1 Boas, p. 606, problem 12.21-13

Verify that the differential equation of problem 11.13 (Homework set # 2, problem 7) is not Fuchsian. Solve it by separation of variables to find the obvious solution y =const. and a second solution in the form of an integral. Show that the second solution is not expandable in a Frobenius series.

The differential equation is

$$y'' + \frac{y'}{x^2} = 0. \tag{1}$$

A differential equation is said *Fuchsian* when, given in the form $y'' + f(x)y' + g''(x)y = 0$, $xf(x)$ and $x^2g(x)$ are not singular in the origin. This is clearly not the case in (1) because $xf(x) = \frac{1}{x}$ is singular for $x \to 0$. By separation of variables we have

$$\frac{dy'}{y'} = -\frac{dx}{x^2} \implies y' = Ce^{-\int dx/x^2} = -Ce^{-1/x} \implies y = -C\int dxe^{-1/x} + \text{const.} \tag{2}$$

We have found two solutions of the equation: one is the constant function, the other one is given by $\int dx\, \exp\left(-1/x\right)$. The latter cannot be expanded as a Frobenius series: if it could, the second solution would be

$$y_2 = x^s \sum_n a_n x^n = \int \sum_n a_n(n+s)x^{n+s-1}\,,$$

so that we could expand the integrand in the previous equation as a power series with a finite number of negative powers of x. This is not the case for $e^{-1/x} = \sum_n \frac{(-1)^n}{n!}x^{-n}$, so the solution cannot be written as a Frobenius series.

2 Boas, p. 612, problem 12.22-5

Solve the Hermite differential equation by power series

$$y'' - 2xy' + 2py = 0 \tag{3}$$

We insert the power series $y = \sum_n a_n x^n$ in to the equation

$$\sum_n n(n-1)a_n x^{n-2} - 2\sum_n na_n x^n + 2p\sum_n a_n x^n = 0 \tag{4}$$

$$\sum_n (n+2)(n+1)a_{n+2}x^n - 2\sum_n na_n x^n + 2p\sum_n a_n x^n = 0 \tag{5}$$

$$(n+2)(n+1)a_{n+2} + (-2n+2p)a_n = 0 \tag{6}$$

which gives the recursion relation $a_{n+2} = \frac{2n-2p}{(n+2)(n+1)}a_n$. We can write the solution as a sum of two series, one dependent on a_0 and one dependent on a_1:

$$y(x) = a_0\left(1 + \frac{-2p}{2}x^2 + \frac{-2p(4-2p)}{4\cdot 3\cdot 2}x^4 + \frac{-2p(4-2p)(8-2p)}{6!}x^6 + \dots\right) + \tag{7}$$

$$+a_1\left(x + \frac{(2-2p)}{3\cdot 2}x^3 + \frac{(2-2p)(6-2p)}{5!}x^5 + \frac{(2-2p)(6-2p)(10-2p)}{7!}x^7 + \dots\right). \tag{8}$$

When p is a non-negative even integer, $p = 2N$, the a_0 series terminates after $N+1$ terms and we have a polynomial of order $2N$; when p is a positive odd integer, $p = 2N+1$, the odd series terminates after $N+1$ terms and we have a polynomial of order $2N+1$.

These are the *Hermite polynomials*. It is conventional to fix a_0 and a_1 by the normalization condition in which the highest power of x of $H_n(x)$ is given by $(2x)^n$. In this convention, the first three Hermite polynomials are given by:

$$H_0(x) = 1\,, \qquad H_1(x) = 2x\,, \qquad H_2(x) = 4x^2 - 2\,. \tag{9}$$

3 Boas, p. 612, problem 12.22-8

In the generating function for the Hermite polynomials $\Phi(x,h) = e^{2xh-h^2}$, expand the exponential and obtain the firs few Hermite polynomials. Verify the identity

$$\frac{\partial^2}{\partial x^2}\Phi - 2x\frac{\partial}{\partial x}\Phi + 2h\frac{\partial}{\partial h}\Phi = 0 \tag{10}$$

and that this proves the polynomials H_n satisfy Hermite equation; then verify that the highest term in H_n is $(2x)^n$.

We have

$$\Phi(x,h) = e^{2xh-h^2} = \sum_n H_n(x)\frac{h^n}{n!} = \sum_k \frac{(2xh-h^2)^k}{k!} = \sum_k\sum_{j=0}^{k} \frac{1}{k!}\frac{k!}{j!(k-j)!}(2xh)^j(-h^2)^{k-j} \tag{11}$$

The first powers in h are:

$$\Phi(x,h) = 1 + 2xh + (4x^2-2)\frac{h^2}{2} + \ldots \tag{12}$$

which we recognize as yielding the first three Hermite polynomials [cf. (9)].

Now we verify (10):

$$\frac{\partial}{\partial x}\Phi = 2h\Phi\,, \qquad \frac{\partial}{\partial h}\Phi = (2x-2h)\Phi\,; \tag{13}$$

$$\frac{\partial^2}{\partial x^2}\Phi - 2x\frac{\partial}{\partial x}\Phi + 2h\frac{\partial}{\partial h}\Phi = 4h^2\Phi - 2x\cdot 2h\Phi + 2h(2x-2h)\Phi = 0. \tag{14}$$

From this and remembering that $\Phi = \sum_n H_n h^n/n!$ we can obtain a differential equation for $H_n(x)$:

$$\frac{\partial^2}{\partial x^2}\Phi - 2x\frac{\partial}{\partial x}\Phi + 2h\frac{\partial}{\partial h}\Phi = \sum_n H_n''(x)\frac{h^n}{n!} - 2x\sum_n H_n'(x)\frac{h^n}{n!} + 2\sum_n H_n(x)\frac{h^n}{(n-1)!} = 0 \tag{15}$$

$$\Longrightarrow\; H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0\,, \tag{16}$$

which is precisely Hermite's equation (3) for integer p.

