1. Consider the differential equation

$$(1-x^2)y'' + p(p-1)y = 0\,, \tag{1}$$

defined on the interval $-1 \leq x \leq 1$, where p is a real number. The points $x = \pm 1$ are regular singular points of the differential equation.

(a) Find the series solution of eq. (1) for arbitrary p. Your answer should consist of a linear combination of two series with arbitrary coefficients.

Since $x = 0$ is not a singular point of eq. (1), we employ a series solution of the form,

$$y(x) = \sum_{n=0}^{\infty} a_n x^n\,.$$

Inserting this into eq. (1) yields:

$$(1-x^2)\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + p(p-1)\sum_{n=0}^{\infty} a_n x^n = 0\,, \tag{2}$$

Noting that:

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n\,,$$

we can rewrite eq. (2) as:

$$\sum_{n=0}^{\infty}\left\{(n+2)(n+1)a_{n+2} + [p(p-1) - n(n-1)]a_n\right\}x^n = 0\,.$$

Thus, we immediate derive the recursion relation,

$$(n+2)(n+1)a_{n+2} = [n(n-1) - p(p-1)]a_n\,, \qquad \text{for } n = 0, 1, 2, 3, \ldots$$

or equivalently,

$$a_{n+2} = \frac{n(n-1) - p(p-1)}{(n+2)(n+1)}\,a_n\,, \qquad \text{for } n = 0, 1, 2, 3, \ldots$$

We shall treat even n and odd n separately. For even n, one determines that

$$\begin{aligned}
a_2 &= -\frac{p(p-1)}{2\cdot 1}\,a_0\,,\\
a_4 &= \frac{2\cdot 1 - p(p-1)}{4\cdot 3}a_2 = \frac{(2-p)(-p)(-1+p)(1+p)}{4!}\,a_0\,,\\
a_6 &= \frac{4\cdot 3 - p(p-1)}{6\cdot 5}a_4 = \frac{(4-p)(2-p)(-p)(-1+p)(1+p)(3+p)}{6!}\,a_0\,,
\end{aligned}$$

and so on. Thus,

$$a_{2n} = \frac{(2n-2-p)\cdots(4-p)(2-p)(-p)(-1+p)(1+p)(3+p)\cdots(2n-3+p)}{(2n)!}\,a_0\,,$$

for $n = 1, 2, 3, \ldots$. For even n, one determines that

$$\begin{aligned}
a_3 &= -\frac{p(p-1)}{3\cdot 2}\,a_1\,, \\
a_5 &= \frac{3\cdot 2 - p(p-1)}{5\cdot 4}a_3 = \frac{(3-p)(1-p)p(2+p)}{5!}\,a_1\,, \\
a_7 &= \frac{5\cdot 4 - p(p-1)}{7\cdot 6}a_5 = \frac{(5-p)(3-p)(1-p)p(2+p)(4+p)}{6!}\,a_1\,,
\end{aligned}$$

and so on. Thus,

$$a_{2n+1} = \frac{(2n-1-p)\cdots(5-p)(3-p)(1-p)p(2+p)(4+p)\cdots(2n-2+p)}{(2n+1)!}\,a_1\,,$$

for $n = 1, 2, 3, \ldots$. Thus, we have obtained the following series solutions for eq. (1),

$$\begin{aligned}
y(x) = a_0 &\left[1 + \sum_{n=1}^{\infty}(2n-2-p)\cdots(4-p)(2-p)(-p)(-1+p)(1+p)(3+p)\cdots(2n-3+p)\frac{x^{2n}}{(2n)!}\right] \\
&+ a_1\left[x + \sum_{n=1}^{\infty}(2n-1-p)\cdots(5-p)(3-p)(1-p)p(2+p)(4+p)\cdots(2n-2+p)\frac{x^{2n+1}}{(2n+1)!}\right]\,,
\end{aligned}$$

where a_0 and a_1 are arbitrary coefficients.

For example, the first few terms of the a_0 series are:

$$1+\frac{p(1-p)x^2}{2!}+\frac{(2-p)(-p)(-1+p)(1+p)x^4}{4!}+\frac{(4-p)(2-p)(-p)(-1+p)(1+p)(3+p)x^6}{6!}+\cdots\,,$$

and the first few terms of the a_1 series are:

$$x+\frac{p(1-p)x^3}{3!}+\frac{(3-p)(1-p)p(2+p)x^5}{5!}+\frac{(5-p)(3-p)(1-p)p(2+p)(4+p)x^7}{7!}+\cdots\,.$$

Note that the solutions above are unchanged under $p \to 1-p$. This is not surprising, since the differential equation is invariant with respect to this change of parameters.

(b) If we demand that a solution obtained in part (a) converges to a finite value at $x = \pm 1$, then this solution *must* be a polynomial of finite degree. This requirement imposes a condition on p. Determined all possible values of p that satisfy this requirement.

