The Laplacian of the inverse distance

1. Poisson's Equation

Consider the laws of electrostatics in cgs units,

$$\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{E}} = 4\pi\rho\,, \qquad \vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{E}} = 0\,, \tag{1}$$

where $\vec{\boldsymbol{E}}$ is the electric field vector and ρ is the local charge density. Since $\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{E}} = 0$, it follows that $\vec{\boldsymbol{E}}$ can be expressed as the gradient of a scalar function. Thus, we define the electric potential as

$$\vec{\boldsymbol{E}} = -\boldsymbol{\nabla}\Phi\,. \tag{2}$$

Note that $\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{E}} = -\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{\nabla}}\Phi = 0$, since the curl of the gradient of any well-behaved scalar function is zero. Combining eqs. (1) and (2) yields

$$\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{E}} = -\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\boldsymbol{\nabla}\Phi = -\vec{\boldsymbol{\nabla}}^{\boldsymbol{2}}\Phi = 4\pi\rho\,.$$

That is,

$$\vec{\boldsymbol{\nabla}}^{\boldsymbol{2}}\Phi = -4\pi\rho\,. \tag{3}$$

This is Poisson's equation (or the inhomogeneous Laplace equation).

Consider a point charge located at the position $\vec{\boldsymbol{r}}_{\boldsymbol{0}}$. Then, we can write

$$\rho(\vec{\boldsymbol{r}}) = q\delta^3(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_{\boldsymbol{0}})\,.$$

The delta function representation of a point charge indicates that no charge exists anywhere other than at the position $\vec{\boldsymbol{r}}_{\boldsymbol{0}}$. Moreover, the total charge contained in the point charge is

$$\int_V \rho(\vec{\boldsymbol{r}})d^3r = q\int_V \delta^3(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_{\boldsymbol{0}}) = q\,, \tag{4}$$

where d^3r is the infinitesimal three-dimensional volume element and V is any finite volume that contains the point $\vec{\boldsymbol{r}}_{\boldsymbol{0}}$.

Given a point charge located at the position $\vec{\boldsymbol{r}}_{\boldsymbol{0}}$, the corresponding electric field is given by Coulomb's law (in cgs units),

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}) = \frac{q(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_{\boldsymbol{0}})}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_{\boldsymbol{0}}|^3} = -q\vec{\boldsymbol{\nabla}}\left(\frac{1}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_{\boldsymbol{0}}|}\right)\,.$$

From this, we can obtain the potential from eq. (2),

$$\Phi(\vec{\boldsymbol{r}}) = \frac{q}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_{\boldsymbol{0}}|}\,.$$

2. The Laplacian of the inverse distance

The inverse distance between the vectors $\vec{\boldsymbol{r}}$ and $\vec{\boldsymbol{r}}_{\mathbf{0}}$ is given by the function,

$$d(\vec{\boldsymbol{r}}) = \frac{1}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_{\mathbf{0}}|}\,.$$

Here, $\vec{\boldsymbol{r}}_{\mathbf{0}}$ is a fixed location (such as the position of a point charge) and $\vec{\boldsymbol{r}}$ is the location of the observer. We shall compute the Laplacian of the inverse distance

$$\vec{\boldsymbol{\nabla}}^{\mathbf{2}} d(\vec{\boldsymbol{r}}) = \vec{\boldsymbol{\nabla}}^{\mathbf{2}}\left(\frac{1}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_{\mathbf{0}}|}\right),$$

where $\vec{\boldsymbol{\nabla}}^{\mathbf{2}}$ involves derivatives with respect to $\vec{\boldsymbol{r}}$, with $\vec{\boldsymbol{r}}_{\mathbf{0}}$ held fixed. It is convenient to define a new variable $\vec{\boldsymbol{R}} \equiv \vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_{\mathbf{0}}$. Then, it follows that $\vec{\boldsymbol{\nabla}}^{\mathbf{2}}_{\boldsymbol{R}} = \vec{\boldsymbol{\nabla}}^{\mathbf{2}}_{\boldsymbol{r}}$, where the subscript indicates the variable employed by the corresponding derivatives.[1] Thus, we evaluate

$$\vec{\boldsymbol{\nabla}}^{\mathbf{2}}\left(\frac{1}{R}\right) = \frac{1}{R^2}\frac{\partial}{\partial R}\left\{R^2\frac{\partial}{\partial R}\left(\frac{1}{R}\right)\right\} = \frac{1}{R^2}\frac{\partial}{\partial R}(-1) = 0\,,$$

where we have performed the computation in spherical coordinates, where

$$\vec{\boldsymbol{\nabla}}^{\mathbf{2}}_{\boldsymbol{R}} = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial}{\partial R}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\,.$$

Of course, the angular derivatives vanish when applied to a radial function. Thus, we have apparently derived the result,

$$\vec{\boldsymbol{\nabla}}^{\mathbf{2}}\left(\frac{1}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_{\mathbf{0}}|}\right) = 0\,. \tag{5}$$

