The Riemann Zeta Function

In this note, I will sketch some of the main properties of the Riemann zeta function, $\zeta(x)$. For $x > 1$, we define

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}\,, \qquad x > 1\,. \tag{1}$$

For $x \leq 1$, this sum diverges. However, we shall see that it is possible to extend the definition of $\zeta(x)$ so that it is valid for all real x (more generally, all complex x).

There is a beautiful relation between the zeta function and the Gamma function. In deriving this relation, we shall establish another possible definition of $\zeta(x)$ [for $x > 1$] in terms of an integral. The trick here is to note the following formula:

$$\int_0^\infty e^{-nt}\, t^{x-1}\, dt = \frac{\Gamma(x)}{n^x}\,.$$

To prove this result, define a new variable $u = nt$. The resulting integral is recognized as the definition of the Gamma function up to an overall factor of n^{-x}. Now, we can do something clever and use the above result to write

$$\frac{1}{n^x} = \frac{1}{\Gamma(x)} \int_0^\infty e^{-nt}\, t^{x-1}\, dt\,.$$

If we now sum both sides of the equation from $n = 1$ to $n = \infty$, we obtain:

$$\begin{aligned}
\zeta(x) &= \frac{1}{\Gamma(x)} \sum_{n=1}^{\infty} \int_0^\infty e^{-nt}\, t^{x-1}\, dt \\
&= \frac{1}{\Gamma(x)} \int_0^\infty \sum_{n=1}^{\infty} e^{-nt}\, t^{x-1}\, dt\,.
\end{aligned} \tag{2}$$

where the interchange of the sum and the integral is justified by the uniform convergence of the sum. The sum over n is just an infinite geometrical series with the $n = 0$ term missing. Thus, using

$$\sum_{n=1}^{\infty} r^n = \frac{1}{1-r} - 1 = \frac{r}{1-r} = \frac{1}{\frac{1}{r} - 1}$$

with $r = e^{-t}$, it follows that

$$\boxed{\zeta(x) = \frac{1}{\Gamma(x)} \int_0^\infty \frac{t^{x-1}}{e^t - 1}\, dt\,, \qquad x > 1\,.} \tag{3}$$

Note that this integral form for $\zeta(x)$ converges for $x > 1$ and diverges for $x \leq 1$ as expected from the convergence properties of eq. (1).

In particular, $\zeta(1) = \infty$. But, it is useful to determine the *behavior* of $\zeta(x)$ as $x \to 1$. Here we shall simply quote the relevant result:

$$\lim_{x\to 1}\left(\zeta(x) - \frac{1}{x-1}\right) = \gamma\,,$$

where γ is Euler's constant. Equivalently,

$$\zeta(x) = \frac{1}{x-1} + \gamma + \mathcal{O}(x-1)\,, \qquad \text{as } x \to 1\,. \tag{4}$$

The proof of this result can be found in the article on the Riemann zeta function that is posted on the class web site.

We shall now define another function that is closely related to the Riemann zeta function:

$$\eta(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^x}\,, \qquad x > 0\,. \tag{5}$$

Note that this series converges for all $x > 0$ by the alternating series test. We can also derive an integral expression for this function by exactly the same procedure as above. This time, the geometric series that we encounter is given by

$$\sum_{n=1}^{\infty}(-1)^{n+1} r^n = 1 - \frac{1}{1+r} = \frac{r}{1+r} = \frac{1}{\frac{1}{r}+1}$$

Thus, we end up with

$$\boxed{\eta(x) = \frac{1}{\Gamma(x)}\int_0^\infty \frac{t^{x-1}}{e^t+1}\,dt\,, \qquad x > 0\,.} \tag{6}$$

I encourage the reader to fill in the missing steps of this derivation. Eqs. (3) and (6) play a central role in the statistical mechanics of a free ideal Bose and Fermi gas, respectively.

