Here is a collection of practice problems covering material from Chapters 3 and 10 of Boas and homework sets 8–10. Along with the first two practice problem sets, this may help you in preparing for the final exam.

1. One of the eigenvalues of the matrix A is $\lambda = 0$. Prove that A^{-1} does not exist.

2. Find the eigenvalues and eigenvectors of

$$M = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix} .$$

Is M diagonalizable?

3. Determine whether the following matrices are diagonalizable. If diagonalizable, indicate whether it is possible to diagonalize the matrix with a unitary similarity transformation.

$$\text{(a) } A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} , \qquad \text{(b) } A = \begin{pmatrix} 1 & -4 & 2 \\ -4 & 1 & -2 \\ 2 & -2 & -2 \end{pmatrix} .$$

4. Eq. (11.27) on p. 154 of Boas is imprecisely worded. The correct statement is that a matrix has real eigenvalues and can be diagonalized by an orthogonal similarity transformation transformation if and only if it is a *real* symmetric matrix. (Boas omitted the qualification that the symmetric matrix must be real.) To see that it is important to be precise, consider the following *complex* symmetric matrix,

$$A = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} .$$

(a) Determine the eigenvalues and eigenvectors of A.

(b) How many linearly independent eigenvectors of A exist?

(c) Is A diagonalizable?

5. Suppose that U is a $n \times n$ matrix whose columns comprise an orthonormal set of column vectors. Prove that U is a unitary matrix. Moreover, show that the rows of U must also comprise an orthonormal set of row vectors. (*HINT:* In this problem, orthonormality must be defined with respect to a *complex* vector space.)

6. Let A be a complex $n \times n$ matrix. Prove that the eigenvalues of $AA^\dagger$ are real and non-negative.

HINT: Let $\vec{\boldsymbol{w}} = A^\dagger \vec{\boldsymbol{v}}$, where $\vec{\boldsymbol{v}}$ is an eigenvector of $AA^\dagger$. Investigate the consequence of the fact that the inner product $\langle \vec{\boldsymbol{w}}\,, \vec{\boldsymbol{w}} \rangle$ is non-negative in a complex Euclidean space.

7. A linear transformation A is represented by the matrix:

$$A = \begin{pmatrix} 1 & -4 & 2 \\ -4 & 1 & -2 \\ 2 & -2 & -2 \end{pmatrix},$$

with respect to the standard basis $\mathcal{B} = \{(1,0,0),(0,1,0),(0,0,1)\}$. Consider a new basis, $\mathcal{B}' = \{(2,-2,1),(1,1,0),(-1,1,4)\}$, where the components of the basis vectors of $\mathcal{B}'$ are given with respect to the standard basis $\mathcal{B}$.

(a) What are the components of the basis vectors of $\mathcal{B}$ when expressed relative to the basis $\mathcal{B}'$?

(b) Determine the matrix representation of A relative to the basis $\mathcal{B}'$.

8. Consider the matrix

$$M = \begin{pmatrix} 0 & b \\ 0 & a \end{pmatrix},$$

where a and b are arbitrary complex numbers.

(a) Compute the eigenvalues of M.

(b) Find a matrix C such that $C^{-1}MC$ is diagonal.

(c) Compute e^M.

HINT: Denote $D = C^{-1}MC$ where D is the diagonal matrix obtained in part (b). Show that

$$e^M = e^{CDC^{-1}} = Ce^D C^{-1}. \tag{1}$$

Employing the results of parts (a) and (b), first evaluate e^D and then use eq. (1) to compute e^M.

(d) Verify that $\det(e^M) = e^{\mathrm{Tr}\, M}$.

9. If A is a diagonalizable matrix, prove that

$$e^{\mathrm{Tr} A} = \det e^A.$$

HINT: First prove this result for a diagonal matrix. Then, try to prove the more general result by diagonalizing A. This result is true for *any* matrix A, but if A is not diagonalizable, a more sophisticated technique is required.

10. A projection operator is a linear transformation P that satisfies: $P^2 = P$.

(a) Suppose that $P \neq \mathbf{I}$ (where $\mathbf{I}$ is the identity operator). Prove that P^{-1} does not exist.

(b) Prove that the only possible eigenvalues of P are $\lambda = 0$ and $\lambda = 1$.

11. Consider the matrices:

$$G = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad K = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}.$$

(a) Show that these are rotation matrices. Are the corresponding rotations proper or improper?

(b) Describe precisely the nature of the rotations produced when G and K act on a vector.

(c) Do the rotations represented by G and K commute? Compare the rotations produced by GK and KG.

12. Consider tensors that live in n-dimensional Euclidean space.

(a) How many independent components does a symmetric second-rank tensor, S_{ij}, possess?

(b) How many independent components does an antisymmetric second-rank tensor, A_{ij}, possess?

13. A product of Levi-Civita ϵ symbols can be expressed in terms of products of Kronecker deltas.

(a) Show that the following determinantal identity is satisfied:

$$\epsilon_{ijk}\epsilon_{\ell mn} = \begin{vmatrix} \delta_{i\ell} & \delta_{im} & \delta_{in} \\ \delta_{j\ell} & \delta_{jm} & \delta_{jn} \\ \delta_{k\ell} & \delta_{km} & \delta_{kn} \end{vmatrix}.$$

HINT: You may find eq. (5.5) on p. 509 of Boas useful.