Next we look at the highest power of x in $H_n(x)$ in the expression (11) of Φ:

$$\Phi(x,h) = \sum_n H_n(x)\frac{h^n}{n!} = \sum_k\sum_{j=0}^{k}\frac{1}{k!}\frac{k!}{j!(k-j)!}(2x)^j(-1)^{k-j}h^{2k-j} \tag{17}$$

for fixed $n = 2k-j$, that is for $j = 2k-n$, we have terms proportional to $(2x)^{2k-n}$. Because the sum over j was between 0 and k, the terms in the sum over k which contribute are those with $\frac{n}{2} < k < n$. Thus, the highest power of x is $2n-n = n$ and the highest power of $H_n(x)$ is $(2x)^n$; one also checks that the combinatorial factor in the front of H_n is $\frac{1}{j!(k-j)!} = \frac{1}{(2k-n)!(n-k)!} = \frac{1}{n!}$ so that we have factorized $\frac{h^n}{n!}$.

Finally, we have proved that Φ generates the Hermite polynomials as this is the only polynomial solution to Hermite's equation (see previous problem 2).

4 Boas, p. 612, problem 12.22-15

Solve the Laguerre differential equation by power series:

$$xy'' + (1-x)y' + py = 0 \tag{18}$$

Inserting the series $y = \sum_n a_n x^n$, we have

$$\sum_n n(n-1)a_n x^{n-1} + (1-x)\sum_n n a_n x^{n-1} + p\sum_n a_n x^n = 0 \tag{19}$$

$$\sum_n \Big[(n+1)na_{n+1} + (n+1)a_{n+1} - na_n + pa_n\Big]x^n = 0 \tag{20}$$

$$(n+1)^2 a_{n+1} = (n-p)a_n \qquad \Longrightarrow \qquad a_{n+1} = \frac{(n-p)}{(n+1)^2}a_n \tag{21}$$

if p is an integer, the coefficient a_{p+1} is zero and the series terminate, that is, the solution is a polynomial. Setting the normalization to $a_0 = 1$, these are called *Laguerre polynomials* $L_n(x)$:

$$L_n(x) = 1 - nx + \frac{-n(1-n)}{(2!)^2}x^2 + \frac{-n(1-n)(2-n)}{(3!)^2}x^3 + \ldots + \frac{-n(1-n)(2-n)\cdots(-1)}{(n!)^2}x^n \tag{22}$$

From this we can read the first few Laguerre polynomials:

$$L_0(x) = 1\,, \qquad L_1(x) = 1-x\,, \qquad L_2(x) = 1 - 2x + \frac{x^2}{2}\,, \qquad L_3(x) = 1 - 3x + \frac{3}{2}x^2 - \frac{1}{6}x^3\,. \tag{23}$$

5 Boas, p. 614, problem 12.22-26

Given the differential equation

$$y'' + \left(\frac{\lambda}{x} - \frac{1}{4} - \frac{l(l+1)}{x^2}\right)y = 0 \tag{24}$$

where $l > 0$ is an integer, find values of λ such that $y \to 0$ for $x \to \infty$ and find the corresponding eigenfunctions.

We write

$$y(x) = x^{l+1}e^{-x/2}v(x)\,, \tag{25}$$

and find a related differential equation for v:

$$y' = (l+1)x^l e^{-x/2}v - \frac{1}{2}x^{l+1}e^{-x/2}v + x^{l+1}e^{-x/2}v'; \tag{26}$$

$$y'' = l(l+1)x^{l-1}e^{-x/2}v - \frac{l+1}{2}x^l e^{-x/2}v + (l+1)x^l e^{-x/2}v' - \tag{27}$$

$$-\frac{l+1}{2}x^l e^{-x/2}v + \frac{1}{4}x^{l+1}e^{-x/2}v - \frac{1}{2}x^{l+1}e^{-x/2}v' + \tag{28}$$

$$+(l+1)x^l e^{-x/2}v' - \frac{1}{2}x^{l+1}e^{-x/2}v' + x^{l+1}e^{-x/2}v''; \tag{29}$$

$$\Longrightarrow -(l+1)x^l v + [2(l+1)x^l - x^{l+1}]v' + x^{l+1}v'' + \lambda x^l v = 0, \tag{30}$$

$$\Longrightarrow xv'' + (2l+2-x)v' + (\lambda - l - 1)v = 0 \tag{31}$$

This has the same form of the equation solved by the associated Laguerre polynomials:

$$xy'' + (k+1-x)y' + ny = 0\,, \qquad y = L_n^k(x) \tag{32}$$

Then, for an integer $\lambda > l$, we there is a polynomial solution of the form $v(x) = L^{2l+1}_{\lambda-l-1}(x)$. The solution to the original equation (24) is

$$y(x) = x^{l+1}e^{-x/2}L^{2l+1}_{\lambda-l-1}(x) \tag{33}$$

We can note that we just solved an eigenvalue problem: we found that for specific values of λ, the equation (24) admit solutions related to the associated Laguerre polynomials.