From the explicit forms of the series solutions obtained in part (a), it is immediately obvious that these solutions are polynomial of finite order if and only if p is an integer. In light of the comment at the end of part (a), one can take p to be a positive integer without loss of generality. For $p = 1$, the a_0 series yields the solution $y(x) = 1$ and the a_1 series yields the solution $y(x) = x$. For $p = 2, 3, 4, \ldots$, there is only one series that is a polynomial of finite order, which has order p. Note that the polynomial solution of degree p is an even function of x if p is even and is an odd function of x if p is odd.

(c) Cast eq. (1) in Sturm-Liouville form and determine the weight function.

The Sturm Liouville form for the differential equation is:

$$\frac{d}{dx}\left[A(x)\frac{dy}{dx}\right] + [\lambda B(x) + C(x)]y = 0\,, \tag{3}$$

where $B(x)$ is called the *weight function*. Eq. (3) is equivalent to

$$A(x)y'' + \frac{dA}{dx}y' + [\lambda B(x) + C(x)]y = 0\,.$$

That is, the coefficient of y' must be the derivative of the coefficient of y''. Thus, to cast eq. (1) in Sturm-Liouville form, we must divide eq. (1) by $1 - x^2$, which yields:

$$y'' + \frac{p(p-1)}{1-x^2}\,y = 0\,.$$

Indeed, the coefficient of y'' is 1 and the coefficient of y' is zero, which satisfies the Sturm-Liouville form. We can then immediately identify $A(x) = 1$, $C(x) = 0$, $\lambda = p(p-1)$, and the weight function,

$$B(x) = \frac{1}{1-x^2}\,. \tag{4}$$

(d) Denote by $\{C_n(x)\}$ the set of nth-order polynomial solutions to eq. (1), where $n = 0, 1, 2, 3, \ldots$. Write down the orthogonality relations satisfied by these polynomials.

Assuming that the theorems on the Sturm-Liouville problem proved in class are applicable to this problem, the orthogonality relations satisfied by the $\{C_n(x)\}$ are:

$$\int_{-1}^{1} B(x)C_m(x)C_n(x)\,dx = \int_{-1}^{1}\frac{dx}{1-x^2}\,C_m(x)C_n(x)\,dx = 0\,, \qquad \text{for } n \neq m\,. \tag{5}$$

It is disturbing to discover that for $C_0(x) = 1$ and $C_1(x) = x$, the integral in eq. (5) does not converge! This observation signals the fact that the Strum-Liouville problem above, which has been formulated under the assumption that the series solution $y(x)$ converges for all $|x| \leq 1$, is a little too singular. In fact, the appropriate condition for this Sturm-Liouville problem is the requirement that:

$$\int_{-1}^{1} B(x)|C_n(x)|^2\,dx = \int_{-1}^{1}\frac{|C_n(x)|^2}{1-x^2}\,dx < \infty\,. \tag{6}$$

One can show that $C_{n+2}(x) = (1-x^2)p_n(x)$ for some nth order polynomial $p_n(x)$ when n is any non-negative integer (see Appendix I for further details). This implies that the integral in eq. (6) converges for all integer values of $n \geq 2$ (whereas it clearly diverges when $n = 0$ or $n = 1$). Thus, the theorems of Sturm-Liouville theory imply that the orthogonality relation given by eq. (5) is only valid for integer values of $n, m \geq 2$.

2. The associated Laguerre polynomial is denoted by $L_n^k(x)$, where n and k are non-negative integers. Evaluate this polynomial at $x = 0$. That is, determine $L_n^k(0)$ as a function of n and k.

We shall use the recursion relation given in eq. (22.28)(b) on p. 611 of Boas,

$$x\frac{d}{dx}L_n^k(x) - nL_n^k(x) + (n+k)L_{n-1}^k(x) = 0\,.$$

Setting $x = 0$, this relation simplifies to:

$$L_n^k(0) = \frac{n+k}{n}L_{n-1}^k(0)\,.$$

Iterating this equation yields,

$$\begin{aligned} L_n^k(0) &= \frac{n+k}{n}L_{n-1}^k(0) = \frac{(n+k)(n-1+k)}{n(n-1)}L_{n-2}^k(0) \\ &= \cdots = \frac{(n+k)(n-1+k)\cdots(1+k)}{n!}L_0^k(0) = \frac{(n+k)!}{n!\,k!}L_0^k(0)\,, \end{aligned} \tag{7}$$

where we have multiplied numerator and denominator by $k!$ to obtain the final result. Finally, we use eq. (22.27) on p. 611 of Boas,

$$L_n^k(x) = \frac{x^{-k}e^x}{n!}\,\frac{d^n}{dx^n}\left(x^{n+k}e^{-x}\right)\,. \tag{8}$$

Setting $n = 0$, eq. (8) reduces to $L_0^k(x) = 1$. Hence, it follows that $L_0^k(0) = 1$, which when applied to eq. (7) yields

$$L_n^k(0) = \frac{(n+k)!}{n!\,k!}\,. \tag{9}$$

Two alternative derivations are presented in Appendix II. One derivation based on the Rodrigues formula [see eq. (22.27) on p. 611 of Boas] is quite simple. Another derivation based on the generating function of the Laguerre polynomials is considerably more involved.

