1

Evaluate the integral

$$\int \frac{\cos\theta}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}d\Omega : \tag{1}$$

The addition theorem for spherical harmonics is

$$P_\ell(\cos\gamma) = \frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{\ell} Y_\ell^m(\theta,\phi)Y_\ell^m(\theta',\phi')^*, \tag{2}$$

where γ is the angle between the vectors characterized by the angles (θ,ϕ) and (θ',ϕ'), and the spherical harmonics are defined by

$$Y_\ell^m(\theta,\phi) = (-1)^m\sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}P_\ell^m(\cos\theta)e^{im\phi}\,, \qquad \int d\Omega Y_\ell^m(\theta,\phi)Y_{\ell'}^{m'}(\theta,\phi)^* = \delta_{ll'}\delta_{mm'} \tag{3}$$

With this we can derive the identity

$$\frac{1}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|} = 4\pi\sum_{\ell=0}^{\infty}\frac{1}{2\ell+1}\frac{r'^\ell}{r^{\ell+1}}\sum_{m=-\ell}^{\ell}Y_\ell^m(\theta,\phi)Y_\ell^m(\theta',\phi')^*\,, \qquad r>r'. \tag{4}$$

In the integral (1) we can express $\cos\theta$ in terms of spherical harmonics: in particular, we have

$$Y_1^0(\theta,\phi) = \sqrt{\frac{3}{4\pi}}P_1^0(\cos\theta)\,, \qquad P_1^0(\cos\theta) = \cos\theta = \sqrt{\frac{4\pi}{3}}Y_1^0(\theta,\phi) \tag{5}$$

Noting that $Y_1^0(\theta,\phi)$ is real, we can write $Y_1^0(\theta,\phi) = Y_1^0(\theta,\phi)^*$. Then we have

$$\int \frac{\cos\theta}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}d\Omega = 4\pi\sqrt{\frac{4\pi}{3}}\sum_{\ell=0}^{\infty}\frac{1}{2\ell+1}\frac{r'^\ell}{r^{\ell+1}}Y_\ell^m(\theta',\phi')^*\sum_{m=-\ell}^{\ell}\int d\Omega\, Y_1^0(\theta,\phi)^*Y_\ell^m(\theta,\phi) \tag{6}$$

$$= 4\pi\sqrt{\frac{4\pi}{3}}\sum_{\ell=0}^{\infty}\frac{1}{2l+1}\frac{r'^\ell}{r^{\ell+1}}Y_l^m(\theta',\phi')\delta_{m0}\delta_{l1} = \frac{4\pi}{3}\sqrt{\frac{4\pi}{3}}\frac{r'}{r^2}Y_1^0(\theta',\phi') = \frac{4\pi r'}{3r^2}\cos\theta' \tag{7}$$

This result holds for $r>r'$. If $r<r'$ we have the same expression but with r,r' interchanged.

2 Boas, p. 658, problem 13.8-4

Do the two-dimensional analogue of the problem in Boas, p.655, Example 1. That is, solve Poisson's equation in two dimensions, with a point charge outside a circle. This is equivalent to the three dimensional problem of an infinite line of charge perpendicular to the plane, outside of an infinite circular cylinder whose axis is parallel to the line charge.

Let the charge be at $x = a$: the charge density is $\rho(x, y) = k\delta(x - a)\delta(y)$, where k is the linear density of the charged line. We want to solve

$$\nabla^2 V = -4\pi\rho \tag{8}$$

with the boundary condition $V(r = R) = 0$. First we find the potential generated by a charged line in a plane perpendicular to the line: applying Gauss's law to a cylinder of radius r and height h centered around the charge, we have

$$\Phi(\mathbf{E}) = \int_S \mathbf{E} \cdot \hat{\mathbf{n}}\, dS = 2\pi r h E = 4\pi Q = 4\pi\lambda h \implies \mathbf{E} = 2\frac{\lambda}{r}\hat{\mathbf{r}} \tag{9}$$

From this we can find the potential V:

$$V(\mathbf{r}) = -\int \mathbf{E} \cdot d\ell = -2\lambda \ln|\mathbf{r}| + C = -\lambda \ln r^2 + C \tag{10}$$

where we have recalled that the potential can always include a constant.