The function $\eta(x)$ is related in a simple way to the Riemann zeta function as follows

$$\begin{aligned}
\eta(x) &= 1 - \frac{1}{2^x} + \frac{1}{3^x} - \frac{1}{4^x} + \cdots \\
&= 1 + \frac{1}{2^x} + \frac{1}{3^x} + \frac{1}{4^x} + \cdots - 2\left[\frac{1}{2^x} + \frac{1}{4^x} + \frac{1}{6^x} + \cdots\right] \\
&= \zeta(x) - \frac{2}{2^x}\left(1 + \frac{1}{2^x} + \frac{1}{3^x} + \cdots\right) \\
&= \zeta(x) - 2^{1-x}\zeta(x) \\
&= (1 - 2^{1-x})\zeta(x)\,.
\end{aligned}$$

However, we must be careful. Although the series definition for $\eta(x)$ [eq. (5)] converges for $x > 0$, this convergence is conditional for $0 < x \leq 1$ and absolute only for $x > 1$. Whereas the rearrangement of an absolutely convergent series does not change its sum, this is no longer true for a conditionally convergent series. Hence, the manipulations just performed are mathematically sound only if $x > 1$. Hence, we conclude that

$$\eta(x) = (1 - 2^{1-x})\zeta(x)\,, \qquad x > 1\,. \tag{7}$$

If we solve eq. (7) for $\zeta(x)$, we can write:

$$\boxed{\zeta(x) = \frac{1}{1 - 2^{1-x}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^x}\,.} \tag{8}$$

This expression is valid for $x > 1$. If we try to set $x = 1$, we find that $\zeta(1) = \infty$ as expected. But, we also notice something remarkable. Eq. (8) seems to make perfect sense for $0 < x < 1$. We know that the sum converges conditionally if $0 < x < 1$. Moreover, there are no singularities in sight. Thus, we shall *extend* the definition of the Riemann zeta function by defining $\zeta(x)$ by eq. (8) for $x > 0$. Eq. (8) is particularly useful for determining the behavior of $\zeta(x)$ as $x \to 1$. For further details, see the Appendix to these notes.

The evaluation of $\zeta(2)$ is a famous problem in mathematics that was first cracked by Leonhard Euler in 1735. Here, I will provide a derivation which is different from the one first discovered by Euler. The derivation presented here employs the following result obtained in problem 1(b) of Homework set #4,

$$S \equiv \sum_{n=1}^{\infty} \frac{\sin n\theta}{n} = \tfrac{1}{2}(\pi - \theta)\,, \qquad \text{where } 0 < \theta < 2\pi\,. \tag{9}$$

This result was derived from the power series expansion for $\mathrm{Ln}(1 - e^{i\theta})$. Using the results given in the class handout entitled, *Theorems About Power Series*, the indefinite integral of the series given in eq. (9) possesses this same radius of convergence as the original series.* Thus, we can integrate S to obtain

$$\sum_{n=1}^{\infty} \frac{\cos n\theta}{n^2} = -\int S d\theta = C - \tfrac{1}{2}\pi\theta + \tfrac{1}{4}\theta^2\,, \qquad \text{where } 0 \leq \theta \leq 2\pi\,, \tag{10}$$

and C is the constant of integration. Notice that the interval of convergence for θ now includes the two endpoints. This follows from Abel's theorem (cf. the class handout cited above), since the sum clearly converges at these two endpoints. By setting $\theta = 0$ in eq. (10), it immediately follows that:

$$C = \sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2)\,. \tag{11}$$

*Although the theorems quoted in the handout were stated for real power series, the same theorems also hold for complex power series.

By setting $\theta = \pi$ in eq. (10) and noting that $\cos(n\pi) = (-1)^n$, it immediately follows that:

$$C - \tfrac{1}{4}\pi^2 = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\eta(2)\,. \tag{12}$$

We can eliminate C from eqs. (11) and (12) to obtain:

$$\zeta(2) + \eta(2) = \tfrac{1}{4}\pi^2\,.$$

Finally, we use eq. (7) to obtain $\eta(2) = \frac{1}{2}\zeta(2)$. Inserting this result into the above equation yields the famous result,

$$\zeta(2) = \frac{\pi^2}{6}\,. \tag{13}$$

By further integrations, one can work out $\zeta(2n)$ for any positive integer n. For example, starting with eq. (10), with $C = \frac{1}{6}\pi^2$ [after using eq. (11)],

$$\sum_{n=1}^{\infty} \frac{\cos n\theta}{n^2} = \tfrac{1}{6}\pi^2 - \tfrac{1}{2}\pi\theta + \tfrac{1}{4}\theta^2\,, \qquad \text{where } 0 \leq \theta \leq 2\pi\,.$$