3. Consider the differential equation,

$$\frac{1}{\sin\theta}\,\frac{d}{d\theta}\left(\sin\theta\frac{dy}{d\theta}\right) + \ell(\ell+1)y = 0\,, \tag{10}$$

whose solution $y(\theta)$ is a function of θ.

(a) Define $x \equiv \cos\theta$. Using the chain rule, show that $y(x)$ satisfies the Legendre differential equation, $(1-x^2)y'' - 2xy' + \ell(\ell+1)y = 0$.

Using the chain rule with $x \equiv \cos\theta$,

$$\frac{d}{d\theta} = \frac{dx}{d\theta}\frac{d}{dx} = -\sin\theta\,\frac{d}{dx}\,.$$

Hence, it follows that:

$$\begin{aligned}
\frac{1}{\sin\theta}\,\frac{d}{d\theta} &= -\frac{d}{dx}\,, \\
\sin\theta\,\frac{dy}{d\theta} &= -\sin^2\theta\frac{dy}{dx} = -(1-x^2)\,\frac{dy}{dx}\,.
\end{aligned}$$

Using these results, eq. (10) takes the following form:

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + \ell(\ell+1)y = 0\,.$$

Expanding out the first term then yields the Legendre differential equation,

$$(1-x^2)y'' - 2xy' + \ell(\ell+1)y = 0\,.$$

(b) Consider the limit in which $\ell \to \infty$ and $\theta \to 0$ in such a way that the product $\ell\theta \equiv z$ is held to a fixed finite value. In this limit, we can approximate $\sin\theta \simeq \theta$ and $\ell(\ell+1) \simeq \ell^2$. Employing these approximations in eq. (10), obtain a new differential equation for $y(\theta)$. You should recognize this differential equation—then write the most general solution of this new differential equation by inspection.

Applying $\sin\theta \simeq \theta$ and $\ell(\ell+1) \simeq \ell^2$ to eq. (10) yields,

$$\frac{1}{\theta}\,\frac{d}{d\theta}\left(\theta\frac{dy}{d\theta}\right) + \ell^2 y = 0\,.$$

Expanding out the first term and multiplying through by θ^2 yields,

$$\theta^2\,\frac{d^2y}{d\theta^2} + \theta\,\frac{dy}{d\theta} + \ell^2\theta^2 y = 0\,. \tag{11}$$

Compare this with Bessel's differential equation,

$$x^2y'' + xy' + (q^2x^2 - p^2)y = 0\,, \tag{12}$$

whose solutions are $y(x) = c_1 J_p(qx) + c_2 N_p(qx)$. Hence, the general solution to eq. (11) is

$$y(\theta) = c_1 J_0(\ell\theta) + c_2 N_0(\ell\theta)\,.$$

(c) Using the result of part (b), determine

$$\lim_{\substack{\ell\to\infty \\ \theta\to 0 \\ \ell\theta = z}} P_\ell(\cos\theta)\,,$$

where $P_\ell(\cos\theta)$ is a Legendre polynomial in the variable $\cos\theta$.

The results of part (b) imply that

$$\lim_{\substack{\ell\to\infty\\ \theta\to 0\\ \ell\theta=z}} P_\ell(\cos\theta) = c_1 J_0(\ell\theta) + c_2 N_0(\ell\theta)\,.$$

To determine c_1 and c_2, consider the limit of $\theta \to 0$ with ℓ very very large but finite (in which case $\ell\theta = z \to 0$). Using the fact that $\cos\theta \to 1$ as $\theta \to 0$, it follows that $P_\ell(\cos\theta) \to P_\ell(1) = 1$. However, as long as ℓ is finite (no matter how large), $\ell\theta \to 0$ as $\theta \to 0$. Hence, it follows that:

$$1 = \lim_{\ell\theta\to 0} \left[c_1 J_0(\ell\theta) + c_2 N_0(\ell\theta)\right].$$

But $J_0(0) = 1$ whereas $N_0(x)$ diverges logarithmically as $x \to 0$. Thus, we conclude that $c_1 = 1$ and $c_2 = 0$. That is,

$$\lim_{\substack{\ell\to\infty\\ \theta\to 0\\ \ell\theta=z}} P_\ell(\cos\theta) = J_0(z)\,. \tag{13}$$

(d) [*EXTRA CREDIT*] Determine $P_\ell^m(\cos\theta)$ in the limit of $\ell \to \infty$ and $\theta \to 0$ with $z = \ell\theta$ held to a fixed finite value, where $P_\ell^m(\cos\theta)$ is an associated Legendre function, which is a bivariate polynomial in the two variables $\sin\theta$ and $\cos\theta$.