However, eq. (5) cannot be strictly true. After all, in section 1, we saw that a point charge produces an electric potential

$$\Phi(\vec{\boldsymbol{r}}) = \frac{q}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_{\mathbf{0}}|}\,,$$

and Poisson's equation implies that $\vec{\boldsymbol{\nabla}}^{\mathbf{2}}\Phi = 4\pi\rho \neq 0$. Indeed, using the results of section 1, given a point charge located at $\vec{\boldsymbol{r}}_{\mathbf{0}}$,

$$\vec{\boldsymbol{\nabla}}^{\mathbf{2}}\Phi(\vec{\boldsymbol{r}}) = q\vec{\boldsymbol{\nabla}}^{\mathbf{2}}\left(\frac{1}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_{\mathbf{0}}|}\right) = -4\pi\rho(\vec{\boldsymbol{r}}) = -4\pi q\delta^3(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_{\mathbf{0}})\,,$$

which yields

$$\vec{\boldsymbol{\nabla}}^{\mathbf{2}}\left(\frac{1}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_{\mathbf{0}}|}\right) = -4\pi\delta^3(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_{\mathbf{0}})\,. \tag{6}$$

[1] It is standard practice to write $\vec{\boldsymbol{\nabla}}^{\mathbf{2}} \equiv \vec{\boldsymbol{\nabla}}^{\mathbf{2}}_{\boldsymbol{r}}$. That is, if no subscript appears, one assumes that derivatives are to be computed with respect to $\vec{\boldsymbol{r}}$.

Comparing eqs. (5) and (6), we see that our previous derivation is correct for all values of $\boldsymbol{\vec{r}} \neq \boldsymbol{\vec{r}_0}$. However for $\boldsymbol{\vec{r}} = \boldsymbol{\vec{r}_0}$, the inverse distance is singular, and hence the explicit computation of the Laplacian given above eq. (5) is suspect.

We can confirm the mathematical correctness of eq. (6) as follows. Again, it is convenient to work with the variable $\boldsymbol{\vec{R}}$, in which case, we have

$$\boldsymbol{\vec{\nabla}^2}\left(\frac{1}{R}\right) = -4\pi\delta^3(\boldsymbol{\vec{R}})\,, \tag{7}$$

where $R \equiv |\boldsymbol{\vec{R}}|$. Let us integrate $\boldsymbol{\vec{\nabla}^2}(1/R)$ over any volume V that contains the origin. We can divide up this volume into two pieces $V = V_a + V_b$, where V_a is a solid sphere of radius a whose center is the origin and V_b is the remaining part of the volume Then,

$$\int_V \boldsymbol{\vec{\nabla}^2}\left(\frac{1}{R}\right) d^3R = \int_{V_a} \boldsymbol{\vec{\nabla}^2}\left(\frac{1}{R}\right) d^3R + \int_{V_b} \boldsymbol{\vec{\nabla}^2}\left(\frac{1}{R}\right) d^3R = \int_{V_a} \boldsymbol{\vec{\nabla}^2}\left(\frac{1}{R}\right) d^3R\,,$$

since $\boldsymbol{\vec{\nabla}^2}(1/R) = 0$ at all points in V_b since the latter excludes the origin. Using the divergence theorem of vector calculus,

$$\int_{V_a} \boldsymbol{\vec{\nabla}^2}\left(\frac{1}{R}\right) d^3R = \int_{V_a} \boldsymbol{\vec{\nabla}}\cdot\boldsymbol{\vec{\nabla}}\left(\frac{1}{R}\right) d^3R = \oint_{S_a} \boldsymbol{\vec{\nabla}}\left(\frac{1}{R}\right)\cdot\boldsymbol{\hat{R}}\,ds\,,$$

where S_a is the closed spherical surface of radius a which constitutes the boundary of V_a. In particular, the outward normal to S_a is the unit radial vector $\boldsymbol{\hat{R}}$. The infinitesimal surface element is $ds = a^2 d\Omega$, where $d\Omega = \sin\theta d\theta d\phi$ is the usual differential solid angle element. Using

$$\boldsymbol{\vec{\nabla}}\left(\frac{1}{R}\right)\bigg|_{R=a} = \boldsymbol{\hat{R}}\frac{\partial}{\partial R}\left(\frac{1}{R}\right)\bigg|_{R=a} = -\frac{1}{a^2}\boldsymbol{\hat{R}}\,,$$

it follows that

$$\int_V \boldsymbol{\vec{\nabla}^2}\left(\frac{1}{R}\right) d^3R = -\oint_{S_A} \boldsymbol{\hat{R}}\cdot\boldsymbol{\hat{R}}\frac{1}{a^2}\,a^2 d\Omega = -\oint_{S_A} d\Omega = -4\pi\,.$$

That is, we have shown that

$$\boldsymbol{\vec{\nabla}^2}\left(\frac{1}{R}\right) = 0 \ \text{ for } R \neq 0 \qquad \text{and} \qquad \int_V \boldsymbol{\vec{\nabla}^2}\left(\frac{1}{R}\right) d^3R = -4\pi\,,$$

for any volume V that contains the origin. The only "function" that satisfies these relations is

$$\boldsymbol{\vec{\nabla}^2}\left(\frac{1}{R}\right) = -4\pi\delta^3(\boldsymbol{\vec{R}})\,,$$

since $\delta^3(\boldsymbol{\vec{R}}) = 0$ for any $\boldsymbol{\vec{R}} \neq 0$ and

$$\int_V \delta^3(\boldsymbol{\vec{R}})\,d^3R = 1\,,$$

for any volume V that contains the origin. Thus, we have confirmed the validity of eq. (7). More generally, we have established the result,

$$\boxed{\boldsymbol{\vec{\nabla}^2}\left(\frac{1}{|\boldsymbol{\vec{r}} - \boldsymbol{\vec{r}_0}|}\right) = -4\pi\delta^3(\boldsymbol{\vec{r}} - \boldsymbol{\vec{r}_0})} \tag{8}$$