Then the potential generated by our charge is

$$V_0(x, y, z) = -\lambda \ln((x - a)^2 + y^2) = -\lambda \ln(r^2 + a^2 - 2ar\cos\theta) \tag{11}$$

where we have expressed our result in terms of polar coordinates. Now we have to add to this another potential generated by an image charge such that the potential is null on the circle, $V(r = R) = 0$. Exactly as in the case of the sphere, the image charge $\lambda' = -\lambda$ will be situated at $x = \frac{R^2}{a}$, $y = 0$. The potential generated by such a charge is

$$V_1 = \lambda \ln((x - R^2/a)^2 + y^2) = \lambda \ln(r^2 + R^4/a^2 - 2rR^2/a\cos\theta) \tag{12}$$

We now check the value of $V = V_0 + V_1$ on the circle:

$$\begin{aligned} V(r = R) &= -\lambda \ln(R^2 + a^2 - 2aR\cos\theta) + \lambda \ln(R^2 + R^4/a^2 - 2R^3/a\cos\theta) \\ &= -\lambda \ln(R^2 + a^2 - 2aR\cos\theta) + \lambda[\ln(R^2/a^2) + \ln(R^2 + a^2 - 2Ra\cos\theta)] \\ &= \lambda \ln(R^2/a^2) \end{aligned} \tag{13}$$

Although $V(r = R)$ obtained above is not zero on the boundary, it is a constant. Thus, to obtain the correct boundary condition, one simply adds a constant to the potential such that $V(r = R) = 0$. This step is permissible since a constant satisfies Laplace's equation. A little algebra yields

$$V(r) = V_0 + V_1 - \lambda \ln(R^2/a^2) = \lambda \ln\left(\frac{R^2 + \frac{r^2}{R^2}a^2 - 2ar\cos\theta}{r^2 + a^2 - 2ar\cos\theta}\right), \qquad \text{for } r \geq R. \tag{14}$$

Note that $V(r = R) = 0$ as required by the boundary condition.

3 Boas, p. 659, problem 13.8-9

Find the following solution of $\nabla^2 u = f$

$$u(\mathbf{r}) = \int G(\mathbf{r}, \mathbf{r}') f(\mathbf{r}') d\tau' + \int u(\mathbf{r}') \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n'} d\sigma' \tag{15}$$

where G is the Green function which is zero on the surface σ.

From Green's second identity we have,

$$\int d\tau(\phi\nabla^2\psi - \psi\nabla^2\phi) = \int d\sigma(\phi\vec{\nabla}\psi - \psi\vec{\nabla}\phi)\cdot\hat{n} \tag{16}$$

If we set $\phi = u(\mathbf{r})$ and $\psi = G(\mathbf{r},\mathbf{r}')$ we have

$$\int d\tau'(u(\mathbf{r}')\nabla^2 G(\mathbf{r},\mathbf{r}') - G(\mathbf{r},\mathbf{r}')\nabla^2 u(\mathbf{r}')) = \int d\sigma'(u(\mathbf{r}')\vec{\nabla}G(\mathbf{r},\mathbf{r}') - G(\mathbf{r},\mathbf{r}')\vec{\nabla}u(\mathbf{r}'))\cdot\hat{n} \tag{17}$$

where on the left hand side, we have $\nabla^2 G(\mathbf{r},\mathbf{r}') = \delta^3(\mathbf{r}-\mathbf{r}')$ and $\nabla^2 u(\mathbf{r}') = f(\mathbf{r}')$, while on the right hand side we have $G(\mathbf{r},\mathbf{r}')|_\sigma = 0$. Noting that

$$\int d\tau' u(\mathbf{r}')\delta^3(\mathbf{r}-\mathbf{r}') = u(\mathbf{r}) \tag{18}$$

and $\partial G/\partial n \equiv \hat{n}\cdot\vec{\nabla}G$, we end up with

$$u(\mathbf{r}) = \int G(\mathbf{r},\mathbf{r}')f(\mathbf{r}')d\tau' + \int u(\mathbf{r}')\frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial n'}d\sigma' \tag{19}$$

4 Boas, p. 662, problem 13.9-4

A semi-infinite bar is initially at a temperature 100 for $0 < x < 1$ and 0 for $x > 1$. Starting at $t = 0$ the $x = 0$ end is maintained at 0 and the sides are insulated. Find the temperature at time t.

We separate the heat flow equation

$$\nabla^2 u = \frac{1}{\alpha^2}\frac{\partial u}{\partial t}\,, \quad u = X(x)T(t) \implies \frac{X''}{X} = \frac{T'}{\alpha^2}T = -k^2 \implies T(t) = e^{-k^2\alpha^2 t}\,,\ X(x) = \left\{\begin{array}{l}\sin kx\\ \cos kx\end{array}\right. . \tag{20}$$