Motivation for the change of variables

To understand the motivation for (25), let us deduce the *behavior* of the solution $y(x)$ to the differential equation (24) in the limit where $x \to 0$ and $x \to \infty$ respectively. First, as $x \to 0$, the term $l(l+1)/x^2$ is much larger than λ/x and $\frac{1}{4}$. Hence, the latter two terms can be neglected, and we examine

$$y'' - \frac{l(l+1)y}{x^2} = 0\,.$$

Multiplying by x^2, we see that this differential equation has the form of an Euler differential equation [cf. Case (d) on p. 434 of Boas]. The solution to this equation is a power law, $y = x^p$. Plugging this into the above equation yields $p(p-1) = l(l+1)$, which has two solutions $p = l+1$ and $p = -l$. We reject $p = -l$ which is negative for positive integer l, as this would correspond to a solution $y(x)$ that is singular (i.e., unbounded) at $x = 0$. Hence, the non-singular behavior of $y(x)$ as $x \to 0$ is $y(x) \sim x^{l+1}$.

As $x \to \infty$, we can neglect the λ/x and $l(l+1)/x^2$ as compared to $\frac{1}{4}$ in (24). Hence, we examine

$$y'' - \tfrac{1}{4}y = 0\,.$$

The solution to this equation is a linear combination of $e^{-x/2}$ and $e^{x/2}$. We reject the latter as it corresponds to a solution $y(x)$ that is singular (i.e., unbounded) as $x \to \infty$. Hence, the non-singular behavior of $y(x)$ as $x \to \infty$ is $y(x) \sim e^{-x/2}$. Combining these two results, it is especially useful to define

$$y(x) = x^{l+1}e^{-x/2}v(x)\,,$$

which embodies both the small x and large x behavior of $y(x)$, assuming that $v(x)$ is well-behaved in these limits. This is precisely the change of variables proposed in (25).

6 Boas, p. 614, problem 12.22-27

In the theory of the hydrogen atom the functions of interest are

$$f_n(x) = x^{l+1}e^{-x/2n}L^{2l+1}_{n-l-1}\left(\frac{x}{n}\right) \tag{34}$$

where n is an integer and so is l, $0 \le l \le n-1$. For $l = 1$, show that

$$f_2(x) = x^2e^{-x/4}\,, \qquad f_3(x) = x^2e^{-x/6}\left(4 - \frac{x}{3}\right)\,, \qquad f_4(x) = x^2e^{-x/8}\left(10 - \frac{5x}{4} + \frac{x^2}{32}\right). \tag{35}$$

We will find the associated Laguerre polynomials starting from the Laguerre polynomials and using

$$L_n^k(x) = (-1)^k\frac{d^k}{dx^k}L_{n+k}(x). \tag{36}$$

We need to find $L_0^3,\ L_1^3,\ L_2^3$; in addition to the first polynomials in (23), we need the other L_n's up to $n = 5$. We make use of the definition (22):

$$L_4(x) = 1 - 4x + 3x^2 - \frac{2}{3}x^3 + \frac{1}{24}x^4 , \qquad L_5(x) = 1 = 5x + 5x^2 - \frac{5}{3}x^3 + \frac{5}{24}x^4 - \frac{1}{120}x^5 \tag{37}$$

$$L_0^3 = -\frac{d^3}{dx^3}L_3(x) = 1 , \qquad L_1^3 = -\frac{d^3}{dx^3}L_4(x) = 4 - x , \qquad L_2^3 = -\frac{d^3}{dx^3}L_5(x) = 10 - 5x + \frac{x^2}{2} \tag{38}$$

Replacing $x \to \frac{x}{n}$ we have

$$f_2(x) = x^2 e^{-x/4} L_0^3\left(\frac{x}{2}\right) = x^2 e^{-x/4} , \tag{39}$$

$$f_3(x) = x^2 e^{-x/6} L_1^3\left(\frac{x}{3}\right) = x^2 e^{-x/6}\left(4 - \frac{x}{3}\right) , \tag{40}$$

$$f_4(x) = x^2 e^{-x/8} L_2^3\left(\frac{x}{4}\right) = x^2 e^{-x/8}\left(10 - \frac{5x}{4} + \frac{x^2}{32}\right) . \tag{41}$$

which is precisely (35).

For fixed l, the functions $f_n(x)$ are an orthogonal set on $(0, \infty)$ (as a consequence of Sturm-Liouville theory). We can verify this with these three functions:

$$\int_0^\infty dx f_2(x) f_3(x) = \int_0^\infty dx x^4 e^{-5x/12}\left(4 - \frac{x}{3}\right) = 4\int_0^\infty dx x^4 e^{-5x/12} - \frac{1}{3}\int_0^\infty dx x^5 e^{-5x/12} = \tag{42}$$

$$= 4\left(\frac{12}{5}\right)^5 \int_0^\infty dy y^4 e^{-y} - \frac{1}{3}\left(\frac{12}{5}\right)^6 \int_0^\infty dy y^5 e^{-y} = \left(\frac{12}{5}\right)^5 \left[4\Gamma(5) - \frac{1}{3}\frac{12}{5}\Gamma(6)\right] = 0$$