We integrate once to get:

$$\sum_{n=1}^{\infty} \frac{\sin n\theta}{n^3} = \tfrac{1}{6}\pi^2\theta - \tfrac{1}{4}\pi\theta^2 + \tfrac{1}{12}\theta^3\,, \qquad \text{where } 0 \leq \theta \leq 2\pi\,.$$

Note that the integration constant vanishes, since both sides of the above equation vanish when $\theta = 0$. Integrating once more and multiplying both sides by -1 yields

$$\sum_{n=1}^{\infty} \frac{\cos n\theta}{n^4} = C' - \tfrac{1}{12}\pi^2\theta^2 + \tfrac{1}{12}\pi\theta^3 - \tfrac{1}{48}\theta^4\,, \qquad \text{where } 0 \leq \theta \leq 2\pi\,,$$

and C' is another constant of integration. We again obtain two equations by setting $\theta = 0$ and $\theta = \pi$ respectively. Eliminating C' (as before), we obtain:

$$\zeta(4) + \eta(4) = \tfrac{1}{48}\pi^4\,.$$

Finally, we use eq. (7) to obtain $\eta(4) = \frac{7}{8}\zeta(4)$. Inserting this result into the above equation yields

$$\zeta(4) = \frac{\pi^4}{90}\,. \tag{14}$$

Further integrations will produce values for $\zeta(6)$, $\zeta(8), \ldots$, although this method does not easily provide a closed-form expression for $\zeta(2n)$ [cf. eq. (17) below]. If you would like to see another method (corresponding to the original method discovered by Euler) that produces a formula for $\zeta(2n)$ for all positive integers n, see the class handout entitled, *Evaluation of $\zeta(2n)$ in terms of Bernoulli numbers.*

So far, we have managed to extend the definition of $\zeta(x)$ to include all $x > 0$, using eq. (8)/ As in the case of the Gamma function, we would like to extend the definition of the Riemann zeta function further to include all real x (and eventually all complex x). This can be done, but it requires methods beyond the scope of this note. One of the key steps makes use of the following remarkable functional relation:

$$\boxed{\zeta(x) = 2^x\, \pi^{x-1}\, \sin\left(\tfrac{1}{2}\pi x\right) \Gamma(1-x)\, \zeta(1-x)\,.} \tag{15}$$

One can prove that this functional relation holds for $0 < x < 1$, where the zeta function on both sides of the equation are defined by eq. (8). The general proof of this relation can be found in the article on the Riemann zeta function that is posted on the class web site. Eq. (15) provides the way to extend the definition of $\zeta(x)$. For values of $x < 0$, the right hand side of eq. (15) is perfectly well behaved. Thus, we can use eq. (15) to define $\zeta(x)$ for all negative values of x.

Using eq. (15), we can work out the value of $\zeta(0)$. First, we multiply eq. (15) by $(1-x)$ and use $(1-x)\Gamma(1-x) = \Gamma(2-x)$. Then we take the limit as $x \to 1$:

$$\lim_{x\to 1}(1-x)\zeta(x) = 2\zeta(0)\,.$$

But, according to eq. (4), $\lim_{x\to 1}(1-x)\zeta(x) = -1$. We conclude that

$$\zeta(0) = -\tfrac{1}{2}\,.$$

This remarkable result plays a significant role in the Casimir effect, which is the phenomenon (predicted by quantum field theory) that two uncharged conducting plates in the vacuum actually attract each other. In a naive version of the theoretical computation, one obtains a result that involves $\lim_{x\to 0}\sum_{n=1}^{\infty}(1/n^x) = \infty$. In the more sophisticated analysis, it turns out that the actual result of the computation involves $\lim_{x\to 0}\zeta(x) = -\frac{1}{2}$. It seems that nature is telling us that:

$$1 + 1 + 1 + 1 + 1 + 1 + \cdots = -\tfrac{1}{2}\,.$$

While we are at it, we can also evaluate $\eta(0) = (1-2^1)\zeta(0) = \frac{1}{2}$. If we interpret this as the $x \to 0$ limit of eq. (5), we would conclude that

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots = \tfrac{1}{2}\,,$$

a result we have seen before when substituting $x = -1$ into the geometric series $(1-x)^{-1} = \sum_{n=0}^{\infty} x^n$. Of course, you should not take these last two equations literally!

