity) in the finite complex z-plane. Thus, one can regard eq. (66) as defining a set of independent single-valued functions for each value of n. Typically, the $n = 0$ case is the most useful, in which case, we would simply define:

$$w = c^z = e^{z \operatorname{Ln} c}, \qquad c \neq 0. \tag{67}$$

This conforms with our definition of the exponential function in section 3 (where $c = e$). Henceforth, we shall employ eq. (67) as the definition of the single-valued generalized exponential function.[§]

Some results for the principal value of the complex power function can be immediately adapted to the generalized exponential function. For example, by an almost identical computation as in eqs. (64) and (65), we find that:

$$\mathrm{Ln}(c^z) = \mathrm{Ln}(e^{z\,\mathrm{Ln}\,c}) = z\,\mathrm{Ln}\,c + 2\pi i N'_c\,, \tag{68}$$

where N'_c is an integer determined by:

$$N'_c \equiv \left[\frac{1}{2} - \frac{\mathrm{Im}\,(z\,\mathrm{Ln}\,c)}{2\pi}\right]. \tag{69}$$

7. Properties of the generalized power function

Let us examine the properties listed in eqs. (23)–(29). Eq. (23) defines the complex power function. It is tempting to write:

$$z^a z^b = e^{a\ln z}e^{b\ln z} = e^{a\ln z + b\ln z} \stackrel{?}{=} e^{(a+b)\ln z} = z^{a+b}\,. \tag{70}$$

However, consider the case of non-integer a and b where $a + b$ is an integer. In this case, eq. (70) cannot be correct since it would equate a multi-valued function $z^a z^b$ with a single-valued function z^{a+b}. In fact, the questionable step in eq. (70) is false:

$$a\ln z + b\ln z \stackrel{?}{=} (a+b)\ln z \qquad \text{[FALSE!!]}. \tag{71}$$

We previously noted that eq. (71) is false in the case of $a = b = 1$ [cf. eq. (8)]. A more careful computation yields:

$$\begin{aligned} z^a z^b &= e^{a\ln z}e^{b\ln z} = e^{a(\mathrm{Ln}\,z+2\pi in)}e^{b(\mathrm{Ln}\,z+2\pi ik)} = e^{(a+b)\mathrm{Ln}\,z}e^{2\pi i(na+kb)}\,,\\ z^{a+b} &= e^{(a+b)\ln z} = e^{(a+b)(\mathrm{Ln}\,z+2\pi ik)} = e^{(a+b)\mathrm{Ln}\,z}e^{2\pi ik(a+b)}\,, \end{aligned} \tag{72}$$

where k and n are arbitrary integers. Hence, z^{a+b} is a subset of $z^a z^b$. Whether the set of values for $z^a z^b$ and z^{a+b} does or does not coincide depends on a and b. However, in general, eq. (24) does not hold.

Similarly,

$$\begin{aligned} \frac{z^a}{z^b} &= \frac{e^{a\ln z}}{e^{b\ln z}} = \frac{e^{a(\mathrm{Ln}\,z+2\pi in)}}{e^{b(\mathrm{Ln}\,z+2\pi ik)}} = e^{(a-b)\mathrm{Ln}\,z}e^{2\pi i(na-kb)}\,,\\ z^{a-b} &= e^{(a-b)\ln z} = e^{(a-b)(\mathrm{Ln}\,z+2\pi ik)} = e^{(a-b)\mathrm{Ln}\,z}e^{2\pi ik(a-b)}\,, \end{aligned} \tag{73}$$

[§]In practice, many textbooks treat the generalized exponential function as a single-valued function, $c^z = e^{z\,\mathrm{Ln}\,c}$, only when c is a positive real number. For any other value of c, the multi-valued function $c^z = e^{z\,\ln c}$ is preferred. We shall not pursue this approach in these notes.