$$\int_0^\infty dx f_2(x) f_4(x) = \int_0^\infty dx x^4 e^{-3x/8}\left(10 - \frac{5x}{4} + \frac{x^2}{32}\right) = 10\left(\frac{8}{3}\right)^5 \Gamma(5) - \frac{5}{4}\left(\frac{8}{3}\right)^6 \Gamma(6) + \frac{1}{32}\left(\frac{8}{3}\right)^7 \Gamma(7) =$$

$$= \left(\frac{8}{3}\right)^5 \Gamma(5)\left[10 - \frac{5}{4}\frac{8}{3}5 + \frac{2}{9}6\cdot 5\right] = 0 \tag{43}$$

$$\int_0^\infty dx f_3(x) f_4(x) = \int_0^\infty dx\, x^4 e^{-7x/24}\left(40 - \frac{25}{3}x + \frac{13}{24}x^2 - \frac{1}{96}x^3\right) = \tag{44}$$

$$= \left(\frac{24}{7}\right)^5 \Gamma(5)\left[40 - \frac{25}{3}\left(\frac{24}{7}\right)5 + \frac{13}{24}\left(\frac{24}{7}\right)^2 6\cdot 5 - \frac{1}{96}\left(\frac{24}{7}\right)^3 7\cdot 6\cdot 5\right] = 0 \tag{45}$$

where the Γ function is defined as $\Gamma(z) = \int_0^\infty dt\, e^{-t} t^{z-1}$ and $\Gamma(z+1) = z\Gamma(z)$.

7 Boas, p. 618, problem 12.23-27

Show that $R = lx - (1 - x^2)D$ and $L = lx + (1 - x^2)D$ where $D = \frac{d}{dx}$ are raising and lowering operators for the Legendre polynomials. More precisely, show that $RP_{l-1} = lP_l$ and $LP_l = lP_{l-1}$:

This is immediate once we recall the recursion relations of the Legendre polynomials:

$$RP_{l-1} = \left[lx - (1 - x^2)\frac{d}{dx}\right]P_{l-1} = lxP_{l-1} - (1 - x^2)P'_{l-1} = lxP_{l-1} - lxP_{l-1} + lP_l = lP_l \tag{46}$$

$$LP_l = \left[lx + (1 - x^2)\frac{d}{dx}\right]P_l = lxP_l + (1 - x^2)P'_l = lxP_l lP_{l-1} - lxP_l = lP_{l-1} \tag{47}$$

Assuming $P_l(1) = 1$, we can find P_0 as the polynomial annihilated by L and then find the other Legendre polynomials using the raising operators:

$$LP_0(x) = 0 \iff (1-x^2)P_0' = 0 \implies P_0 = \text{const} = 1, \tag{48}$$

$$P_1(x) = RP_0 = x - (1-x^2)\frac{d}{dx}1 = x\,, \qquad P_2(x) = \frac{1}{2}RP_1 = \frac{1}{2}(2x^2 - (1-x^2)) = \frac{1}{2}(3x^2-1). \tag{49}$$

Note that the choice of constant such that $P_0(x) = 1$ is a convention. Once this convention has been chosen, the normalization of the other Legendre polynomials is fixed and determined by applying the raising operators.

8 Boas, p. 620-621, problem 13.1-2

(a) Show that the expression $u(x,t) = \sin(x - vt)$ satisfies the wave equation. Show that, in general, $u = f(x - vt)$ and $u = f(x + vt)$ satisfy the wave equation.

The wave equation is

$$\nabla^2 u - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}u = 0 \tag{50}$$

For a one-dimensional problem, $\nabla = \frac{\partial}{\partial x}$ and this admits the solution $u(x,t) = \sin(x - vt)$:

$$\frac{\partial}{\partial x}u = \cos(x-vt)\,, \qquad \frac{\partial^2}{\partial x^2}u = -u\,, \qquad \frac{\partial^2}{\partial t^2} = -v^2 u \qquad \implies \nabla^2 u - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}u = 0 \tag{51}$$

More generally, one can see that any function (that has a second derivative) $u(x,t) = f(x \pm vt)$ satisfies (50):

$$\xi_\pm = x \pm vt\,, \qquad \frac{\partial}{\partial x} = \frac{\partial \xi_\pm}{\partial x}\frac{\partial}{\partial \xi_\pm} = \frac{\partial}{\partial \xi_\pm}\,, \qquad \frac{\partial}{\partial t} = \frac{\partial \xi_\pm}{\partial t}\frac{\partial}{\partial \xi_\pm} = \pm v\frac{\partial}{\partial \xi_\pm} \tag{52}$$

$$\nabla^2 u - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}u = f''(\xi_\pm) - \frac{1}{v^2}(\pm v)^2 f''(\xi_\pm) = 0 \tag{53}$$

where the equation holds separately for ξ_+ and ξ_-. $f(x - vt)$ represents an excitation moving in the positive x direction and $f(x + vt)$ an excitation moving in the opposite direction.

(b) Show that $u(r,t) = \frac{1}{r}f(r - vt)$ and $u(r,t) = \frac{1}{r}f(r + vt)$ satisfy the wave equation in spherical coordinates.