From eq. (8), it is clear that $\zeta(x) \neq 0$ for any value of $x > 0$. For negative values of x, we can examine eq. (15). In this case, the right hand side of eq. (15) can vanish only when $\sin(\pi x/2) = 0$.† For $x < 0$, we have $\sin(\pi x/2) = 0$ when $\frac{1}{2}\pi x = -n\pi$ $(n = 1, 2, 3, \ldots)$. Hence, we conclude that:

$$\zeta(-2n) = 0\,, \qquad n = 1, 2, 3, \ldots$$

†For the case of $x = 0$, we note that $\lim_{x\to 0}\sin(\pi x/2)\zeta(1-x) = -\pi/2$, which implies again that $\zeta(0) = -\frac{1}{2}$. You should be able to derive this limit from the results presented above.

These are the only zeros of the zeta function for real values of x. When the definition of the zeta function is extended to complex values z, Riemann conjectured that the only other zeros of the Riemann zeta function occur for $z = \frac{1}{2} + iy$ (where y is real). An infinite number of zeros appear for various discrete values of y. This conjecture is called the Riemann zeta hypothesis and is the most famous outstanding unsolved problem in mathematics (now that Fermat's last theorem has been proven). The proof of this hypothesis would have very profound results for the behavior of prime numbers. Remarkably, there seems to be an amazing connection with physics as well. The distribution of the discrete values of y corresponding to the zeros of $\zeta(z)$ for $z = \frac{1}{2} + y$ seems to mirror precisely the distribution of energy levels of certain complex quantum systems.

It would be remiss not to mention the value of the zeta function at other integer values. There is a deep connection between the zeta function and the Bernoulli numbers.‡ As a result, the following relation holds:

$$\zeta(1-2n) = -\frac{B_{2n}}{2n}\,, \qquad n = 1, 2, 3, \ldots\,, \tag{16}$$

which provides the value of the zeta function evaluated at negative odd integers in terms of the Bernoulli numbers B_{2n}. Using this result and eq. (15), one can now easily derive the value of the zeta function evaluated at the positive even integers:

$$\zeta(2n) = \frac{(-1)^{n+1}(2\pi)^{2n}B_{2n}}{2(2n)!}\,, \qquad n = 1, 2, 3, \ldots \tag{17}$$

In deriving this result, I inserted $x = 1 - 2n$ in eq. (15) and used $2n\,\Gamma(2n) = \Gamma(2n+1) = (2n)!$ and $\sin(\frac{1}{2}\pi(1-2n)) = (-1)^n$. As a test, you should put $n = 1$ in eq. (17) and find $\zeta(2) = \frac{1}{4}(2\pi)^2 B_2 = \pi^2/6$. Similarly, for $n = 2$ we obtain $\zeta(4) = \pi^4/90$. Of course these reproduce the explicit results obtained in eqs. (13) and (14).

The only integers we have not yet considered are the positive odd integers. You might think that we could get a formula for $\zeta(2n+1)$ using eq. (15). However, given that $\zeta(-2n) = 0$, eq. (15) with $x = 2n$ (for positive integers n) simply reads $0 = 0$, and we fail to extract a value for $\zeta(2n+1)$. In fact, there are no known formulae for $\zeta(2n+1)$ in terms of powers of π or any other known constants. Whether such formulae could ever be discovered is an open question, although there is strong evidence that no such formulae exist. Until 1978, no one knew for sure whether the numbers $\zeta(3)$, $\zeta(5)$, $\zeta(7), \ldots$ were rational or irrational. In 1978, Roger Apéry stunned the world of mathematics by presenting new ideas for proof that $\zeta(3)$ was irrational. The formal proof was completed and published the following year. Unfortunately, his proof did not generalize to $\zeta(2n+1)$ for all $n > 1$.