The associated Legendre function satisfies the differential equation,

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dy}{d\theta}\right) + \left[\ell(\ell+1) - \frac{m^2}{\sin^2\theta}\right] y = 0\,. \tag{14}$$

If ℓ is a non-negative integer and m is an integer such that $-\ell \leq m \leq m$, then the solution to eq. (14) is $y(\theta) = P_\ell^m(\cos\theta)$, which is a polynomial in the two variables $\sin\theta$ and $\cos\theta$. Using the results of part (a), it is easy to check that eq. (14) is equivalent to eq. (10.1) on p. 583 of Boas. Applying the same approximations as in part (b), namely $\sin\theta \simeq \theta$ and $\ell(\ell+1) \simeq \ell^2$, the above equation reduces to:

$$\theta^2\frac{d^2y}{d\theta^2} + \theta\frac{dy}{d\theta} + \left(\ell^2\theta^2 - m^2\right) y = 0\,.$$

Comparing with eq. (12), it follows that

$$y(\theta) = c_1 J_m(\ell\theta) + c_2 N_m(\ell\theta)\,.$$

Following the analysis of part (c), we must set $c_2 = 0$ since $N_m(x)$ diverges as $x \to 0$. Moreover, by comparing the small argument behavior $J_m(x)$ with the behavior of $P_\ell^m(\cos\theta)$ as $\theta \to 0$, one can show that $c_1 = \ell^m$, assuming that m is held fixed in the large ℓ limit (details of this computation are given in Appendix III). Hence,

$$\lim_{\substack{\ell\to\infty\\ \theta\to 0\\ \ell\theta=z}} P_\ell^m(\cos\theta) = \ell^m J_m(z)\,.$$

4. A string of length ℓ is fixed at $x = 0$ and $x = \ell$. At time $t = 0$, the string has an initial displacement $y(x, t = 0) = x(\ell - x)$, whereas the velocity of all points on the string at $t = 0$ is zero.

(a) Find the displacement $y(x, t)$ as a function of x and t.

In class, we showed that the general solution to the wave equation in $1 + 1$ dimensions, subject to the boundary condition that $y(0, t) = y(\ell, t) = 0$ is given by:

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right) \cos\left(\frac{n\pi vt}{\ell}\right) ,$$

after taking into account that the velocity of all points on the string at $t = 0$ is zero. We now apply the final initial condition that $y(x, t = 0) = x(\ell - x)$, which implies that:

$$x(\ell - x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right) .$$

This is a Fourier sine series. The coefficients b_n is obtained by the standard procedure,

$$b_n = \frac{2}{\ell}\int_0^{\ell} x(\ell - x) \sin\left(\frac{n\pi x}{\ell}\right) dx = \frac{4\ell^2\left[1 - (-1)^n\right]}{n^3\pi^3} = \begin{cases} 0\,, & \text{for even } n\,, \\ \dfrac{8\ell^2}{n^3\pi^3}\,, & \text{for odd } n\,. \end{cases}$$

Thus,

$$y(x,t) = \frac{8\ell^2}{\pi^3} \sum_{\text{odd } n} \frac{1}{n^3} \sin\left(\frac{n\pi x}{\ell}\right) \cos\left(\frac{n\pi vt}{\ell}\right) , \tag{15}$$

where "odd n" indicates that the sum is taken over $n = 1, 3, 5, \ldots$.

(b) Determine the fundamental frequency of the vibrations and the frequencies of the higher harmonics that appear in the motion of the string.

The frequencies f can be determined by examining the time dependence in $\cos(2\pi f t)$. The fundamental frequency corresponds to $n = 1$ and is given by $f_1 = v/(2\ell)$. Only the harmonics corresponding to odd n contribute in eq. (15), with corresponding frequencies,

$$f_n = \frac{nv}{2\ell}\,, \qquad \text{for } n = 1, 3, 5, \ldots .$$

The absence of the harmonics corresponding to even n is easy to understand. Note that the initial displacement of the string is an even function with respect to $x \to \ell - x$ (that is, its shape is symmetric about the midpoint $x = \frac{1}{2}\ell$). This is also a feature of all the harmonics corresponding to odd n. In contrast, the harmonics corresponding to $n = 2, 4, 6, \ldots$ are odd functions with respect to $x \to \ell - x$.

APPENDIX: Further commentary on Problems 1, 2 and 3

I. More details on the polynomials of problem 1

The differential equation given by eq. (1) is a special case of the Gegenbauer differential equation,

$$(1-x^2)y'' - (2\alpha+1)xy' + p(p+2\alpha)y = 0\,, \tag{16}$$

where the *index* α is a fixed real parameter. If we cast this differential equation into Sturm-Liouville form,

$$\frac{d}{dx}\left[(1-x^2)^{\alpha+\frac{1}{2}}\,\frac{dy}{dx}\right] + p(p+2\alpha)(1-x^2)^{\alpha-\frac{1}{2}}\,y = 0\,,$$

then we can read off the resulting weight function $B(x)$ [which multiplies the eigenvalue, $\lambda = n(n+2\alpha)$],

$$B(x) = (1-x^2)^{\alpha-\frac{1}{2}}\,.$$

We define the Sturm-Liouville problem by imposing the condition

$$\int_{-1}^{1} B(x)|y(x)|^2 = \int_{-1}^{1}(1-x^2)^{\alpha-\frac{1}{2}}|y(x)|^2\,dx < \infty\,. \tag{17}$$

