INSTRUCTIONS: This is a one-hour exam. During the exam, you may refer to the textbook (Boas), the class handouts (including solution sets to the homeworks and the practice problems and exam) or your own personal notes. Collaboration with your neighbor is strictly forbidden.

In answering the questions, it is not sufficient to simply give a final result. You must provide the intermediate steps needed to arrive at the final solution in order to get full credit. *However, if you are employing any result already obtained in class or in the textbook, you do not need to re-derive it.* The point value of each problem is indicated in brackets. The total number of points available is 100, corresponding to 10 points for each part.

1. [30] Evaluate the following two integrals:

$$\text{(a)}\quad I \equiv \int_0^\infty e^{-x^3}\,dx\,, \qquad\qquad \text{(b)}\quad J(a) \equiv \int_0^a \frac{dx}{\sqrt{a^6 - x^6}}\,.$$

Simplify your resulting expressions as much as possible.

HINT: By an appropriate change of variables, you can transform each integral into a more recognizable form.

(c) Using only dimensional analysis *without* evaluating the integral, show that $J(a) = a^p J(1)$ for some suitably chosen power p. Determine p and check that this is consistent with your answer in part (b).

2. [20] Suppose that a and b are two different real numbers (both are finite and are independent of the variable x).

(a) Use Stirling's approximation to find the *behavior* of

$$\ln\Gamma(x+a) - \ln\Gamma(x+b)\,,$$

as $x \to \infty$.

(b) Using the result of part (a), evaluate the limit,

$$\lim_{x\to\infty} x^{b-a}\frac{\Gamma(x+a)}{\Gamma(x+b)}\,.$$

3. [20] (a) The matrix X satisfies the equation,

$$X = AX + B\,, \tag{1}$$

where

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 3 \end{pmatrix}\,. \tag{2}$$

Determine the matrix X.

HINT: Let the matrix elements of X be unknown variables. Using eq. (1), write out the set of linear equations for the unknown variables. Then solve these equations. You can use any method you prefer to solve these equations (but I highly recommend that you employ the simplest possible approach to determine the unknown quantities).

(b) Given the matrices A and B of eq. (2), compute $(\mathbf{I} - A)^{-1}$, where $\mathbf{I}$ is the 3×3 identity matrix. Then, evaluate the matrix product $(\mathbf{I} - A)^{-1}B$. The result should be the same as the result of part (a). Explain.

4. [30] Consider the 3×3 matrix $C = [c_{ij}]$ whose matrix elements are given by $c_{ij} = a_i b_j$, where the numbers a_1, a_2, a_3 and b_1, b_2 and b_3 are all nonzero numbers.

(a) What is the rank of C?

(b) Evaluate $\det C$. [*HINT:* You should be able to deduce the correct result without an explicit computation.]

(c) Consider the possible solutions to the following system of equations

$$C\vec{\boldsymbol{v}} = 0\,, \tag{3}$$

where the matrix C is defined at the beginning of this problem. Determine the maximal number of linearly independent vectors $\vec{\boldsymbol{v}}$ that satisfy eq. (3).

NOTE: If you feel more comfortable with numbers rather than letters, I will allow you to solve this problem with $a_1 = 1$, $a_2 = 3$, $a_3 = 5$, $b_1 = 2$, $b_2 = 4$ and $b_3 = 6$, but with a 10 point penalty. Personally, I think that this problem is easier to do without assigning particular values to the a_i and b_i, but the choice is yours.