‡Bernoulli numbers were introduced in the class handout entitled, *Taylor series expansions.* A derivation of eqs. (16) and (17) can be found in the class handout entitled, *Evaluation of* $\zeta(2n)$ *in terms of Bernoulli numbers.* I have also posted a nice introduction to the Bernoulli numbers on the class web site.

APPENDIX: The behavior of $\zeta(x)$ as $x \to 1$

In order to determine the behavior of $\zeta(x)$ as $x \to 1$, we will make use of eq. (8). Let $x = 1 + \epsilon$, where ϵ is a very small parameter. We shall expand eq. (8) about $\epsilon = 0$. The first step is to expand $(1 - 2^{1-x})^{-1}$. Putting $1 - x = -\epsilon$,

$$\frac{1}{1 - 2^{1-x}} = \frac{1}{1 - 2^{-\epsilon}} = \frac{1}{1 - e^{-\epsilon \ln 2}}\,.$$

Expanding out the exponential in the denominator of the last expression,

$$\begin{aligned}
\frac{1}{1 - 2^{1-x}} &\simeq \frac{1}{1 - (1 - \epsilon \ln 2 + \frac{1}{2}\epsilon^2 \ln^2 2)} \\
&\simeq \frac{1}{\epsilon \ln 2}\left(\frac{1}{1 - \frac{1}{2}\epsilon \ln 2}\right) \\
&\simeq \frac{1}{\epsilon \ln 2}\left(1 + \tfrac{1}{2}\epsilon \ln 2\right)\,. \qquad (18)
\end{aligned}$$

The second step is to expand out the factor $n^x = n^{1+\epsilon}$ that appears in the denominator of the summand in eq. (8).

$$\frac{1}{n^{1+\epsilon}} = \frac{1}{n \cdot n^{\epsilon}} = \frac{1}{n e^{\epsilon \ln n}}\,.$$

Working to first order in ϵ, we expand the exponential and then expand the resulting expression as follows:

$$\frac{1}{n e^{\epsilon \ln n}} \simeq \frac{1}{n(1 + \epsilon \ln n)} \simeq \frac{1}{n}\left(1 - \epsilon \ln n\right)\,.$$

It then follows that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{1+\epsilon}} &\simeq \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\left[1 - \epsilon \ln n\right] \\
&= \ln 2 - \epsilon \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n} \ln n + \mathcal{O}(\epsilon^2)\,, \qquad (19)
\end{aligned}$$

where we have identified the well known expansion, $\ln 2 = \sum_{n=1}^{\infty} (-1)^{n+1}/n$.

Applying the results of eqs. (18) and (19) to eq. (8), we find

$$\begin{aligned}
\zeta(1+\epsilon) &= \frac{1}{\epsilon \ln 2}\left[1 + \tfrac{1}{2}\epsilon \ln 2\right]\left[\ln 2 - \epsilon \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n} \ln n + \mathcal{O}(\epsilon^2)\right] \\
&= \frac{1}{\epsilon} + \tfrac{1}{2}\ln 2 - \frac{1}{\ln 2} \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n} \ln n + \mathcal{O}(\epsilon)\,.
\end{aligned}$$

Putting back $x = 1 + \epsilon$, and writing $(-1)^{n+1} = -(-1)^n$, we arrive at our final result:

$$\boxed{\zeta(x) = \frac{1}{x-1} + \tfrac{1}{2}\ln 2 + \frac{1}{\ln 2}\sum_{n=2}^{\infty}\frac{(-1)^n}{n}\ln n + \mathcal{O}(x-1)\,, \qquad \text{as } x \to 1\,.} \tag{20}$$

Comparing eq. (20) with eq. (4), we learn that:

$$\gamma = \tfrac{1}{2}\ln 2 + \frac{1}{\ln 2}\sum_{n=2}^{\infty}\frac{(-1)^n}{n}\ln n\,,$$

which also provides us with the sum of the alternating series:

$$\sum_{n=2}^{\infty}\frac{(-1)^n}{n}\ln n = \ln 2\left[\gamma - \tfrac{1}{2}\ln 2\right]\,.$$