The Laplacian operator in spherical coordinates is

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial}{\partial\varphi}\right) + \frac{1}{r^2\sin^2\varphi}\frac{\partial^2}{\partial\theta^2}. \tag{54}$$

If we are looking for solutions independent of ϕ, θ, only the first term contributes; the wave equation becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)u(r,t) - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}u(r,t) = 0 \tag{55}$$

As before, if we insert the coordinates $\xi_\pm = r \pm vt$ we see that $u(r,t) = \frac{1}{r}f(\xi_\pm)$ is a solution for any f:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\frac{1}{r}f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(-f + rf'\right) = -\frac{1}{r^2}f' + \frac{1}{r^2}f' + \frac{1}{r}f'', \tag{56}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)u(r,t) - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}u(r,t) = \frac{1}{r}f''(\xi_\pm) - \frac{1}{v^2}(\pm v)^2\frac{1}{r}f'' = 0. \tag{57}$$

These functions represent spherical waves radially coming out of (or into) the origin.

9 Boas, p. 626, problem 13.2-4

Solve the semi-infinite plate problem if the bottom edge of width 30 is held at

$$T = \begin{cases} x, & 0 < x < 15, \\ 30 - x, & 15 < x < 30, \end{cases} \tag{58}$$

and the other sides are at 0°C.

The temperature T inside the plate satisfies Laplace's equation with the boundary conditions given by (58). Solving by separation of variables, we have

$$\nabla^2 T(x,y) = 0\,, \qquad T(x,y) = X(x)Y(y) = \left\{ \begin{array}{c} e^{ky} \\ e^{-ky} \end{array} \right\} \left\{ \begin{array}{c} \sin kx \\ \cos kx \end{array} \right\} \tag{59}$$

Note that the boundary heat distribution (58) does not influence the form of the solution inside the plate; it will only select a different solution of that form.

We now apply the boundary condition:

- since $T \to 0$ as $y \to \infty$, no solution of the form e^{ky} is acceptable;
- similarly, as $T(0,y) = 0$, no solution with $\cos kx$ can exist;
- $T(30,y) = 0$ so that $\sin 30k = 0$, or $k = \frac{n\pi}{30}$; The solution must have the form

$$T(x,y) = \sum_n c_n e^{-n\pi y/30} \sin \frac{n\pi x}{30}; \tag{60}$$

- Finally, we must reproduce the temperature distribution (58) for $y = 0$:

$$T(x,0) = \sum_n c_n \sin \frac{n\pi x}{30} = \begin{cases} x, & 0 < x < 15, \\ 30 - x, & 15 < x < 30 \end{cases}; \tag{61}$$

we can find the coefficient c_n's because this is a Fourier series:

$$c_n = \frac{2}{30}\int_0^{30} T(x,0)\sin\frac{n\pi x}{30} = \frac{2}{30}\int_0^{15} x \sin\frac{n\pi x}{30}dx + \frac{2}{30}\int_{15}^{30}(30-x)\sin\frac{n\pi x}{30}dx = \tag{62}$$

$$= 60\int_0^{1/2} ydy \sin n\pi y + 60\int_{1/2}^{1}(1-y)dy\sin n\pi y = \tag{63}$$

$$= 60\left[-\frac{1}{n\pi}y\cos n\pi y\Big|_0^{1/2} + \frac{1}{n\pi}\int_0^{1/2}\cos n\pi y dy - \frac{1}{n\pi}(1-y)\cos n\pi y\Big|_{1/2}^{1} - \frac{1}{n\pi}\int_{1/2}^{1}\cos n\pi y dy\right] \tag{64}$$

$$= \frac{60}{n^2\pi^2}2\sin\frac{n\pi}{2} \tag{65}$$

For even $n = 2k$ that tells us $c_2 k = 0$, while for odd $n = 2k+1$ we have $c_n = \frac{120}{n^2\pi^2}(-1)^k$.

We can now write the solution to Laplace's equation with the boundary condition (58):

$$T(x,y) = \sum_k \frac{120(-1)^k}{(2k+1)^2\pi^2} e^{-\frac{(2k+1)\pi y}{30}} \sin\frac{(2k+1)\pi x}{30} \tag{66}$$

The plots for the temperature distribution are in figure 1.

