The volume and surface area of an n-dimensional hypersphere

An n-dimensional hypersphere of radius R consists of the locus of points such that the distance from the origin is less than or equal to R. A point in an n-dimensional Euclidean space is designated by $(x_1\,,\,x_2\,,\,\ldots\,,\,x_n)$. In equation form, the hypersphere corresponds to the set of points such that

$$x_1^2 + x_2^2 + \cdots + x_n^2 \leq R^2\,. \tag{1}$$

To compute the volume of this hypersphere, we simply integrate the infinitesimal volume element $dV = dx_1 dx_2 \cdots dx_n$ over the region of n-dimensional space indicated by eq. (1). We wish to compute this volume $V_n(R)$. Explicitly,

$$V_n(R) = \int \cdots \int_{x_1^2+x_2^2+\cdots+x_n^2 \leq R^2} dx_1 dx_2 \cdots dx_n = C_n R^n\,. \tag{2}$$

The factor of R^n is a consequence of dimensional analysis.

The surface "area" of the n-dimensional hypersphere defined by eq. (1) will be denoted by $S_{n-1}(R)$. The surface of the hypersphere corresponds to the locus of points such that $x_1^2 + x_2^2 + \cdots + x_n^2 = R^2$. We can construct the volume $V_n(R)$ by adding infinitely thin spherical shells of radius $0 \leq r \leq R$. In equation form, this reads:

$$V_n(R) = \int_0^R S_{n-1}(r)\, dr\,. \tag{3}$$

It follows from the fundamental theorem of calculus that

$$S_{n-1}(R) = \frac{dV_n(R)}{dR} = nC_n R^{n-1}\,, \tag{4}$$

where we have used eq. (2). Thus, the only remaining task is to compute C_n.

In order to obtain a better intuition on the meaning of C_n, let us equate the two expressions we have for $V_n(R)$, namely eq. (2) and eq. (3). In the latter, $S_{n-1}(r)$ is determined by eq. (4). Thus,

$$\int \cdots \int_{x_1^2+x_2^2+\cdots+x_n^2 \leq R^2} dx_1 dx_2 \cdots dx_n = nC_n \int_0^R r^{n-1}\, dr\,. \tag{5}$$

Eq. (5) is simply the evaluation of an n-dimensional integral either in rectangular co-ordinates or hyper-spherical co-ordinates. In n-dimensions, we would write

$$dx_1\, dx_2 \cdots dx_n = r^{n-1}\, dr\, d\Omega_{n-1}\,, \tag{6}$$

where $d\Omega_{n-1}$ contains all of the angular factors. For example, for $n=2$, $d\Omega_1 = d\theta$; for $d=2$, $d\Omega_2 = \sin\theta\, d\theta\, d\phi$; *etc.* One could explicitly define the $n-1$ angular variables in n-dimensions. However, if we are integrating over a spherically symmetric function, then we will simply integrate over $d\Omega_{n-1}$. Comparing eqs. (5) and (6), we see that:

$$\int \cdots \int d\Omega_{n-1} = nC_n\,. \tag{7}$$

I will present a trick for computing C_n without ever explicitly parameterizing the angles of the hyper-spherical coordinate system. Consider the function

$$f(x_1\,,\, x_2\,,\, \ldots\,,\, x_n) = e^{-(x_1^2+x_2^2+\cdots+x_n^2)} = e^{-r^2}\,,$$

If we integrate this function over the full n-dimensional space in both rectangular and hyper-spherical coordinates, we obtain:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-(x_1^2+x_2^2+\cdots+x_n^2)} = \int_0^\infty r^{n-1}\,dr \int d\Omega_{n-1} e^{-r^2}\,.$$

Since the integrand on the right hand side depends only on r (there is no angular dependence), we can immediately perform the integral over $d\Omega_{n-1}$. Using eq. (7),

$$\int_{-\infty}^{\infty} e^{-x_1^2}\,dx_1 \int_{-\infty}^{\infty} e^{-x_2^2}\,dx_2 \,\cdots\, \int_{-\infty}^{\infty} e^{-x_n^2}\,dx_n = nC_n \int_0^\infty r^{n-1}\,e^{-r^2}\,dr\,. \tag{8}$$

All the integrals above can be evaluated exactly:

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}\,,$$

$$\int_0^\infty r^{n-1}\,e^{-r^2}\,dr = \frac{1}{2}\Gamma\left(\frac{n}{2}\right)\,.$$

Inserting these results into eq. (8), we obtain:

$$\pi^{n/2} = C_n\,\frac{n}{2}\,\Gamma\left(\frac{n}{2}\right) = C_n\,\Gamma\left(1+\frac{n}{2}\right)\,,$$

where we used the property of the Gamma function, $x\Gamma(x) = \Gamma(x+1)$, at the final step. Solving for C_n, we obtain

$$\boxed{C_n = \frac{\pi^{n/2}}{\Gamma\left(1+\frac{n}{2}\right)}\,.} \tag{9}$$

Although we chose a particular function $f(x_1\,,\, x_2\,,\, \ldots\,,\, x_n)$ to obtain the final result for C_n, it is clear that the value of C_n is independent of this function. After all, the defining equation for C_n [eq. (7)] makes no reference to any function. As a check, let us evaluate $n\,C_n$ for $n=2$ and $n=3$:

$$2\,C_2 = \frac{2\pi}{\Gamma(2)} = 2\pi = \int_0^{2\pi} d\theta\,,$$

$$3\,C_3 = \frac{3\pi^{3/2}}{\Gamma\left(\frac{5}{2}\right)} = 4\pi = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta\,.$$

In the last computation, we used $\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) = \frac{3}{4}\sqrt{\pi}$. Indeed, we have produced the correct values for the integration over the angular factors in two and three dimensions.