(b) Set $k = \ell$ in part (a) and sum the resulting expression from $k = 1$ to 3. Show that the result coincides with eq. (5.8) on p. 510 of Boas.

14. In three dimensional space, the components of the vector cross product (in rectangular coordinates) is defined as

$$(\vec{\boldsymbol{b}} \times \vec{\boldsymbol{c}})_i = \sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon_{ijk} b_j c_k \,,$$

where ϵ_{ijk} is the Levi-Civita symbol.

(a) Using the formula for the determinant given on p. 509 of Boas, prove that:

$$\vec{\boldsymbol{a}} \cdot (\vec{\boldsymbol{b}} \times \vec{\boldsymbol{c}}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} .$$

(b) Using the properties of the determinant, prove that

$$\begin{aligned} &\text{(i)} \quad \vec{\boldsymbol{a}} \cdot (\vec{\boldsymbol{b}} \times \vec{\boldsymbol{c}}) = \vec{\boldsymbol{b}} \cdot (\vec{\boldsymbol{c}} \times \vec{\boldsymbol{a}}) = \vec{\boldsymbol{c}} \cdot (\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}}) \,, \\ &\text{(ii)} \quad \vec{\boldsymbol{a}} \cdot (\vec{\boldsymbol{b}} \times \vec{\boldsymbol{c}}) = (\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}}) \cdot \vec{\boldsymbol{c}} \,, \qquad \text{[interchange of the dot and cross]} \,, \\ &\text{(iii)} \quad \vec{\boldsymbol{a}} \cdot (\vec{\boldsymbol{b}} \times \vec{\boldsymbol{c}}) = -\vec{\boldsymbol{c}} \cdot (\vec{\boldsymbol{b}} \times \vec{\boldsymbol{a}}) \,. \end{aligned}$$

15. Using the methods of tensor algebra, one can express the vector cross and dot products as:

$$(\boldsymbol{A} \times \boldsymbol{B})_i = \epsilon_{ijk} A_j B_k \,, \qquad \boldsymbol{A} \cdot \boldsymbol{C} = A_i B_i \,,$$

Employ these methods in deriving the two vector identities below.

(a) Prove Lagrange's identity:

$$(\boldsymbol{A} \times \boldsymbol{B}) \cdot (\boldsymbol{C} \times \boldsymbol{D}) = (\boldsymbol{A} \cdot \boldsymbol{C})(\boldsymbol{B} \cdot \boldsymbol{D}) - (\boldsymbol{A} \cdot \boldsymbol{D})(\boldsymbol{B} \cdot \boldsymbol{C}) \,.$$

(b) Define $(\boldsymbol{XYZ}) \equiv \boldsymbol{X} \cdot (\boldsymbol{Y} \times \boldsymbol{Z})$. Then, prove that:

$$(\boldsymbol{A} \times \boldsymbol{B}) \times (\boldsymbol{C} \times \boldsymbol{D}) = (\boldsymbol{ABD})\boldsymbol{C} - (\boldsymbol{ABC})\boldsymbol{D} \,.$$

16. Suppose that $T_{\alpha\beta}$ are the components of a second-rank tensor and V_β are the components of a vector. Assume that α and β can take on the values 1, 2 and 3.

(a) If $T_{\alpha\beta} V_\beta = 0$, for *all* vectors of a three-dimensional vector space, then prove that the tensor $T_{\alpha\beta} = 0$ (i.e. all the components of the tensor vanish, which implies that the tensor in question is the zero tensor).

(b) Suppose that $T_{\alpha\beta} V_\beta = 0$ holds for a particular non-zero vector. Can you still conclude that $T_{\alpha\beta} = 0$? If no, then suppose $T_{\alpha\beta} V_\beta = 0$ holds for N different non-zero vectors. What is the minimum value of N required in order to conclude that $T_{\alpha\beta} = 0$?

BONUS PROBLEM (just for fun):

On the second midterm exam, I posed a question with three parts. In fact the original question had six parts. See if you can complete the final three parts.

17. Consider the 3×3 matrix $C = [c_{ij}]$ whose matrix elements are given by $c_{ij} = a_i b_j$, where the numbers a_1, a_2, a_3 and b_1, b_2 and b_3 are all nonzero numbers.

(a) What is the rank of C?

(b) Evaluate $\det C$. [*HINT:* You should be able to deduce the correct result without an explicit computation.]

(c) Consider the possible solutions to the following system of equations

$$C\vec{\boldsymbol{v}} = 0\,, \tag{2}$$

where the matrix C is defined at the beginning of this problem. Determine the maximal number of linearly independent vectors $\vec{\boldsymbol{v}}$ that satisfy eq. (2).

(d) Based on the results of part (c), determine the eigenvalues of C.

HINT: To avoid some messy algebra, think carefully about what the results of part (c) tell you about the eigenvalues of C. Then, using the value of $\operatorname{Tr} C$, you should be able to determine all the eigenvalues of C.

(e) Assuming that C possesses at least one nonzero eigenvalue, is C diagonalizable? Justify your answer.

(f) Determine the rank, determinant, eigenvalues and eigenvectors of an $n \times n$ matrix $C = [c_{ij}]$, where $c_{ij} = a_i b_j$. Assuming that C possesses at least one nonzero eigenvalue, is C diagonalizable? What is the relation between the rank and the number of linearly independent eigenvalues corresponding to the zero eigenvalues?