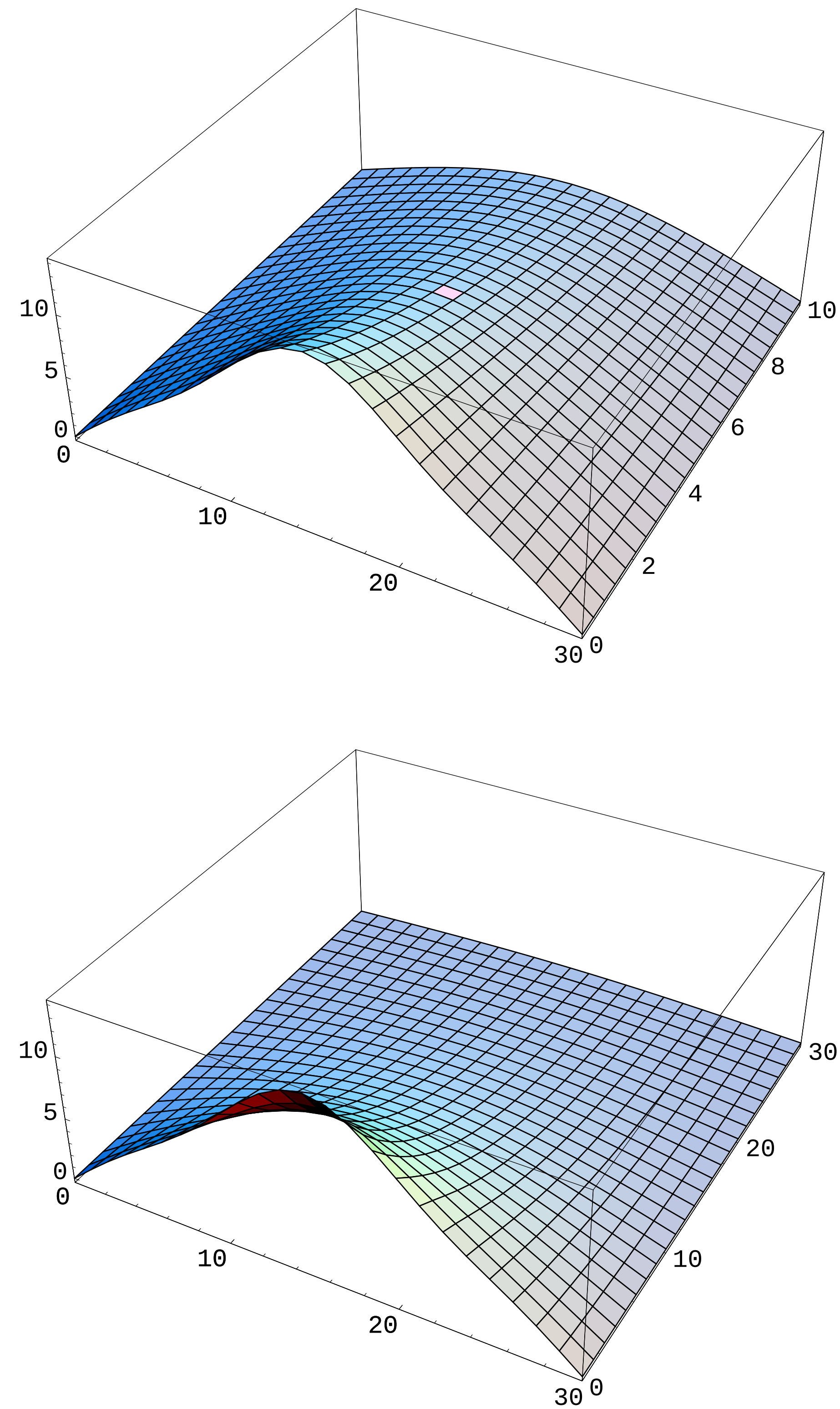

Figure 1: Temperature distribution (66), with the sum truncated at $k = 2$

10 Boas, p. 627, problem 13.2-6

Show that the series

$$T = \frac{400}{\pi} \sum_k \frac{1}{2k+1} e^{-\frac{(2k+1)\pi y}{10}} \sin \frac{(2k+1)\pi x}{10} \tag{67}$$

can be summed to get

$$T = \frac{200}{\pi} \arctan\left(\frac{\sin(\pi x/10)}{\sinh(\pi y/10)}\right) . \tag{68}$$

We have $\sin x = \frac{1}{2i}(e^{ix} - e^{-ix}) = \operatorname{Im} e^{ix}$, then

$$T = \frac{400}{\pi} \sum_{n \text{ odd}} \frac{1}{n} e^{-n\pi y/10} \operatorname{Im} e^{in\pi x/10} = \frac{400}{\pi} \operatorname{Im} \sum_{n \text{ odd}} \frac{1}{n} e^{in\pi(x+iy)/10} = \frac{400}{\pi} \operatorname{Im} \sum_{n \text{ odd}} \frac{z^n}{n} , \tag{69}$$

where $z \equiv e^{i\pi(x+iy)/10}$ and "n odd" means that we sum over $n = 1, 3, 5, \ldots$. To evaluate this sum, recall that

$$\operatorname{Ln}(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n , \qquad \text{for } |z| \leq 1,\ z \neq -1 , \tag{70}$$

where z is a complex number and Ln is the principal value of the complex logarithm.[1] We would like to write:

$$\begin{aligned}\operatorname{Ln}\left(\frac{1+z}{1-z}\right) &= \operatorname{Ln}(1+z) - \operatorname{Ln}(1-z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (-z)^n \\ &= 2 \sum_{n \text{ odd}} \frac{z^n}{n} , \qquad \text{for } |z| \leq 1,\ z \neq \pm 1 . \end{aligned} \tag{71}$$

However, recall that $\operatorname{Ln}(z_1/z_2) = \operatorname{Ln} z_1 - \operatorname{Ln} z_1$ is valid only when $-\pi < \operatorname{Arg} z_1 - \operatorname{Arg} z_2 \leq \pi$, where $\operatorname{Arg} z$ is the principal value of the argument of the complex number z (as discussed at great length in the Physics 116A handout entitled, *The complex logarithm, exponential and power functions*). Nevertheless, it is straightforward to check that for $z_1 = 1 + z$ and $z_2 = 1 - z$, this condition is satisfied when $|z| \leq 1$ and $z \neq \pm 1$. Hence, it follows that:

$$T = \frac{200}{\pi} \operatorname{Im} \operatorname{Ln}\left(\frac{1 + e^{i\pi(x+iy)/10}}{1 - e^{i\pi(x+iy)/10}}\right) = \frac{200}{\pi} \operatorname{Arg}\left(\frac{1 + e^{i\pi(x+iy)/10}}{1 - e^{i\pi(x+iy)/10}}\right) ,$$

where we have used $\operatorname{Ln} z = \operatorname{Ln} |z| + i\operatorname{Arg} z$ for the principal value of the complex logarithm.