If $\alpha > -\frac{1}{2}$, then any polynomial solution to eq. (16) (if it exists) will satisfy eq. (17). Polynomial solutions of finite order n exist if $p = n$ is a non-negative integer.* The corresponding eigenfunction of the Sturm-Liouville problem is denoted by

$$y(x) = C_n^{\alpha}(x)\,, \qquad \text{for } n = 0, 1, 2, 3, \ldots, \tag{18}$$

which are called the Gegenbauer polynomials. The set of Gegenbauer polynomials depends on the parameter index α. Indeed, if $\alpha = \frac{1}{2}$ then we recognize eq. (16) as the Legendre differential equation, so that

$$C_n^{1/2}(x) = P_n(x)\,. \tag{19}$$

For $\alpha > -1/2$, the theorems of Sturm-Liouville theory are applicable and yield the following orthogonality relations,

$$\int_{-1}^{1} C_n^{\alpha}(x)\,C_m^{\alpha}(x)\,(1-x^2)^{\alpha-\frac{1}{2}}\,dx = 0\,, \qquad \text{for } n \neq m \text{ and } n, m = 0, 1, 2, 3, \ldots \tag{20}$$

Moreover, for any choice of the index $\alpha > -\frac{1}{2}$, the set of Gegenbauer polynomials $\{C_n^{\alpha}(x)\}$, for $n = 0, 1, 2, 3, \ldots$, is a complete set of linearly independent functions on the space of functions that satisfies eq. (17).

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*When p is a non-negative integer, the second linearly independent solution to eq. (16) is non-polynomial as it corresponds to a series solution of eq. (16) that does not terminate. Moreover, this series does not converge at $x = \pm 1$, which implies that eq. (17) does not hold for this solution. When p is not a non-negative integer, then both linearly independent solutions to eq. (16) are non-terminating series, and neither solution satisfies eq. (17).

In formulating problem 1, I chose $\alpha = -1/2$ in order to eliminate the term in eq. (1) proportional to y' (thereby simplifying the analysis needed to derive the series solution). The polynomials that were denoted in problem 1 by $C_n(x)$ are proportional to the Gegenbauer polynomials $C_n^{-1/2}(x)$. However, in making this choice, eq. (20) was placed in jeopardy due to possible convergence problems in eqs. (17) and (20). Indeed, in problem 1 we found that $C_0(x) = 1$ and $C_1(x) = x$, which do not satisfy eq. (17). However, I asserted below eq. (6) that $C_{n+2}(x) = (1 - x^2)p_n(x)$ for some nth order polynomial for any non-negative integer n, and the $\{C_{n+2}(x)\}$ satisfy the orthogonality relations specified by eq. (5). To prove these assertions, we define a new variable z such that

$$y(x) = (1 - x^2)z(x)\,.$$

Inserting this change of variables into eq. (1), where $p = n$ is some non-negative integer, the following differential equation for z is obtained:

$$(1 - x^2)z'' - 4xz' + r(r + 3)z = 0\,, \qquad \text{where } r = n - 2\,.$$

Comparing with eq. (11), we conclude that

$$z = C_r^{3/2}(x)\,,$$

which is a polynomial of degree r. In terms of $y = C_n^{-1/2}(x)$ (with $n = r + 2$), we conclude that[†]

$$C_{r+2}^{-1/2}(x) = (1 - x^2)C_r^{3/2}(x)\,, \qquad \text{for } r = 0, 1, 2, 3, \ldots, \tag{21}$$

which confirms the assertion that $C_{n+2}^{-1/2}(x)$ is the product of $1 - x^2$ and a polynomial of degree n for any non-negative integer n.

The orthogonality relation satisfied by $\{C_n^{3/2}(x)\}$ is obtained from eq. (20),

$$\int_{-1}^{1} C_n^{3/2}(x)\, C_m^{3/2}(x)\, (1 - x^2)\, dx = 0\,, \qquad \text{for } n \neq m \text{ and } n, m = 0, 1, 2, 3, \ldots$$

Using eq. (21), it follows that

$$\int_{-1}^{1} C_n^{-1/2}(x) C_m^{-1/2}(x) \frac{dx}{1 - x^2} = 0\,, \qquad \text{for } n \neq m \text{ and } n, m = 2, 3, 4, 5, \ldots,$$

which verifies the claim made at the end of the solution to problem 1.

The relation given by eq. (21) is more general. Starting from eq. (16), we can change variables by defining

$$y(x) = (1 - x^2)^{\frac{1}{2} - \alpha}\, z(x)\,, \qquad p = r - 2\alpha + 1\,.$$

The resulting differential equation is given by:

$$(1 - x^2)z'' - x[2(1 - \alpha) + 1]z' + r[r + 2(1 - \alpha)]z = 0\,. \tag{22}$$

[†]In eq. (21), we have omitted an overall multiplicative constant that depends on the normalization convention used to define the Gegenbauer polynomials. This constant is not required for the present discussion.

