1. Consider the function

$$f(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)}\, x^{\alpha-1} e^{-x/\beta}\,, \tag{1}$$

where $0 < x < \infty$, and α and β are positive constants.

Define the family of integrals

$$I_n \equiv \int_0^\infty x^n f(x)\, dx\,,$$

where n is a non-negative integer.*

(a) Evaluate the integral I_n for arbitrary n. Your answer should be expressed in terms of functions that depend on α, β and n.

(b) For the cases of I_1 and I_2, simplify the corresponding result obtained in part (a). Then, compute the quantity $I_2 - (I_1)^2$. Your final answers should be expressed in terms of simple elementary functions of α and β.

(c) Suppose that the integration variable x has dimensions of length. What are the dimension of α and β required for the consistency of dimensional analysis? Determine the dimension of I_n as a function of n. Then, check that your results for I_1 and I_2 in part (b) exhibit the correct dimensions of length.

2. If n is a positive integer, then the double factorial is defined as:

$$(2n)!! \equiv 2{\cdot}4{\cdot}6\cdots(2n)\,, \qquad (2n-1)!! \equiv 1{\cdot}3{\cdot}5\cdots(2n-1)\,.$$

(a) Express $(2n)!!$ in terms of the gamma function. [*HINT:* Factor out a 2 from each term in the definition of $(2n)!!$]

(b) Express $(2n-1)!!$ in terms of a ratio of two gamma functions. [*HINT:* First evaluate the product $(2n)!!(2n-1)!!$, and then use the result of part (a)]

(c) Find the leading *behavior* of the ratio

$$r_n \equiv \frac{(2n-1)!!}{(2n)!!}\,, \qquad \text{as } n \to \infty\,.$$

*In statistics, eq. (1) is known as the gamma distribution, which is sometimes employed to describe the probability that a random variable x lies within an interval on the positive real axis. In this case, I_1 is the mean of the distribution and $I_2 - (I_1)^2$ is the variance.

3. Consider the matrix

$$M = \begin{pmatrix} x & 1 & 0 & 1 \\ 1 & x & 1 & 0 \\ 0 & 1 & x & 1 \\ 1 & 0 & 1 & x \end{pmatrix}.$$

(a) Find all values of x for which the inverse of M does *not* exist. [*HINT:* You do not have to compute M^{-1} to answer this question.]

(b) Determine the rank of M as a function of x. [*HINT:* Consider the explicit form of M for the values of x found in part (a).]

4. In the following problem, c is some number (which may be complex). Note that parts (b) and (c) of this problem are independent of part (a).

(a) For what values of c (if any) will the following equations,

$$x + y = cx\,,$$

$$-x + y = cy\,,$$

have non-trivial (i.e. non-zero) solutions?

(b) For what values of c (if any) will the following equations,

$$x + y + z = 6\,,$$

$$x + cy + cz = 2\,,$$

have either a unique solution, an infinite number of solutions, or no solutions.

(c) Determine the values of x, y and z that solve the equations of part (b) as a function of c, assuming solutions exist. If an infinite number of solutions exist, write the solution set for (x, y, z) that indicates all the possible solutions.