3. Solutions to Poisson's Equation

We wish to solve Poisson's equation, eq. (3), given a known charge distribution $\rho(\vec{\boldsymbol{r}})$ that is nonzero over some finite volume of space, subject to boundary conditions (typically taken to be Dirichlet, in which Φ is specified over some closed surface or Neumann where $\vec{\boldsymbol{E}} = -\vec{\boldsymbol{\nabla}}\Phi$ is specified over some closed surface). The solution will take the form,

$$\Phi(\vec{\boldsymbol{r}}) = \Phi_p(\vec{\boldsymbol{r}}) + \Phi_c(\vec{\boldsymbol{r}})\,, \tag{9}$$

where $\Phi_p(\vec{\boldsymbol{r}})$ is a particular solution to the Poisson equation and $\Phi_c(\vec{\boldsymbol{r}})$ is the (complementary) solution to the Laplace equation, $\vec{\boldsymbol{\nabla}}^{\boldsymbol{2}}\Phi_c(\vec{\boldsymbol{r}}) = 0$. In defining the particular solution, we shall impose the condition that

$$\lim_{r\to\infty} \Phi_p(\vec{\boldsymbol{r}}) = 0\,, \tag{10}$$

which can be viewed as a boundary condition that states that $\Phi_p(\vec{\boldsymbol{r}})$ vanishes on the surface of a sphere of radius r in the limit of $r \to \infty$. Then

$$\Phi_p(\vec{\boldsymbol{r}}) = \int \frac{\rho(\vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|}\, d^3r'\,, \tag{11}$$

where the integration is taken over all of three-dimensional space. To prove that $\Phi_p(\vec{\boldsymbol{r}})$ satisfies Poisson's equation subject to eq. (10), we first note that as $r \to \infty$, we have $|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'| = r[1 + \mathcal{O}(1/r)]$ so that

$$\lim_{r\to\infty} \Phi_p(\vec{\boldsymbol{r}}) = \lim_{r\to\infty} \int \frac{\rho(\vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|}\, d^3r' = \lim_{r\to\infty} \frac{1}{r}\int \rho(\vec{\boldsymbol{r}}\,')d^3r' + \mathcal{O}\left(\frac{1}{r^2}\right) = \lim_{r\to\infty} \frac{q}{r} + \mathcal{O}\left(\frac{1}{r^2}\right) = 0\,, \tag{12}$$

where we have used eq. (4), under the assumption that the charge distribution is restricted to a finite region of space. Next, we compute the Laplacian of $\Phi_p(\vec{\boldsymbol{r}})$,

$$\begin{aligned}\vec{\boldsymbol{\nabla}}^{\boldsymbol{2}}\Phi_p(\vec{\boldsymbol{r}}) &= \vec{\boldsymbol{\nabla}}^{\boldsymbol{2}} \int \frac{\rho(\vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|}\, d^3r' = \int \rho(\vec{\boldsymbol{r}}\,')\vec{\boldsymbol{\nabla}}^{\boldsymbol{2}}\left(\frac{1}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|}\right) d^3r' \\ &= -4\pi \int \rho(\vec{\boldsymbol{r}}\,')\delta^3(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,')d^3r' = -4\pi\rho(\vec{\boldsymbol{r}})\,,\end{aligned} \tag{13}$$

where we have used eq. (8). Note that $\vec{\boldsymbol{\nabla}}^{\boldsymbol{2}}$ involves derivatives with respect to $\vec{\boldsymbol{r}}$, so that in applying the Laplacian, the variable $\vec{\boldsymbol{r}}\,'$ (which is a dummy integration variable) is treated as being fixed. Thus, we have verified that $\Phi_p(\vec{\boldsymbol{r}})$ is a solution to Poisson's equation.

Indeed, $\Phi_p(\vec{\boldsymbol{r}})$ is the unique solution to Poisson's equation, which is valid at all points in space, subject to eq. (10). More general boundary value problems would involve solving Poisson's equation in a restricted region of space, V. In this case, we must specify the boundary conditions on the closed surface S of V. The solution is then given by:

$$\Phi(\vec{\boldsymbol{r}}) = \Phi_c(\vec{\boldsymbol{r}}) + \int_V \frac{\rho(\vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|}\, d^3r'\,, \tag{14}$$

where $\Phi_c(\vec{\boldsymbol{r}})$ is a solution to Laplace's equation, which is chosen such that the boundary conditions are satisfied when applied to the *complete* solution to the problem, $\Phi(\vec{\boldsymbol{r}})$.

4. The inverse Laplacian and the Green function

Consider the solution to Poisson's equation, which is valid at all points in space, subject to eq. (10),

$$\vec{\nabla}^2 \Phi(\vec{r}) = -4\pi\rho(\vec{r}) \,, \tag{15}$$

where $\rho(\vec{r})$ is nonzero only over some finite region in space. In fact, this last assumption is stronger than is necessary. It is sufficient to assume that $\rho(\vec{r}) \to 0$ as $r \to \infty$ fast enough such that the volume integral of $\rho(\vec{r})$ over all space converges. Then, as discussed in Section 3, the solution to Poisson's equation is unique. That is, the solution to Poisson's equation is given by eq. (14) with $\Phi_c(\vec{r}) = 0$. Equivalently, $\Phi(\vec{r}) = \Phi_p(\vec{r})$, where $\Phi_p(\vec{r})$ is given by eq. (11).