where k and n are arbitrary integers. Hence, z^{a-b} is a subset of z^a/z^b. Whether the set of values z^a/z^b and z^{a-b} does or does not coincide depends on a and b. However, in general, eq. (25) does not hold. Setting $a=b$ in eq. (73) yields the expected result:

$$z^0 = 1\,, \qquad z \neq 0$$

for any non-zero complex number z. Setting $a=0$ in eq. (73) yields the set equality:

$$z^{-b} = \frac{1}{z^b}\,, \tag{74}$$

i.e., the set of values for z^{-b} and $1/z^b$ coincide. Thus, eq. (26) is satisfied. Note, however, that

$$z^a z^{-a} = e^{a\ln z}e^{-a\ln z} = e^{a(\ln z - \ln z)} = e^{a\ln 1} = e^{2\pi i k a}\,,$$

where k is an arbitrary integer. Hence, if a is a non-integer, then $z^a z^{-a} \neq 1$ for $k \neq 0$. This is not in conflict with the set equality given in eq. (74) since there always exists at least one value of k (namely $k=0$) for which $z^a z^{-a} = 1$.

To show that eq. (27) can fail, we use eq. (48) to conclude that

$$(z^a)^b = (e^{a\ln z})^b = e^{b\ln(e^{a\ln z})} = e^{b(a\ln z + 2\pi i k)} = e^{ba\ln z}e^{2\pi i b k} = z^{ab}e^{2\pi i b k}\,, \tag{75}$$

where k is an arbitrary integer. If we employ the principal value of the generalized power function, defined in eq. (63), then eqs. (61) and (75) yield,

$$(z^a)^b = Z^{ab}e^{2\pi i n a b}e^{2\pi i b k}\,, \tag{76}$$

$$z^{ab} = Z^{ab}e^{2\pi i n a b}\,, \tag{77}$$

where both k and n are arbitrary integers. Thus, z^{ab} is a subset of $(z^a)^b$. Note that the elements of z^{ab} and $(z^a)^b$ coincide if $a = \pm 1$ and/or b is an integer.[¶] In contrast, if $z=a=b=i$, then $z^{ab} = -i$, whereas eq. (75) yields,

$$(i^i)^i = i^{i\cdot i}e^{-2\pi k} = i^{-1}e^{-2\pi k} = -ie^{-2\pi k}\,, \qquad k = 0\,, \pm 1\,, \pm 2\,, \cdots\,. \tag{78}$$

On the other hand, eqs. (28) and (29) are satisfied by the multi-valued power function, since

$$(z_1 z_2)^a = e^{a\ln(z_1 z_2)} = e^{a(\ln z_1 + \ln z_2)} = e^{a\ln z_1}e^{a\ln z_2} = z_1^a z_2^a\,,$$

$$\left(\frac{z_1}{z_2}\right)^a = e^{a\ln(z_1/z_2)} = e^{a(\ln z_1 - \ln z_2)} = e^{a\ln z_1}e^{-a\ln z_2} = z_1^a z_2^{-a}\,.$$

[¶]Other special cases exist in which the elements of z^{ab} and $(z^a)^b$ coincide. For example, if $a = 3/2$ and $b = 1/2$, then one can check that allowing for all possible integer values of n and k in eqs. (76) and (77) yields $(z^a)^b = z^{ab} = \{Z^{ab}, -Z^{ab}, iZ^{ab}, -iZ^{ab}\}$.

We now repeat the above analysis for the principal value of the power function, $Z^c = e^{c\,\mathrm{Ln}\,z}$. In this case, the results are somewhat reversed from the case of the multi-valued power function. In particular, eqs. (24)–(26) are satisfied, whereas eqs. (27)–(29) may be violated. For example, for the single-valued power function,

$$Z^a Z^b = e^{a\mathrm{Ln}\,z} e^{b\mathrm{Ln}\,z} = e^{(a+b)\mathrm{Ln}\,z} = Z^{a+b}\,, \tag{79}$$

$$\frac{Z^a}{Z^b} = \frac{e^{a\mathrm{Ln}\,z}}{e^{b\mathrm{Ln}\,z}} = e^{(a-b)\mathrm{Ln}\,z} = Z^{a-b}\,, \tag{80}$$

$$Z^a Z^{-a} = e^{a\mathrm{Ln}\,z} e^{-a\mathrm{Ln}\,z} = 1\,. \tag{81}$$

Setting $a = b$ in eq. (80) yields $Z^0 = 1$ (for $z \neq 0$) as expected.