To evaluate the argument of the expression above, it is convenient to rewrite the complex number in $a + ib$ form,

$$\frac{1 + e^{i\pi(x+iy)/10}}{1 - e^{i\pi(x+iy)/10}} = \frac{(1 + e^{i\pi(x+iy)/10})(1 - e^{-i\pi(x-iy)/10})}{(1 - e^{i\pi(x+iy)/10})(1 - e^{-i\pi(x-iy)/10})} = \frac{1 - e^{-\pi y/5} + 2ie^{-\pi y/10} \sin(\pi x/10)}{1 + e^{-\pi y/5} - 2e^{-\pi y/10} \cos(\pi x/10)} .$$

In the Physics 116A handout entitled, *The argument of a complex number*, I show that if $a > 0$ then $\operatorname{Arg}(a + ib) = \operatorname{Arctan}(b/a)$, where Arctan is the principal value of the arctangent function. In the present application, we have:

$$a = \frac{1 - e^{-\pi y/5}}{1 + e^{-\pi y/5} - 2e^{-\pi y/10} \cos(\pi x/10)} , \qquad b = \frac{2e^{-\pi y/10} \sin(\pi x/10)}{1 + e^{-\pi y/5} - 2e^{-\pi y/10} \cos(\pi x/10)} . \tag{72}$$

[1] Note that since the right hand side of (70) is a single-valued function, the left hand side must be single-valued as well. Choosing $z = 0$ yields $\operatorname{Ln} 1 = 0$ as expected for the principal value.

Since $y \geq 0$, we shall treat $y = 0$ and $y > 0$ separately. When $y > 0$, it follows that $a > 0$, since the numerator of a is positive and the denominator of a is

$$1 + e^{-\pi y/5} - 2e^{-\pi y/10}\cos(\pi x/10) \geq (1 - e^{-\pi y/10})^2 \geq 0\,,$$

after noting that $|\cos(\pi x/10)| \leq 1$. Hence,

$$\text{Arg}\left(\frac{1 + e^{i\pi(x+iy)/10}}{1 - e^{i\pi(x+iy)/10}}\right) = \text{Arctan}\left(\frac{b}{a}\right) = \text{Arctan}\left(\frac{2e^{-\pi y/10}\sin(\pi x/10)}{1 - e^{-\pi y/5}}\right) = \text{Arctan}\left(\frac{\sin(\pi x/10)}{\sinh(\pi y/10)}\right)\,,$$

where in the last step, we used the fact that:

$$\frac{2e^{-\pi y/10}}{1 - e^{-\pi y/5}} = \frac{2}{e^{\pi y/10}(1 - e^{-\pi y/5})} = \frac{2}{e^{\pi y/10} - e^{-\pi y/10}} = \frac{1}{\sinh(\pi y/10)}\,.$$

Hence, we conclude that:

$$T = \frac{200}{\pi}\text{Arctan}\left(\frac{\sin(\pi x/10)}{\sinh(\pi y/10)}\right)\,. \tag{73}$$

In the case of $y = 0$ and $0 < x < 10$, we have[2]

$$a = 0\,, \qquad b = \frac{\sin(\pi x/10)}{1 - \cos(\pi x/10)} = \cot(\pi x/20) > 0\,.$$

If $a = 0$ and $b > 0$, then it follows that $\text{Arg}(a + bi) = \frac{1}{2}\pi$. Hence, in (73), if $y = 0$ and $0 < x < 10$, the arctangent is equal to $\frac{1}{2}\pi$ and we find $T = 100$, which is the boundary condition for the bottom of the rectangular plate. Finally, we can use (73) to calculate $T(5, 5) = 26.096^{\circ}$C.

11 Boas, p. 627, problem 13.2-13

Find the steady state temperature distribution in a rectangular plate covering the area $0 < x < 10$, $0 < y < 20$ if the two adjacent sides along the axes are held at temperatures $T = x$ and $T = y$ and the other two sides at 0°C.

The solution to Laplace's equation is always the same, but this time we have different boundary conditions.

$$\nabla^2 T(x, y) = 0\,, \qquad T(x, y) = X(x)Y(y) = \begin{Bmatrix} \sinh ky \\ \cosh ky \end{Bmatrix} \begin{Bmatrix} \sin kx \\ \cos kx \end{Bmatrix}\,, \tag{74}$$

$$T(x, 0) = x\,, \qquad T(0, y) = y\,, \qquad T(x, 20) = T(10, y) = 0. \tag{75}$$

Here we have substituted the exponentials in y with the hyperbolic sine and cosine. We can do it because these are linear combinations of the exponentials and still solutions to Laplace's equation.

Now, because Laplace's equation is a linear differential equation, the sum of two solutions is still a solution; then we will find a solution T_1 satisfying the boundary condition $T_1(x, 0) = x$, $T_1(0, y) = 0$ and one T_2 satisfying $T_2(10, y) = 0$, $T_2(0, y) = y$ and add them together; the sum will satisfy Laplace's equation and the boundary conditions (75).