We are finally ready to compute the volume and surface "area" of an n-dimensional hypersphere. Inserting eq. (9) into eqs. (2) and (4), we find:

$$\boxed{V_n(R) = \frac{\pi^{n/2}\,R^n}{\Gamma\left(1+\frac{n}{2}\right)}}$$

$$S_{n-1}(R) = \frac{n\pi^{n/2}\,R^{n-1}}{\Gamma\left(1+\frac{n}{2}\right)}\,.$$

The expression for $S_n(R)$ can be slightly simplified by writing $\Gamma\left(1+\frac{n}{2}\right) = \frac{n}{2}\,\Gamma\left(\frac{n}{2}\right)$. This yields

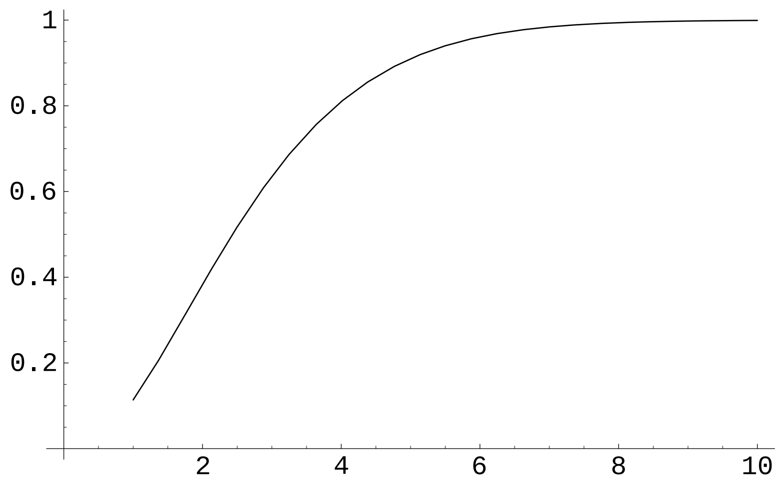
$$\boxed{S_{n-1}(R) = \frac{2\pi^{n/2}\,R^{n-1}}{\Gamma\left(\frac{n}{2}\right)}}\,.$$

You should check that you get the expected results for $n=2$ and $n=3$. For example, $V_3(R) = \frac{4}{3}\pi R^3$ and $S_2(R) = 4\pi R^2$.

Finally, it is amusing to note that $\lim_{n\to\infty} V_n(R) = 0$ and $\lim_{n\to\infty} S_n(R) = 0$. To understand this peculiar behavior, consider a hypercube in n-dimensional space measuring one unit on each side. The total volume of this hypercube is 1. We can fit a hypersphere of diameter 1 (or radius $\frac{1}{2}$) inside the hypercube such that the surface of the hypersphere just touches each of the walls of the hypercube. Then $1 - V_n(\frac{1}{2})$ is the volume inside the cube but outside the hypersphere.

In particular, as n becomes large, $1 - V_n(\frac{1}{2})$ rapidly approaches 1, which is consistent with the assertion that $\lim_{n\to\infty} V_n(R) = 0$. This simply means that as the number of dimensions becomes larger and larger, the amount of space outside the hypersphere (but inside the cube) is become relatively more and more important. This is already happening as you go from 2 to 3 dimensions. So you can check your intuition by inscribing a circle in a unit square and a sphere in a unit cube and computing the total volume in three dimensions (area in two dimensions) outside the sphere (circle) but inside the cube (square). If you take the ratio of volumes (areas) of the sphere (circle) to that of the cube (square), this ratio actually *decreases* as you go from 2 dimensions to 3 dimensions!

To illustrate the behavior described above, I have plotted $1 - V_n(\frac{1}{2})$ on the y-axis *vs.* the dimension n on the x-axis for values of $n = 1, 2, 3, \ldots, 10$ in the figure below.



Indeed, $1 - V_n(\frac{1}{2})$ rapidly approaches 1 as n becomes large, which is equivalent to the statement that $\lim_{n\to\infty} V_n(R) = 0$.

A related problem examines the volume of a hypersphere of radius 1. (In contrast, the radius was equal to 1/2 in the previous discussion above.) One can study the behavior of $V_n(1)$ as n increases. Unlike the graph shown above, $V_n(1)$ is an increasing function of n for small values of n. However, it quickly reaches a maximum, after which it steadily decreases to zero as n becomes large. This corresponds to problem 7 on homework set #5, so I will leave the rest to you.