Under the stated conditions above, it is tempting to derive the solution to Poisson's equation by introducing the inverse Laplacian, $\vec{\nabla}^{-2}$. Operating with the inverse Laplacian on eq. (15) yields,

$$\vec{\nabla}^{-2}\vec{\nabla}^2 \Phi(\vec{r}) = -4\pi\vec{\nabla}^{-2}\rho(\vec{r}) \,.$$

Clearly, one should define $\vec{\nabla}^{-2}\vec{\nabla}^2$ to be the identity operator, in which case we would conclude that

$$\Phi(\vec{r}) = -4\pi\vec{\nabla}^{-2}\rho(\vec{r}) \,. \tag{16}$$

Comparing this with $\Phi(\vec{r}) = \Phi_p(\vec{r})$, where $\Phi_p(\vec{r})$ is given by eq. (11), it follows that we should identify

$$\boxed{\vec{\nabla}^{-2}\rho(\vec{r}) = -\frac{1}{4\pi}\int \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, d^3r'} \tag{17}$$

Plugging eq. (17) back into eq. (16) yields

$$\Phi(\vec{r}) = \int \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, d^3r' \,, \tag{18}$$

as expected.

The definition of the inverse Laplacian given in eq. (17) shows that this operator acts nonlocally. That is, the value of $\vec{\nabla}^{-2}\rho(\vec{r})$ at the point $\vec{r}$ depends on $\rho(\vec{r}\,')$ evaluated at all points in space. This should not be surprising to you. After all, the antiderivative of calculus is an integral! More importantly, the definition of the inverse Laplacian requires an assumption about the space of functions on which it acts. In the present case, we have required that the space of functions should only include twice differentiable functions that vanish sufficiently fast at infinity. To check that the definition of the inverse Laplacian given in eq. (17) is sensible, we perform the following two computations:

$$\begin{aligned}\vec{\nabla}^2\vec{\nabla}^{-2}\rho(\vec{r}) &= -\frac{1}{4\pi}\vec{\nabla}^2\int \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, d^3r' = -\frac{1}{4\pi}\int \rho(\vec{r}\,')\vec{\nabla}^2\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right) d^3r' \\ &= \int \rho(\vec{r}\,')\delta^3(\vec{r}-\vec{r}\,')d^3r' = \rho(\vec{r}) \,,\end{aligned}$$

and

$$\vec{\boldsymbol{\nabla}}^{-2}\left[\vec{\boldsymbol{\nabla}}^{2}\rho(\vec{\boldsymbol{r}})\right]=-\frac{1}{4\pi}\int\frac{\vec{\boldsymbol{\nabla}}'^{2}\rho(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}\,d^3r'=-\frac{1}{4\pi}\int\rho(\vec{\boldsymbol{r}}')\vec{\boldsymbol{\nabla}}'^{2}\left(\frac{1}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}\right)\,d^3r'$$

$$=\int\rho(\vec{\boldsymbol{r}}')\delta^3(\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}')d^3r'=\rho(\vec{\boldsymbol{r}})\,,\qquad(19)$$

where $\vec{\boldsymbol{\nabla}}'^{2}$ is the Laplacian that involves derivatives with respect to $\vec{\boldsymbol{r}}'$. Note that in deriving eq. (19) we integrated by parts twice, and in each case we dropped the surface term at infinity (which is valid since ρ and $\vec{\boldsymbol{\nabla}}\rho$ vanish at infinity by assumption). Thus, we have shown that eq. (17), subject to restrictions on $\rho(\vec{\boldsymbol{r}})$ at infinity, satisfies

$$\vec{\boldsymbol{\nabla}}^{2}\vec{\boldsymbol{\nabla}}^{-2}\rho(\vec{\boldsymbol{r}})=\vec{\boldsymbol{\nabla}}^{-2}\vec{\boldsymbol{\nabla}}^{2}\rho(\vec{\boldsymbol{r}})=\rho(\vec{\boldsymbol{r}})\,,$$

which confirms that both $\vec{\boldsymbol{\nabla}}^{2}\vec{\boldsymbol{\nabla}}^{-2}$ and $\vec{\boldsymbol{\nabla}}^{2}\vec{\boldsymbol{\nabla}}^{-2}$ are equivalent to the identity operator.

The inverse Laplacian can also be used to determine the Green function of Poisson's equation. First we assume that the potential vanishes sufficiently fast at infinity, as discussed below eq. (10). We define the Green function $G(\vec{\boldsymbol{r}},\vec{\boldsymbol{r}}')$ to be the solution of

$$\vec{\boldsymbol{\nabla}}^{2}G(\vec{\boldsymbol{r}},\vec{\boldsymbol{r}}')=-4\pi\delta^3(\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}')\,.\qquad(20)$$

The factor of -4π is conventional (although not the convention adopted by Boas). Then,

$$G(\vec{\boldsymbol{r}},\vec{\boldsymbol{r}}')=-4\pi\vec{\boldsymbol{\nabla}}^{-2}\delta^3(\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}')=\int\frac{\delta^3(\vec{\boldsymbol{r}}''-\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}''|}d^3r''=\frac{1}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}\,.\qquad(21)$$

