INSTRUCTIONS: This is a three-hour exam. During the exam, you may refer to the textbook, the class handouts (including solution sets to homework and practice problems) or your own personal notes. Collaboration with your neighbor is strictly forbidden.

In answering the questions, it is not sufficient to simply give a final result. You must provide the intermediate steps needed to arrive at the final solution in order to get full credit. *However, if you are employing any result already obtained in class or in the textbook, you may simply cite the source. You do not need to re-derive it.*

1. Consider the differential equation

$$x^3 y'' + x(x+1)y' - y = 0\,. \tag{1}$$

(a) Explain why the method of Frobenius fails for this problem.

(b) Make an inspired guess, and assume that a solution to eq. (1) exists of the form

$$y(x) = \sum_{n=0}^{\infty} \frac{c_n}{x^n}\,.$$

Determine the coefficients c_n. Then sum the series and identify $y(x)$ as a well known function.

2. Consider the Sturm-Liouville problem:

$$x^2 y'' + xy' + \lambda y = 0\,, \tag{2}$$

subject to the boundary conditions: $y(1) = y(b) = 0$, where $b > 1$.

(a) Eq. (2) is an Euler differential equation that can be solved exactly. Find all the eigenvalues λ and the corresponding eigenfunctions $y_\lambda(x)$.

(b) Rewrite eq. (2) in Sturm-Liouville form:

$$\frac{d}{dx}\left[A(x)y'\right] + \left[\lambda B(x) + C(x)\right] y = 0\,.$$

Then, using a theorem proven in class (whose proof you need not repeat here), write down the orthogonality relation satisfied by the eigenfunctions found in part (a).

(c) Determine the normalization constant of the eigenfucntions such that the result given in part (b) is an orthonormality relation.

3. Water at 100° is flowing though a long cylindrical pipe of radius 1 rapidly enough so that we may assume that the temperature is 100° at all points. At $t = 0$, the water is turned off and the surface of the pipe is maintained at 40° from then on (neglect the wall thickness of the pipe). Find the temperature distribution in the water as a function of r and t.

4. The electric potential, $\Phi(\vec{\boldsymbol{r}})$, due to an electric charge density $\rho(\vec{\boldsymbol{r}})$ is given by

$$\Phi(\vec{\boldsymbol{r}}) = \int \frac{\rho(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|}\, dV' \,.$$

Consider the charge distribution:

$$\rho(r', \theta', \phi') = \begin{cases} \rho_0 \cos\theta', & \text{for } 0 \le r' < R\,, \\ 0, & \text{for } r' > R\,, \end{cases}$$

where θ' is measured with respect to a fixed z-axis, R is a fixed radius, and ρ_0 is a constant.

(a) Evaluate $\Phi(\vec{\boldsymbol{r}})$ assuming that $\vec{\boldsymbol{r}}$ points along the z-axis, for all $r > R$. (Note that $r \equiv |\vec{\boldsymbol{r}}|$ and $r' \equiv |\vec{\boldsymbol{r}}'|$.)

HINT: Use spherical coordinates in evaluating the integral ($dV' = r'^2 dr' d\cos\theta' d\phi'$). Expand $1/|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|$ in a series of Legendre polynomials. If you also express $\rho(\vec{\boldsymbol{r}}')$ as a Legendre polynomial, then the integration over $\cos\theta'$ is trivial!

(b) Assuming that $\varphi(\vec{\boldsymbol{r}}) \to 0$ as $r \to \infty$, write down a general solution to Laplace's equation for $\varphi(\vec{\boldsymbol{r}})$ in the region $r > R$. (In this region, there is no electric charge, so $\Phi(\vec{\boldsymbol{r}})$ satisfies $\vec{\boldsymbol{\nabla}}^{\mathbf{2}}\,\Phi = 0$.) Use spherical co-ordinates (r, θ, ϕ); your answer should have the form of an expansion in spherical harmonics (summed over ℓ and m). At this point, the expansion coefficients are undetermined.

(c) Argue from the azimuthal symmetry of the problem that $\Phi(r, \theta, \phi)$ must be independent of ϕ. Conclude that the general solution for $\Phi(r, \theta, \phi)$ has the form of an expansion over Legendre polynomials (summed over ℓ).

(d) Now for the slick part. In part (a), you computed $\Phi(\vec{\boldsymbol{r}})$ assuming that $\vec{\boldsymbol{r}}$ points along the z-axis, for all $r > R$. Using the expansion obtained in part (c), set $\theta = 0$ and determine the expansion coefficients by comparing like powers of r. Write down your final solution for $\Phi(\vec{\boldsymbol{r}})$, which is now valid for all $\vec{\boldsymbol{r}}$ such that $r > R$.

5. Suppose it is known that 1% of the population have a certain kind of cancer. It is also known that a test for this kind of cancer is positive in 99% of the people who have it but is also positive in 2% of the people who do not have it. What is the probability that a person who tests positive has a cancer of this type?

6. Let x be a continuous random variable such that x is non-negative. The probability density is given by

$$f(x) = ce^{-x/\lambda},$$

where λ is a positive constant.

(a) Determine the value of the constant c.

(b) Compute the expectation value $E(x)$.

(c) Compute the variance $\mathrm{Var}(x)$.

(d) What is the probability that x falls within three standard deviations of the mean?

(e) Evaluate the cumulative distribution function $F(x)$.