Remarkably, eq. (22) is Gegenbauer's differential equation with index $1-\alpha$ [cf. eq. (16)]. Moreover, eq. (17) yields

$$\int_{-1}^{1} (1-x^2)^{\frac{1}{2}-\alpha} |z(x)|^2 \, dx < \infty \,,$$

which means that $z(x)$ has polynomial solutions of finite degree if r is a non-negative integer. It follow that

$$C^{\alpha}_{r+1-2\alpha}(x) = (1-x^2)^{\frac{1}{2}-\alpha} \, C^{1-\alpha}_{r}(x) \,, \qquad (23)$$

up to an overall multiplicative constant that depends on the normalization convention used in defining the relevant solution of eq. (16). Indeed, if α is a half integer $(\pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \ldots)$, then eq. (23) relates the Gegenbauer polynomials of two different indicies when both r and $r+1-2\alpha$ are non-negative integers.[‡] Inserting $\alpha = -\frac{1}{2}$ into eq. (23) recovers our previous result given in eq. (21). It is notable that for the case of $\alpha = \frac{1}{2}$, when the Gegenbauer polynomials coincide with the Legendre polynomials as indicated in eq. (19), the relation given in eq. (23) is a tautology. For other half integer choices of α, the Gegenbauer polynomials are related to derivatives of the Legendre polynomials,[§]

$$C_n^{r+\frac{1}{2}}(x) = \frac{1}{(2r-1)!!} \, \frac{d^r}{dx^r} \, P_{n+r}(x) \,, \qquad (24)$$

where n and r are non-negative integers. Combining this result with eq. (23) yields

$$C^{\frac{1}{2}-r}_{n+2r}(x) = \frac{(1-x^2)^r}{(2r-1)!!} \, \frac{d^r}{dx^r} \, P_{n+r}(x) \,. \qquad (25)$$

Eqs. (24) and (25) provide explicit representations for the Gegenbauer polynomials of half integer index. In particular, for $r=1$ eq. (25) yields

$$C^{-1/2}_{n+2}(x) = (1-x^2)\frac{d}{dx} P_{n+1}(x) \,, \qquad \text{for } n = 0, 1, 2, 3, \ldots \qquad (26)$$

Indeed, one can explicitly check that the polynomials $C_{n+2}(x)$ obtained in problem 1 are given by eq. (26), up to an overall constant of proportionality that depends on the normalization convention[¶] used in defining the Gegenbauer polynomials $C_n^{\alpha}(x)$.

[‡] For any other choice of r and/or α, eq. (23) relates solutions of eq. (16) in which at least one (and perhaps both) are non-polynomial, and hence not eigenfunctions of the Sturm-Liouville problem of interest.

[§] See e.g., E.T. Whittaker and G.N. Watson, *A Course of Modern Analysis*, 4th Edition (Cambridge University Press, Cambridge, UK, reissued in 1996), p. 329.

[¶] In the standard normalization convention, the Gegenbauer polynomials $C_n^{\alpha}(x)$ is exhibited by the following generalization of the Rodrigues formula for the Legendre polynomials:

$$C_n^{\alpha}(x) = \frac{(-1)^n}{2^n \, n!} \, \frac{\Gamma(n+2\alpha)\,\Gamma(\alpha+\frac{1}{2})}{\Gamma(2\alpha)\,\Gamma(n+\alpha+\frac{1}{2})} \, (1-x^2)^{\frac{1}{2}-\alpha} \frac{d^n}{dx^n} \, (1-x^2)^{n+\alpha-\frac{1}{2}} \,,$$

for $\alpha \neq 0$. The case of $\alpha = 0$ requires a different normalization convention.

II. Two alternative methods for solving problem 2

In this Appendix, we shall present two alternative methods for computing $L_n^k(0)$. The first alternative method starts with Rodrigues formula [cf. eq. (22.27) on p. 611 of Boas],

$$L_n^k(x) = \frac{x^{-k}e^x}{n!}\,\frac{d^n}{dx^n}(x^{n+k}e^{-x})\,.$$

Applying the Leibniz rule discussed in Chapter 12, Section 3 of Boas, it is clear that as $x \to 0$,

$$\begin{aligned}\frac{d^n}{dx^n}(x^{n+k}e^{-x}) &= e^{-x}\left[\frac{d^n}{dx^n}(x^{n+k}) + \mathcal{O}(x^{k+1})\right] \\ &= e^{-x}\left[(n+k)(n+k-1)\cdots(k+1)x^k + \mathcal{O}(x^{k+1})\right]\,,\end{aligned}$$

since when the derivative acts on e^{-x} it just produces $-e^{-x}$ and does not reduce the power of x. Multiplying this result by $x^{-k}e^x/n!$ then yields

$$L_n^k(x) = \frac{(n+k)(n+k-1)\cdots(k+1)}{n!} + \mathcal{O}(x)\,.$$

Taking the limit of $x \to 0$, we end up with

$$L_n^k(0) = \frac{(n+k)(n+k-1)\cdots(k+1)}{n!} = \frac{(n+k)(n+k-1)\cdots(k+1)k!}{n!\,k!} = \frac{(n+k)!}{n!\,k!}\,,$$

in agreement with eq. (9).