Thus, the inverse Laplacian provides a very quick derivation of the Green function. The interpretation of the Green function is clear—it is the potential that arises due to the presence of a point charge located at $\vec{\boldsymbol{r}}'$. The utility of the Green function is that it can be used to construct the potential for an arbitrary charge density via

$$\Phi(\vec{\boldsymbol{r}})=\int G(\vec{\boldsymbol{r}},\vec{\boldsymbol{r}}')\rho(\vec{\boldsymbol{r}}')d^3r'\,,\qquad(22)$$

since eq. (22) implies that $\Phi(\vec{\boldsymbol{r}})$ satisfies the Poisson equation, i.e.,

$$\vec{\boldsymbol{\nabla}}^{2}\Phi(\vec{\boldsymbol{r}})=\int\rho(\vec{\boldsymbol{r}})\vec{\boldsymbol{\nabla}}^{2}G(\vec{\boldsymbol{r}},\vec{\boldsymbol{r}}')=-4\pi\int\rho(\vec{\boldsymbol{r}})\delta^3(\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}')=-4\pi\rho(\vec{\boldsymbol{r}})\,.$$

Another interpretation of the Green function can be ascertained from eq. (21). The Dirac delta function is the function space analog of the Kronecker delta δ_{ij}. Thus, the Dirac delta function is an infinite dimensional matrix corresponding to the identity matrix, where $\delta^3(\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}')$ are the matrix elements of this infinite dimensional matrix. Apart from the overall factor of -4π (which is a matter of convention), $G(\vec{\boldsymbol{r}},\vec{\boldsymbol{r}}')$ are the matrix elements of the infinite dimensional matrix that represents the inverse Laplacian.

In more general boundary value problems, one must solve Poisson's equation in a restricted region of space, V. In this case, we must specify the boundary conditions on the closed surface S of V. The corresponding Green function is still a solution to eq. (20), but it now must also satisfy the relevant boundary conditions. Thus, in analogy to eq. (14), the Green function takes the form

$$G(\vec{\boldsymbol{r}}, \vec{\boldsymbol{r}}') = F(\vec{\boldsymbol{r}}, \vec{\boldsymbol{r}}') + \frac{1}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|}\,, \tag{23}$$

where $F(\vec{\boldsymbol{r}}, \vec{\boldsymbol{r}}')$ is a solution to Laplace's equation that is adjusted in order that $G(\vec{\boldsymbol{r}}, \vec{\boldsymbol{r}}')$ satisfy the relevant boundary conditions. In this case, eq. (22) yields

$$\Phi(\vec{\boldsymbol{r}}) = \int_V G(\vec{\boldsymbol{r}}, \vec{\boldsymbol{r}}')\rho(\vec{\boldsymbol{r}}')d^3r' = \int_V F(\vec{\boldsymbol{r}}, \vec{\boldsymbol{r}}')\rho(\vec{\boldsymbol{r}}')d^3r' + \int_V \frac{\rho(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|}d^3r'\,.$$

Comparing with eq. (14), we identify

$$\Phi_c(\vec{\boldsymbol{r}}) = \int_V F(\vec{\boldsymbol{r}}, \vec{\boldsymbol{r}}')\rho(\vec{\boldsymbol{r}}')d^3r'\,.$$

The inverse Laplacian was defined in eq. (17) under the assumption that it acts on functions (defined at all points in space) that vanish sufficiently fast at infinity. In contrast, if the functions are defined only in a restricted region of space V, then the inverse Laplacian is ill-defined unless one imposes boundary conditions on the closed surface S of V. This can be understood as follows. If we solve Laplace's equation inside V, we find non-trivial solutions, denoted by $\Phi_c(\vec{\boldsymbol{r}})$ in eq. (9). That is, the Laplacian possesses an eigenfunction $\Phi_c(\vec{\boldsymbol{r}})$ with corresponding eigenvalue equal to zero. This immediately implies that $\vec{\boldsymbol{\nabla}}^{-2}$ is ill-defined; otherwise one would obtain eq. (17) instead of the correct result given in eq. (14).[2] This means that eq. (16) does not determine $\Phi(\vec{\boldsymbol{r}})$ uniquely. This is not surprising, as we have not yet specified the boundary conditions on S. However, once we specify these conditions, $\Phi(\vec{\boldsymbol{r}})$ is uniquely determined. This means that the definition of $\vec{\boldsymbol{\nabla}}^{-2}$ [which generalizes eq. (17)] becomes well-defined. This is not surprising, since we know that the form of the Green function depends in detail on the boundary conditions that are applied, which determines $F(\vec{\boldsymbol{r}}, \vec{\boldsymbol{r}}')$ as indicated in eq. (23).

5. An application of the inverse Laplacian

In this section, we provide an interesting application of the inverse Laplacian in proving the Helmholtz decomposition of a vector field $\vec{\boldsymbol{V}}(\vec{\boldsymbol{r}})$ that exists in all of space. We assume that $\vec{\boldsymbol{V}}(\vec{\boldsymbol{r}})$ vanishes sufficiently fast as $r \to \infty$. The Helmholtz theorem states that the following decomposition is unique,

$$\vec{\boldsymbol{V}} = \vec{\boldsymbol{V}}_{\mathbf{tr}} + \vec{\boldsymbol{V}}_{\mathbf{long}}\,, \qquad \text{where } \vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{V}}_{\mathbf{tr}} = 0 \text{ and } \vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{V}}_{\mathbf{long}} = 0\,. \tag{24}$$

Any vector $\vec{\boldsymbol{V}}_{\mathbf{tr}}$ that satisfies $\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{V}}_{\mathbf{tr}} = 0$ is called solenoidal or transverse. Any vector $\vec{\boldsymbol{V}}_{\mathbf{long}}$ that satisfies $\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{V}}_{\mathbf{long}} = 0$ is called irrotational or longitudinal.[3]

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[2]Consider the analogous case of a finite dimensional operator and its matrix representation M. If M has a zero eigenvalue, then its determinant vanishes (since $\det M$ is the product of its eigenvalues), in which case M^{-1} is ill-defined.