We first look at the boundary condition $T_1(x, 20) = 0$, $T_1(0, y) = 0, T_1(10, y) = 0$:

$$T(0, y) = T(10, y) = 0 \implies X = \sin\frac{n\pi x}{10}\,, \qquad T(x, 20) = 0 \implies Y = \sinh k(20 - y) \tag{76}$$

[2]We do not consider the points $x = y = 0$ or $x = 10$, $y = 0$ since the temperature is not well defined at these two points on the boundary of the rectangular plate.

so that the solution takes the form

$$T_1 = \sum_n A_n \sinh \frac{n\pi}{10}(20-y) \sin \frac{n\pi x}{10}. \tag{77}$$

By applying the last condition $T_1(x,0) = x$ we find the coefficients A_n:

$$A_n \sinh 2n\pi = \frac{2}{10}\int_0^{10} dx\, x \sin \frac{n\pi x}{10} = \frac{2}{10}\left[-\frac{10}{n\pi} x \cos \frac{n\pi x}{10}\bigg|_0^{10} + \frac{10}{n\pi}\int_0^{10} \cos \frac{n\pi x}{10} dx \right] = \tag{78}$$

$$= \frac{2}{10}\left[-\frac{10}{n\pi} 10(-1)^n \right] = \frac{20}{n\pi}(-1)^{n+1} \tag{79}$$

To find the other solution we solve the same equation with boundary conditions $T_2(10,y) = 0$, $T_2(0,y) = y$, $T_2(x,20) = T_2(x,0) = 0$; this goes in the same way we just did so we can give the answer exchanging the roles of x and y (and being careful about the different sides' lengths):

$$T_2 = \sum_n B_n \sinh \frac{n\pi}{20}(10-x) \sin \frac{n\pi y}{20}. \tag{80}$$

$$B_n \sinh \frac{n\pi}{2} = \frac{2}{20}\left[-\frac{20}{n\pi} 20(-1)^n \right] = \frac{40}{n\pi}(-1)^{n+1} \tag{81}$$

The solution satisfying the original boundary conditions (75) is then

$$T(x,y) = \frac{20}{\pi}\sum_n \frac{(-1)^{n+1}}{n \sinh 2n\pi} \sinh \frac{n\pi}{10}(20-y) \sin \frac{n\pi x}{10} + \tag{82}$$

$$+ \frac{40}{\pi}\sum_n \frac{(-1)^{n+1}}{n \sinh n\pi/2} \sinh \frac{n\pi}{20}(10-x) \sin \frac{n\pi y}{20} \tag{83}$$

12 Boas, p. 627-628, problem 13.2-14

The heat flow across an edge is proportional to the derivative along the direction normal to that edge, $\partial T/\partial n$. For a plate with an insulated edge, the boundary condition is that the heat flow along that edge is zero. Find the steady-state temperature of a semi-infinite plate of width 10cm where the two long edges are insulated, the far end is at 0°C and the bottom edge is at $T(x,0) = x - 5$.

We solve Laplace's equation with this boundary condition:

$$\nabla^2 T(x,y) = 0\,, \qquad T(x,y) = X(x)Y(y) = \begin{Bmatrix} e^{ky} \\ e^{-ky} \end{Bmatrix} \begin{Bmatrix} \sin kx \\ \cos kx \end{Bmatrix}, \tag{84}$$

$$T(x,0) = x - 5\,, \qquad \lim_{y\to\infty} T(x,y) = 0\,, \qquad \frac{\partial T}{\partial x}(0,y) = \frac{\partial T}{\partial x}(10,y) = 0\,; \tag{85}$$

Because of the second condition, we eliminate the solution e^{ky}. The others give:

$$\frac{\partial T}{\partial x}(0,y) \propto \begin{Bmatrix} \cos kx \\ \sin kx \end{Bmatrix} = 0 \qquad \Longrightarrow \quad \text{no } \sin kx \text{ terms in } T \tag{86}$$

$$\frac{\partial T}{\partial x}(10,y) = 0 \implies \sin k10 = 0 \implies k = \frac{n\pi}{10} \tag{87}$$

So the solution takes the form

$$T(x,y) = \sum_{n=1} b_n e^{-n\pi y/10} \cos \frac{n\pi x}{10} \tag{88}$$

The coefficients are given by

$$\begin{aligned} b_n &= \frac{2}{10}\int_0^{10} dx\,(x-5)\cos\frac{n\pi x}{10} = \frac{2}{10}\frac{10^2}{\pi^2}\int_0^{\pi}(x-\frac{\pi}{2})\cos nx dx = \frac{20}{\pi^2}\left[\frac{1}{n}(x-\frac{\pi}{2})\sin nx\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin nx\right] \\ &= \frac{20}{n^2\pi^2}((-1)^n-1) = -\frac{40}{n^2\pi^2} \quad \text{for } n \text{ odd} \end{aligned} \tag{89}$$

Finally, the solution is

$$T = -\frac{40}{\pi^2}\sum_{odd\ n}\frac{1}{n^2}e^{-n\pi y/10}\cos\frac{n\pi x}{10}. \tag{90}$$

We eliminated the solution e^{ky} because we wanted $T \to 0$ for $y \to \infty$. If we require T to stay finite (not necessarily zero) for $y \to \infty$ we can admit the solution $e^{ky}|_{k=0}$, or equivalently admit $n = 0$ in the sum (88).

We can solve the same equation with this new boundary condition and the source $f(x) = x$. With respect to the previous boundary condition, we have $f(x) = f_{old}(x) + 5$; then we can use the solution to the previous case and add a solution that respects the boundary condition $T(x, 0) = 5$. A constant respects all the required boundary condition, then the solution to this new problem is

$$T = 5 - \frac{40}{\pi^2}\sum_{odd\ n}\frac{1}{n^2}e^{-n\pi y/10}\cos\frac{n\pi x}{10}. \tag{91}$$