Eq. (27) may be violated since eq. (55) implies that

$$(Z^c)^b = (e^{c\mathrm{Ln}\,z})^b = e^{b\,\mathrm{Ln}(e^{c\mathrm{Ln}\,z})} = e^{b(c\mathrm{Ln}\,z+2\pi i N_c)} = e^{bc\,\mathrm{Ln}\,z}\, e^{2\pi i b N_c} = Z^{cb}\, e^{2\pi i b N_c}\,,$$

where N_c is an integer determined by eq. (65). As an example, if $z = b = c = i$, eq. (65) gives $N_c = 0$, which yields the principal value of $(i^i)^i = i^{i\cdot i} = i^{-1} = -i$. However, in general $N_c \neq 0$ is possible in which case $(Z^c)^b \neq Z^{cb}$ [i.e., eq. (27) does not hold] unless bN_c is an integer. For example, if z is real and negative and $c = -1$, then $N_c = 1$ and $(Z^{-1})^b = Z^{-b} e^{2\pi i b}$ [which provides a resolution for the paradox of eq. (37)]. That is, if z is real and negative then $(Z^{-1})^b \neq Z^{-b}$ unless b is an integer.

Eqs. (28) and (29) may also be violated since eqs. (56) and (57) imply that

$$(Z_1 Z_2)^a = e^{a\mathrm{Ln}(z_1 z_2)} = e^{a(\mathrm{Ln}\,z_1+\mathrm{Ln}\,z_2+2\pi i N_+)} = Z_1^a Z_2^a\, e^{2\pi i a N_+}\,, \tag{82}$$

$$\left(\frac{Z_1}{Z_2}\right)^a = e^{a\mathrm{Ln}(z_1/z_2)} = e^{a(\mathrm{Ln}\,z_1-\mathrm{Ln}\,z_2+2\pi i N_-)} = \frac{Z_1^a}{Z_2^a}\, e^{2\pi i a N_-}\,, \tag{83}$$

where the integers $N_\pm$ are determined from eq. (13).

8. Properties of the generalized exponential function

The generalized exponential function, $w = c^z$ ($c \neq 0$), is a single-valued function defined by eq. (67). Using this definition and the properties of the complex exponential function e^z, one can quickly check whether eqs. (31)–(36) hold in the complex plane. The proof of eqs. (31)–(33) is nearly identical to the one given in eqs. (79)–(81):

$$c^{z_1} c^{z_2} = e^{z_1\mathrm{Ln}\,c} e^{z_2\mathrm{Ln}\,c} = e^{(z_1+z_2)\mathrm{Ln}\,c} = c^{z_1+z_2}\,, \tag{84}$$

$$\frac{c^{z_1}}{c^{z_2}} = \frac{e^{z_1\mathrm{Ln}\,c}}{e^{z_2\mathrm{Ln}\,c}} = e^{(z_1-z_2)\mathrm{Ln}\,c} = c^{z_1-z_2}\,, \tag{85}$$

$$c^z c^{-z} = e^{z\mathrm{Ln}\,c} e^{-z\mathrm{Ln}\,c} = 1\,. \tag{86}$$

However, eq. (34) does not generally hold. Using eq. (68),

$$(c^{z_1})^{z_2} = e^{z_2\,\mathrm{Ln}(c^{z_1})} = e^{z_2(z_1\,\mathrm{Ln}\,c + 2\pi i N'_c)} = e^{z_2 z_1\,\mathrm{Ln}\,c}\,e^{2\pi i z_2 N'_c} = c^{z_1 z_2}\,e^{2\pi i z_2 N'_c}\,, \quad (87)$$

where N'_c is determined by eq. (69) [with z replaced by z_1].