[3]The terminology transverse and longitudinal implicit in eq. (24) arises from the study of vector waves of the form $\vec{\boldsymbol{V}}(\vec{\boldsymbol{r}}, t) = \vec{\boldsymbol{E}}_0 e^{i\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{r}}-i\omega t}$. Then, $\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{V}}_{\mathbf{tr}} = 0$ implies that $\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{V}}_{\mathbf{tr}} = 0$, which means that $\vec{\boldsymbol{V}}_{\mathbf{tr}}$ is *transverse* to the direction of the wave (which propagates along $\vec{\boldsymbol{k}}$). Likewise, $\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{V}}_{\mathbf{long}} = 0$ implies that $\vec{\boldsymbol{k}}\times\vec{\boldsymbol{V}}_{\mathbf{long}} = 0$, which means that $\vec{\boldsymbol{V}}_{\mathbf{long}}$ is *longitudinal*, i.e. parallel to the direction of the wave.

To prove that the decomposition given in eq. (24) exists, we can directly construct $\vec{\boldsymbol{V}}_{\mathbf{tr}}$ and $\vec{\boldsymbol{V}}_{\mathbf{long}}$ with the help of the inverse Laplacian. I claim that

$$\vec{\boldsymbol{V}}_{\mathbf{long}} = \vec{\boldsymbol{\nabla}}\vec{\boldsymbol{\nabla}}^{-2}(\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{V}})\,. \tag{25}$$

First, observe that $\vec{\boldsymbol{V}}_{\mathbf{long}} = \vec{\boldsymbol{\nabla}}\psi$, where $\psi \equiv \vec{\boldsymbol{\nabla}}^{-2}(\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{V}})$. Thus $\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{V}}_{\mathbf{long}} = \vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{\nabla}}\psi = 0$, as required. Next, we use eqs. (24) and (25) to determine $\vec{\boldsymbol{V}}_{\mathbf{tr}}$,

$$\vec{\boldsymbol{V}}_{\mathbf{tr}} = \vec{\boldsymbol{V}} - \vec{\boldsymbol{V}}_{\mathbf{long}} = \vec{\boldsymbol{V}} - \vec{\boldsymbol{\nabla}}\vec{\boldsymbol{\nabla}}^{-2}(\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{V}})\,. \tag{26}$$

We now check that

$$\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{V}}_{\mathbf{tr}} = \vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\left[\vec{\boldsymbol{V}} - \vec{\boldsymbol{\nabla}}\vec{\boldsymbol{\nabla}}^{-2}(\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{V}})\right] = \vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{V}} - \vec{\boldsymbol{\nabla}}^{2}\vec{\boldsymbol{\nabla}}^{-2}(\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{V}}) = \vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{V}} - \vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{V}} = 0\,,$$

as required. This completes the proof that the Helmholtz decomposition exists.

Using the definition of the inverse Laplacian given by eq. (17), it follows from eq. (25) that

$$\vec{\boldsymbol{V}}_{\mathbf{long}}(\vec{\boldsymbol{r}}) = -\frac{1}{4\pi}\vec{\boldsymbol{\nabla}}\int\frac{\vec{\boldsymbol{\nabla}}'\boldsymbol{\cdot}\vec{\boldsymbol{V}}(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}\,d^3r'\,,$$

where the integral is taken over all space. We can also determine $\vec{\boldsymbol{V}}_{\mathbf{tr}}$ from eq. (26)

$$\vec{\boldsymbol{V}}_{\mathbf{tr}}(\vec{\boldsymbol{r}}) = \vec{\boldsymbol{V}}(\vec{\boldsymbol{r}}) - \vec{\boldsymbol{V}}_{\mathbf{long}}(\vec{\boldsymbol{r}}) = \vec{\boldsymbol{V}}(\vec{\boldsymbol{r}}) + \frac{1}{4\pi}\vec{\boldsymbol{\nabla}}\int\frac{\vec{\boldsymbol{\nabla}}'\boldsymbol{\cdot}\vec{\boldsymbol{V}}(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}\,d^3r'\,. \tag{27}$$

In the Appendix [cf. eq. (39)], we demonstrate that an equivalent expression for $\vec{\boldsymbol{V}}_{\mathbf{tr}}$ is given by:

$$\vec{\boldsymbol{V}}_{\mathbf{tr}} = -\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{\nabla}}^{-2}(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{V}})\,. \tag{28}$$

which implies that an equivalent form to eq. (27) is given by:

$$\vec{\boldsymbol{V}}_{\mathbf{tr}}(\vec{\boldsymbol{r}}) = \frac{1}{4\pi}\vec{\boldsymbol{\nabla}}\times\int\frac{\vec{\boldsymbol{\nabla}}'\times\vec{\boldsymbol{V}}(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}\,d^3r'\,. \tag{29}$$

Hence, the Helmholtz decomposition of a vector field, defined over all space, is unique and is given by

$$\vec{\boldsymbol{V}} = \vec{\boldsymbol{\nabla}}\vec{\boldsymbol{\nabla}}^{-2}(\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{V}}) - \vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{\nabla}}^{-2}(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{V}})\,,$$

or equivalently by:

$$\vec{\boldsymbol{V}}(\vec{\boldsymbol{r}}) = -\frac{1}{4\pi}\vec{\boldsymbol{\nabla}}\int\frac{\vec{\boldsymbol{\nabla}}'\boldsymbol{\cdot}\vec{\boldsymbol{V}}(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}\,d^3r' + \frac{1}{4\pi}\vec{\boldsymbol{\nabla}}\times\int\frac{\vec{\boldsymbol{\nabla}}'\times\vec{\boldsymbol{V}}(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}\,d^3r'\,, \tag{30}$$

where the integration is taken over all space. It should be emphasized that eq. (30) is an *identity* for any vector field $\vec{\boldsymbol{V}}(\vec{\boldsymbol{r}})$ which vanishes sufficiently fast at infinity.