The second alternative method employs the generating function of the Laguerre polynomials [cf. eq. (22.23) on p. 610 of Boas],

$$\Phi(x,h) = \frac{e^{-xh/(1-h)}}{1-h} = \sum_{n=0}^{\infty} L_n(x)h^n\,. \tag{27}$$

The associated Laguerre polynomials are defined by eq. (22.25) on p. 610 of Boas,

$$L_n^k(x) = (-1)^k\frac{d^k}{dx^k}L_{n+k}(x)\,. \tag{28}$$

If we differentiate both sides of eq. (27) k times respect to the variable x, then we obtain

$$\frac{\partial^k}{\partial x^k}\left(\frac{e^{-xh/(1-h)}}{1-h}\right) = \frac{(-1)^k\,h^k}{(1-h)^{k+1}}\,e^{-xh/(1-h)} = \sum_{n=0}^{\infty} h^n\,\frac{d^k}{dx^k}L_n(x)\,. \tag{29}$$

Note that the above sum actually starts at $n=k$, since $L_n(x)$ is an nth order polynomial so that

$$\frac{d^k}{dx^k}L_n(x) = 0\,, \qquad \text{for } n = 0, 1, 2, \ldots, k-1\,.$$

Moving the factor of $(-1)^k h^k$ to the right hand side of eq. (29),

$$\frac{e^{-xh/(1-h)}}{(1-h)^{k+1}} = \sum_{n=k}^{\infty} h^{n-k}(-1)^k \frac{d^k}{dx^k} L_n(x)\,.$$

Redefining the summation variable, $n \to n+k$, the sum now starts from $n=0$,

$$\frac{e^{-xh/(1-h)}}{(1-h)^{k+1}} = \sum_{n=0}^{\infty} h^{n}(-1)^k \frac{d^k}{dx^k} L_{n+k}(x)\,.$$

Using eq. (28), we obtain the generating function for the associated Laguerre polynomials,

$$\frac{e^{-xh/(1-h)}}{(1-h)^{k+1}} = \sum_{n=0}^{\infty} L_n^k(x) h^n\,.$$

Finally, setting $x=0$ yields

$$\frac{1}{(1-h)^{k+1}} = \sum_{n=0}^{\infty} L_n^k(0) h^n\,. \tag{30}$$

We now expand the left hand side of eq. (30) in a Taylor series,

$$\frac{1}{(1-h)^{k+1}} = \sum_{n=0}^{\infty} \frac{h^n}{n!}\,\frac{d^n}{dh^n}\left(\frac{1}{(1-h)^{k+1}}\right)\Bigg|_{h=0}\,, \tag{31}$$

and compute

$$\frac{d^n}{dh^n}\left(\frac{1}{(1-h)^{k+1}}\right)\Bigg|_{h=0} = \frac{(k+1)(k+2)\cdots(k+n)}{(1-h)^{k+n+1}}\Bigg|_{h=0} = \frac{(k+n)!}{k!}\,.$$

Hence, eq. (31) yields

$$\frac{1}{(1-h)^{k+1}} = \sum_{n=0}^{\infty} \frac{(k+n)!}{n!\,k!} h^n\,. \tag{32}$$

Comparing the results of eqs. (30) and (32), we can immediately read off

$$L_n^k(0) = \frac{(k+n)!}{n!\,k!} h^n\,,$$

in agreement with the result obtained in eq. (9).

III. More details on the extra credit part of problem 3

In the solutions to the part (d) of problem 3, we showed that for any non-negative integer ℓ and integer m (where $-\ell \le m \le \ell$),

$$\lim_{\substack{\ell\to\infty \\ \theta\to 0 \\ \ell\theta = z}} P_\ell^m(\cos\theta) = c_1 J_m(z)\,. \tag{33}$$

The constant c_1 is determined by comparing the small argument behavior $J_m(z)$ with the behavior of $P_\ell^m(\cos\theta)$ as $\theta \to 0$, with ℓ very large and finite so that $z = \ell\theta \to 0$ as $\theta \to 0$. In this Appendix, the details of this final step of the analysis are provided.

It is convenient to first assume that $m \geq 0$. The leading behavior of $J_m(x)$ as $x \to 0$ [see p. 604 of Boas] is,

$$J_m(z) = \frac{z^m}{2^m m!} + \mathcal{O}(z^{m+2})\,. \tag{34}$$

The leading behavior of $P_\ell^m(\cos\theta)$ as $\theta \to 0$ can be obtained from a formula given in problem 12–10.9 on p. 584 of Boas,

$$P_\ell^m(x) = (-1)^m \frac{(\ell+m)!}{(\ell-m)!}\, \frac{(1-x^2)^{-m/2}}{2^\ell\, \ell!}\, \frac{d^{\ell-m}}{dx^{\ell-m}}\, (x^2-1)^\ell\,. \tag{35}$$

When $\theta \to 0$, we have $x = \cos\theta \to 1$, in which case it follows that for $0 \leq m \leq \ell$,

$$\frac{d^{\ell-m}}{dx^{\ell-m}}\,(x^2-1)^\ell = \ell(\ell-1)(\ell-2)\cdots(m+1)(2x)^{\ell-m}(x^2-1)^m + \mathcal{O}((x^2-1)^{m+1})\,,$$

after repeated differentiation. Multiplying numerator and denominator by $m!$ then yields

$$\frac{d^{\ell-m}}{dx^{\ell-m}}\,(x^2-1)^\ell = \frac{\ell!}{m!}(2x)^{\ell-m}(x^2-1)^m + \mathcal{O}((x^2-1)^{m+1})\,.$$

