Here is a collection of practice problems suitable for the second midterm exam.

1. Here are some quickies involving the gamma and beta functions. Do not use a calculator or a numerical table. Instead, invoke an appropriate gamma function property when needed.

(a) Evaluate $\Gamma(-\frac{5}{2})$.

(b) Evaluate $\lim\limits_{x\to 0} \Gamma(2x)/\Gamma(x)$.

(c) Evaluate $B(\frac{1}{4}, \frac{3}{4})$ in terms of some well known numerical constants.

2. Evaluate $\psi(\frac{1}{2})$, where $\psi(x) \equiv d\ln\Gamma(x)/dx$.

HINT: The simplest way to solve this problem is to make use of the duplication formula for the gamma function (which you derived in problem 2 of homework set #5).

3. A professor posts the final grades by listing the students by the last four digits of their social security numbers. But, is this a good idea? In this problem, you will compute the probability that at least two students have social security numbers such that the last four digits are identical.

Suppose the class has n students. The probability that the last four digits of all the social security numbers of students in the class are distinct is given by:

$$P_0 = 1 \cdot \frac{N-1}{N} \cdot \frac{N-2}{N} \cdots \frac{(N-n+1)}{N},$$

where N is number of possible values for the last four digits of a social security number.

(a) Show that P_0 can be rewritten as a ratio of two factorials divided by N^n.

(b) Using Stirling's approximation, and assuming that $N \gg n \gg 1$ and $N - n \gg 1$, derive a simple formula for $\ln P_0$ as a function of n and N. In your final expression, you may drop terms of $\mathcal{O}(1/N^2)$.

(c) Given P_0, the probability that at least two social security numbers belonging to students of the class possess the same last four digits is $P = 1 - P_0$. Using the approximation obtained in part (b), find the value of n such that in a class of n students or more, the probability that two students will have social security numbers with the same last four digits is greater than 1/2.

4. Using the properties of the beta and gamma functions, evaluate the following integral in terms of *elementary* functions,

$$\int_0^\infty \frac{dx}{(1+x)x^p},$$

assuming that p is a real variable that satisfies $0 < p < 1$.

5. Use Stirling's formula to evaluate the following two limits:

$$\text{(a)}\ \lim_{n\to\infty} \frac{\Gamma(n+\frac{3}{2})}{\sqrt{n}\,\Gamma(n+1)}, \qquad\qquad \text{(b)}\ \lim_{n\to\infty} \frac{(n!)^{1/n}}{n}.$$

6. Consider the function defined by the following integral:

$$f(x) = \int_0^\infty \frac{e^{-t}\,dt}{1+xt}, \qquad \text{for } x \geq 0\,. \tag{1}$$

(a) Evaluate the integral by the following method. First, expand $(1+xt)^{-1}$ as a power series. Then, insert the resulting series into eq. (1) and integrate term by term. Write the final result for $f(x)$ in summation notation.

(b) Prove that the sum obtained in part (a) diverges for all $x \neq 0$. This indicates that the derivation in part (a) involves an illegal mathematical step (otherwise, the resulting sum would have been convergent for some non-zero interval of x). Can you identify the illegal step?

(c) Define a new integration variable, $u = 1 + xt$. Show that $f(x)$ can be expressed in terms of the exponential integral,

$$E_1(y) = \int_1^\infty \frac{e^{-uy}}{u}\,du\,,$$

where $y = 1/x$. Using the asymptotic series for $E_1(x)$ derived in problem 9 on homework set 2, obtain an asymptotic series for $f(x)$ as $x \to 0$. How does this result compare with the series obtained in part (a)?

7. Given the following matrix:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix},$$

compute the inverse matrix A^{-1}.

8. Consider the system of equations:

$$\begin{aligned} 5x + 2y + z &= 2 \\ x + y + 2z &= 1 \\ 3x - 3z &= 0\,. \end{aligned}$$

(a) What is the augmented matrix for this system of equations?

(b) Solve this system of equations using Gaussian elimination.

(c) What is the rank of the augmented matrix of part (b)?

(d) Remove the third equation above, and solve the new system of two equations and three unknowns using Gaussian elimination. What is the rank of the corresponding augmented matrix?

9. Evaluate the following determinant by hand:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{vmatrix}\,.$$

HINT: Simplify the matrix by employing appropriate row and/or column operations.

10. In special relativity, the space-time coordinates of two inertial frames moving at relative constant velocity v are related by

$$\begin{aligned} x' &= \gamma(x - vt)\,, \\ t' &= \gamma(t - vx/c^2)\,, \end{aligned}$$

where $\gamma \equiv (1 - v^2/c^2)^{-1/2}$ and c is the velocity of light.

(a) Rewrite this system of equations in matrix form.

(b) Using Cramer's rule, solve for x and t in terms of x' and t'.

11. Consider the following matrices:

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 3 & -1 & 0 \\ 0 & 5 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 3 & -1 & 0 \end{pmatrix},$$

Compute AB, BA, det A, det B, det AB and det BA. Verify that $AB \neq BA$ and det $AB = (\det A)(\det B)$.

12. Let A be a 3×3 matrix. Assume that $A \neq 0$. The determinant of A is denoted by $\det A$.

(a) Is the equation $\det(3A) = 3 \det A$ true or false? Explain.

(b) Suppose that $\det A = 1$. Let B be a matrix obtained from A by permuting the order of the rows so that the first row of A is the second row of B, the second row of A is the third row of B and the third row of A is the first row of B. (This is called a *cyclic permutation.*) What is the value of $\det B$?

(c) Suppose that the 3×3 matrix $A \neq 0$ but $\det A = 0$. What can you say about the rank of A?

BONUS PROBLEM (just for fun):

13. Consider the following definition of the Bernoulli numbers B_n:

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}\,.$$

To simplify the notation, let us define $b_n \equiv B_n/n!$. Then it follows that:

$$x = (e^x - 1)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(1 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots\right).$$

(a) If two polynomials are equal, then all their coefficients must be equal (can you justify this assertion?). Using this result, show that you get a system of linear equations for the b_n.

(b) Consider the first n linear equations thus obtained. Solve these equations using Cramer's rule and show that:

$$B_n = n!b_n = (-1)^n n! \begin{vmatrix} 1/2! & 1 & 0 & \cdots & 0 \\ 1/3! & 1/2! & 1 & \cdots & 0 \\ 1/4! & 1/3! & 1/2! & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1/(k+1)! & 1/k! & 1/(k-1)! & \cdots & 1/2! \end{vmatrix}\,.$$

(c) Check the formula given in part (b) by computing B_1, B_2 and B_3.