Another version of the Helmholtz decomposition states that under the same conditions as indicated above, $\vec{\boldsymbol{V}}(\vec{\boldsymbol{r}})$ can be rewritten in the form:

$$\vec{\boldsymbol{V}}(\vec{\boldsymbol{r}}) = -\vec{\boldsymbol{\nabla}}\Phi(\vec{\boldsymbol{r}}) + \vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}})\,.$$

Using eq. (30), we can read off the functions $\Phi(\vec{\boldsymbol{r}})$ and $\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}})$,

$$\Phi(\vec{\boldsymbol{r}}) = \frac{1}{4\pi}\int \frac{\vec{\boldsymbol{\nabla}}'\boldsymbol{\cdot}\vec{\boldsymbol{V}}(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}\,d^3r'\,, \qquad \vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}) = \frac{1}{4\pi}\int \frac{\vec{\boldsymbol{\nabla}}'\times\vec{\boldsymbol{V}}(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}\,d^3r'\,.$$

For example, if $\vec{\boldsymbol{V}}(\vec{\boldsymbol{r}}) = \vec{\boldsymbol{E}}(\vec{\boldsymbol{r}})$ is the electrostatic field, then $\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{E}} = 0$ so that $\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}) = 0$. Using $\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{E}} = 4\pi\rho$, the above result for $\Phi(\vec{\boldsymbol{r}})$ reduces to eq. (18) as expected.

APPENDIX: Proof of eq. (28)

In this appendix, I provide additional details to the discussion of the Helmholtz decomposition, $\vec{\boldsymbol{V}} = \vec{\boldsymbol{V}}_{\bf tr} + \vec{\boldsymbol{V}}_{\bf long}$ [cf. eq. (24)]. In Section 5, we showed that

$$\vec{\boldsymbol{V}}_{\bf long} = \vec{\boldsymbol{\nabla}}\vec{\boldsymbol{\nabla}}^{\bf -2}(\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{V}})\,, \qquad \vec{\boldsymbol{V}}_{\bf tr} = \vec{\boldsymbol{V}} - \vec{\boldsymbol{V}}_{\bf long}\,. \tag{31}$$

Here, I will derive an alternative expression for $\vec{\boldsymbol{V}}_{\bf tr}$,

$$\vec{\boldsymbol{V}}_{\bf tr} = -\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{\nabla}}^{\bf -2}(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{V}})\,. \tag{32}$$

To derive this result, we need to establish two identities. First, for any vector function $\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}})$ that vanishes sufficiently fast at infinity,

$$\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{\nabla}}^{\bf -2}(\vec{\boldsymbol{A}}) = \vec{\boldsymbol{\nabla}}^{\bf -2}(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{A}})\,. \tag{33}$$

To prove this result, we employ the definition of the inverse Laplacian given in eq. (17), which yields[4]

$$-4\pi\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{\nabla}}^{\bf -2}(\vec{\boldsymbol{A}}) = \vec{\boldsymbol{\nabla}}\times\int\frac{\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}d^3r' = -\int\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}')\times\vec{\boldsymbol{\nabla}}\left(\frac{1}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}\right)d^3r'\,. \tag{35}$$

Noting that

$$\vec{\boldsymbol{\nabla}}\left(\frac{1}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}\right) = -\vec{\boldsymbol{\nabla}}'\left(\frac{1}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}\right)\,, \tag{36}$$

it follows that

$$\vec{\boldsymbol{\nabla}}\times\int\frac{\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}d^3r' = \int\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}')\times\vec{\boldsymbol{\nabla}}'\left(\frac{1}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}\right)d^3r'\,,$$

To evaluate this last integral, we make use of the identity

$$\vec{\boldsymbol{\nabla}}'\times\left(\frac{\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}\right) = \frac{1}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}\vec{\boldsymbol{\nabla}}'\times\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}') - \vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}')\times\vec{\boldsymbol{\nabla}}'\left(\frac{1}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}\right)\,,$$

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[4] In obtaining eq. (35), we have used

$$\vec{\boldsymbol{\nabla}}\times\left(\psi\vec{\boldsymbol{B}}\right) = \psi\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{B}} - \vec{\boldsymbol{B}}\times\vec{\boldsymbol{\nabla}}\psi\,. \tag{34}$$

In the application to eq. (35), $\psi \equiv 1/|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|$ and $\vec{\boldsymbol{B}} \equiv \vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}')$. Since the latter is independent of $\vec{\boldsymbol{r}}$, it follows that $\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{B}} = 0$.

so that

$$\int \vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}') \times \vec{\boldsymbol{\nabla}}' \left(\frac{1}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|}\right) d^3r' = \int \frac{\vec{\boldsymbol{\nabla}}' \times \vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|} d^3r' - \int \vec{\boldsymbol{\nabla}}' \times \left(\frac{\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|}\right) d^3r' \,.$$