The case of $c = e$ is noteworthy. Eq. (87) reduces to:

$$(e^{z_1})^{z_2} = e^{z_1 z_2} e^{2\pi i z_2 N'_e}\,, \qquad N'_e \equiv \left[\frac{1}{2} - \frac{\mathrm{Im}\ z_1}{2\pi}\right]. \quad (88)$$

If $z_2 = n$ where n is any integer, then $e^{2\pi i n N'_e} = 1$ (since N'_e is an integer by definition of the bracket notation). Thus, we recover eq. (40).

Let us test eq. (88) by substituting $z_1 = -i\pi$. Then, $N'_e = 1$ and hence

$$(e^{-\pi i})^z = e^{i\pi z}\,.$$

This result may seem strange, but it is a consequence of our definition of the generalized exponential function, $c^z = e^{z\,\mathrm{Ln}\,c}$, which employs the principal value of the logarithm. Indeed

$$(e^{-i\pi})^z = (-1)^z = e^{z\,\mathrm{Ln}(-1)} = e^{i\pi z}\,,$$

since $\mathrm{Ln}(-1) = i\pi$. We conclude that eq. (34) can be violated, even for the ordinary exponential function.

Ultimately, the real difficulty with $(c^{z_1})^{z_2}$ is that it is simultaneously a generalized exponential function and a generalized power function. Thus, if z_2 is a non-integer, it may be more convenient to treat $(c^{z_1})^{z_2}$ as a multi-valued function. That is, in this latter convention, we treat the generalized exponential function $c^{z_1} = e^{z_1\,\mathrm{Ln}\,c}$ as a single-valued function (using the principal value definition of the logarithm in the exponent), whereas we treat the generalized power function $(c^{z_1})^{z_2} = e^{z_2\ \ln(c^{z_1})}$ as a possible multi-valued function:

$$(c^{z_1})^{z_2} = e^{z_2\ \ln(c^{z_1})} = e^{z_2 \ln(e^{z_1\,\mathrm{Ln}\,c})} = e^{z_2(z_1\,\mathrm{Ln}\,c + 2\pi i k)} = e^{z_2 z_1\,\mathrm{Ln}\,c}\,e^{2\pi i z_2 k} = c^{z_1 z_2}\,e^{2\pi i k z_2}\,,$$

where k is an arbitrary integer [see eq. (48)]. In particular, for z_2 a non-integer, $(c^{z_1})^{z_2}$ is a multi-valued function, with branches corresponding to different choices of k. For example, for $c = z_1 = z_2 = i$, we recover eq. (78). One might be tempted to call the $k = 0$ branch the principal value of $(c^{z_1})^{z_2}$, in which case eq. (34) would be valid. Clearly, we must define our conventions carefully if we wish to manipulate expressions involving exponentials of exponentials.

Finally, eqs. (35) and (36) may be violated. The calculation is nearly identical to the one given in eqs. (82) and (83):

$$(ab)^z = e^{z\mathrm{Ln}(ab)} = e^{z(\mathrm{Ln}\ a + \mathrm{Ln}\ b + 2\pi i N_+)} = a^z b^z e^{2\pi i z N_+}\,,$$

$$\left(\frac{a}{b}\right)^z = e^{z\mathrm{Ln}(a/b)} = e^{z(\mathrm{Ln}\ a - \mathrm{Ln}\ b + 2\pi i N_-)} = \frac{a^z}{b^z} e^{2\pi i z N_-}\,,$$

where the integers $N_\pm$ are determined from eq. (13) [with z_1 and z_2 replaced by a and b, respectively]. If $\mathrm{Re}\ a > 0$ and $\mathrm{Re}\ b > 0$, then $N_\pm = 0$, and eqs. (35) and (36) are satisfied.