Inserting this result back into eq. (35) and using $(-1)^m(x^2-1)^m = (1-x^2)^m$, one obtains:

$$P_\ell^m(x) = \frac{(\ell+m)!}{(\ell-m)!}\, \frac{x^{\ell-m}(1-x^2)^{m/2}}{2^m\, m!} + \mathcal{O}((1-x^2)^{(m+2)/2})\,.$$

Noting that $x = \cos\theta \simeq 1$ and $1 - x^2 = \sin^2\theta \simeq \theta^2$ in the small angle approximation,

$$P_\ell^m(\cos\theta) = \frac{(\ell+m)!}{(\ell-m)!}\, \frac{\theta^m}{2^m m!} + \mathcal{O}(\theta^{m+2})\,, \qquad \text{for } 0 \leq m \leq \ell\,. \tag{36}$$

Using eqs. (33), (34) and (36), and setting $z = \ell\theta$, it follows that

$$\frac{(\ell+m)!}{(\ell-m)!}\, \frac{\theta^m}{2^m m!} = c_1 \frac{\ell^m \theta^m}{2^m m!}\,,$$

which yields

$$c_1 = \frac{(\ell+m)!}{\ell^m\,(\ell-m)!}\,. \tag{37}$$

Hence, it follows that for $m \geq 0$,

$$\lim_{\substack{\ell\to\infty \\ \theta\to 0 \\ \ell\theta = z}} P_\ell^m(\cos\theta) = \frac{(\ell+m)!}{\ell^m\,(\ell-m)!} J_m(z)\,. \tag{38}$$

If m is held fixed and finite as $\ell \to \infty$, then we can simplify eq. (38) by employing Stirling's approximation $n! \sim n^n e^{-n}\sqrt{2\pi n}$ [cf. eq. (11.1) on p. 552 of Boas] to obtain

$$\frac{(\ell+m)!}{(\ell-m)!} = \ell^{2m}\left[1 + \mathcal{O}\left(\frac{1}{\ell}\right)\right]\,. \tag{39}$$

Inserting this result into eq. (37) yields $c_1 \simeq \ell^m$, in which case

$$\lim_{\substack{\ell\to\infty \\ \theta\to 0 \\ \ell\theta=z}} P_\ell^m(\cos\theta) = \ell^m\, J_m(z)\,. \tag{40}$$

For negative values of m, we use the result of problem 12–10.8 on p. 584 of Boas,

$$P_\ell^{-m}(x) = (-1)^m\, \frac{(\ell-m)!}{(\ell+m)!}\, P_\ell^m(x)\,.$$

Plugging this result into eq. (38) yields (for $m \geq 0$),

$$\lim_{\substack{\ell\to\infty \\ \theta\to 0 \\ \ell\theta=z}} P_\ell^{-m}(\cos\theta) = (-1)^m \ell^{-m}\, J_m(z)\,. \tag{41}$$

Using $J_{-m}(z) = (-1)^m J_m(z)$ [cf. eq. (13.2) on p. 590 of Boas], we see that eq. (41) coincides with eq. (40) when we take $m \to -m$.[‖] That is, eq. (40) is valid for any finite integer value of m (either non-negative or negative). For $m = 0$, we recover the result of eq. (13).

References for further study:

For more details on the Gegenbauer polynomials treated in Appendix I, see e.g., Chapter 6 of Nico M. Temme, *Special Functions: An Introduction to the Classical Functions of Mathematical Physics* (John Wiley & Sons, Inc., New York, 1996) or Chapter 17 of Earl David Rainville, *Special Functions* (Chelsea Publishing Company, Bronx, NY, 1971).

The Laguerre polynomials, treated in Appendix II, play a critical role in the quantum mechanical treatment of the hydrogen atom. One of my favorite sources for information about the special functions most often used in quantum mechanics can be found in Appendix B of Albert Messiah, *Quantum Mechanics*, Volume I (John Wiley & Sons, Inc., New York, 1961). This appendix also reviews the main properties of the Hermite polynomials, Legendre polynomials, spherical Bessel functions and the spherical harmonics.

The technique employed in Appendix III (and in problem 3) is the most elementary method for obtaining eq. (40). However, there are more direct methods that use the hypergeometric function representations of the Legendre polynomial and the Bessel functions. These techniques go beyond the scope of this course. But, if you are interested in pursuing this, you can consult G.N. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd Edition (Cambridge University Press, Cambridge, UK, reissued 1995), pp. 155–156. Watson's treatise is known in the community of applied mathematicians as the Bessel function bible. Generalizations of eq. (40) can be found in an advanced textbook by Gabor Szegö, *Orthogonal Polynomials* (American Mathematical Society, Providence, RI, 1939), pp. 192–198.

[‖] In contrast to the case of positive m, eq. (40) is valid for negative m even when $|m|$ is of $\mathcal{O}(\ell)$.