We can convert the last integral to a surface integral evaluated at the surface at infinity (denoted by S_∞ below) using the analog of the divergence theorem for the curl:

$$\int \vec{\boldsymbol{\nabla}}' \times \left(\frac{\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|}\right) d^3r' = \oint_{S_\infty} \frac{\hat{\boldsymbol{r}}' \times \vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|} r'^2 d\Omega \,.$$

Assuming that $\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}')$ vanishes fast enough at infinity, the surface integral vanishes. In this case,

$$\int \vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}') \times \vec{\boldsymbol{\nabla}}' \left(\frac{1}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|}\right) d^3r' = \int \frac{\vec{\boldsymbol{\nabla}}' \times \vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|} d^3r' \,.$$

What we have just done here is integrated by parts and dropped the surface terms, which vanish under the stated conditions. Hence,

$$-4\pi \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{\nabla}}^{-2}(\vec{\boldsymbol{A}}) = \int \frac{\vec{\boldsymbol{\nabla}}' \times \vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|} d^3r' = -4\pi \vec{\boldsymbol{\nabla}}^{-2}(\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}) \,,$$

after again using the definition of the inverse Laplacian given in eq. (17). Thus, we have verified eq. (33) as promised.

Second, for any scalar function $\psi(\vec{\boldsymbol{r}})$ that vanishes sufficiently fast at infinity,

$$\vec{\boldsymbol{\nabla}} \vec{\boldsymbol{\nabla}}^{-2}(\psi) = \vec{\boldsymbol{\nabla}}^{-2}(\vec{\boldsymbol{\nabla}}\psi) \,. \tag{37}$$

This result can also be proven using eq. (36) and an integration by parts,[5]

$$-4\pi \vec{\boldsymbol{\nabla}} \vec{\boldsymbol{\nabla}}^{-2}(\psi) = \vec{\boldsymbol{\nabla}} \int \frac{\psi(\vec{\boldsymbol{r}})}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|} d^3r' = \int \frac{\vec{\boldsymbol{\nabla}}'\psi}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|} d^3r' = -4\pi \vec{\boldsymbol{\nabla}}^{-2}(\vec{\boldsymbol{\nabla}}\psi) \,,$$

which confirms eq. (37). For completeness, we note a third identity, not needed in the computations of this Appendix, which can be proven by a similar technique,

$$\vec{\boldsymbol{\nabla}} \boldsymbol{\cdot} \vec{\boldsymbol{\nabla}}^{-2}(\vec{\boldsymbol{A}}) = \vec{\boldsymbol{\nabla}}^{-2}(\vec{\boldsymbol{\nabla}} \boldsymbol{\cdot} \vec{\boldsymbol{A}}) \,. \tag{38}$$

With the two identities eqs. (33) and (37) in hand, we can now verify eq. (32):[6]

$$\begin{aligned} \vec{\boldsymbol{V}}_{\mathbf{tr}} &= -\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{\nabla}}^{-2}(\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{V}}) = -\vec{\boldsymbol{\nabla}}^{-2}\left[\vec{\boldsymbol{\nabla}} \times (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{V}})\right] = \vec{\boldsymbol{\nabla}}^{-2}\left[\vec{\boldsymbol{\nabla}}^2 \vec{\boldsymbol{V}} - \vec{\boldsymbol{\nabla}}(\vec{\boldsymbol{\nabla}} \boldsymbol{\cdot} \vec{\boldsymbol{V}})\right] \\ &= \vec{\boldsymbol{V}} - \vec{\boldsymbol{\nabla}}^{-2}\left[\vec{\boldsymbol{\nabla}}(\vec{\boldsymbol{\nabla}} \boldsymbol{\cdot} \vec{\boldsymbol{V}})\right] = \vec{\boldsymbol{V}} - \vec{\boldsymbol{\nabla}} \vec{\boldsymbol{\nabla}}^{-2}(\vec{\boldsymbol{\nabla}} \boldsymbol{\cdot} \vec{\boldsymbol{V}}) = \vec{\boldsymbol{V}} - \vec{\boldsymbol{V}}_{\mathbf{long}} \,, \end{aligned} \tag{39}$$

after using eq. (33) at step 2 and eq. (37) at the penultimate step. Thus, eq. (28) is confirmed.

[5] The key step here is the identity $\vec{\boldsymbol{\nabla}}(\phi\psi) = \phi \vec{\boldsymbol{\nabla}}\psi + \psi \vec{\boldsymbol{\nabla}}\phi$, where $\phi = 1/(|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|)$, and the analog of the divergence theorem for the gradient,

$$\int \vec{\boldsymbol{\nabla}} \left(\frac{\psi(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|}\right) d^3r' = \int_{S_\infty} \frac{\psi(\vec{\boldsymbol{r}}')\hat{\boldsymbol{r}}'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|} r^2 d\Omega \,,$$

which vanishes if $\psi(\vec{\boldsymbol{r}}')$ vanishes sufficiently fast at infinity.

[6] In this derivation, we have employed the vector identity, $\vec{\boldsymbol{\nabla}} \times (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{V}}) = \vec{\boldsymbol{\nabla}}(\vec{\boldsymbol{\nabla}} \boldsymbol{\cdot} \vec{\boldsymbol{V}}) - \vec{\boldsymbol{\nabla}}^2 \vec{\boldsymbol{V}}$.