Since $u(0) = 0$, we discard the cosines and we can then look for a solution of the form

$$u(x,t) = \int_0^\infty B(k)e^{-\alpha^2k^2t}\sin kx\,dk \tag{21}$$

For $t = 0$ we must satisfy the initial condition $u(x,0) = 100\Theta(1-x)$ where Θ is the step function. Then we have

$$u(x,0) = \int B(k)\sin kx\,dk \qquad \implies \qquad B(k) = \frac{2}{\pi}\int_0^\infty u(x,0)\sin kx\,dx = \frac{200}{\pi}\int_0^1 \sin kx\,dx =$$
$$= \frac{200}{k\pi}(1-\cos k) \tag{22}$$

Finally, our solution is

$$u(x,t) = \frac{200}{\pi}\int_0^\infty \frac{1-\cos k}{k}e^{-k^2\alpha^2t}\sin kx\,dk \tag{23}$$

5 Boas, p. 662, problem 13.9-5

A long wire occupying the x axis is initially at rest. Find the displacement if the $x = 0$ end is oscillated so that

$$y(0,t) = 2\sin 3t. \tag{24}$$

The initial and boundary conditions of the wave equation $\frac{\partial^2}{\partial x^2}y = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}y$ are

$$y(x,0) = 0\,,\ \dot{y}(x,0) = 0\,,\ y(0,t) = 2\sin 3t\ ; \tag{25}$$

Let $Y(x,p)$ be the Laplace transform of $y(x,t)$:

$$Y(x,p) = \int_0^\infty u(x,t)e^{-pt}dt \tag{26}$$

By using the properties of the Laplace transform, we have

$$L(\tfrac{\partial^2 y}{\partial t^2}) = p^2Y - py(x,0) - \dot{y}(x,0) = p^2Y(x,p) \tag{27}$$

$$L(\tfrac{\partial^2 y}{\partial x^2}) = \frac{\partial^2}{\partial x^2}Y(x,p) \tag{28}$$

The wave equation then becomes

$$\frac{\partial^2}{\partial x^2}Y(x,p) = \frac{p^2}{v^2}Y(x,p) \qquad \Longrightarrow \qquad Y(x,p) = c\,e^{\pm px/v} \tag{29}$$

Now we need to transform the boundary condition:

$$y(0,t) = 2\sin 3t \implies Y(0,p) = \int_0^\infty y(0,t)e^{-pt}dt = \frac{6}{p^2+9} = c \tag{30}$$

This fixes the multiplication constant in the solution (29). As the wire is described as "long", we can assume that the far end will be unperturbed for all the time so that $y(x,t) \overset{x\to\infty}{\to} 0$ and $Y(x,t) \overset{x\to\infty}{\to} 0$; then only the negative sign in the exponential will be allowed, and our solution is

$$Y(x,p) = \frac{6}{p^2+9}e^{-px/v} \tag{31}$$

To find the displacement of the wire we now have to inverse transform this:

$$y(x,t) = \mathcal{L}^{-1}\left[\frac{6}{p^2+9}e^{-px/v}\right] = \begin{cases} 2\sin 3(t-\frac{x}{v}), & x < vt, \\ 0, & x > vt. \end{cases} \tag{32}$$

We see that the excitation is moving along the x-axis with speed v.

6 Boas, p. 665, problem 13.10-28

Find the steady-state temperature in a semi-infinite plate covering the region $x > 0,\ 0 < y < 1$, if the edges along the axes are insulated and the top edge is held at

$$u(x,1) = \begin{cases} 100, & 0 < x < 1, \\ 0, & x > 1. \end{cases} \tag{33}$$

The heat equation is

$$\nabla^2 u(x,y) = 0\,, \quad u = X(x)Y(y) \quad \Longrightarrow \quad \frac{X''}{X} = -\frac{Y''}{Y} = -k^2 \Longrightarrow X(x) = \begin{cases} \sin kx \\ \cos kx \end{cases}, Y(y) = \begin{cases} \sinh ky \\ \cosh ky \end{cases} \tag{34}$$

Because the edges along the axes are insulated, we have $\frac{\partial}{\partial x}u(0,y) = \frac{\partial}{\partial y}u(x,0) = 0$, which means that only the cosine and hyperbolic cosine are allowed. Then we write our solution as a Fourier integral

$$u(x,y) = \int_0^\infty B(k) \cos kx \cosh ky \, dk \tag{35}$$

We can find $B(k)$ imposing the boundary condition (33):

$$u(x,1) = \int B(k) \cos kx \cosh k \, dk\,, \tag{36}$$

which implies that

$$B(k) \cosh k = \frac{2}{\pi} \int_0^\infty u(x,1) \cos kx \, dx = \frac{200}{\pi} \int_0^1 \cos kx \, dx = \frac{200}{k\pi} \sin k. \tag{37}$$

We can write down the solution as an integral:

$$u(x,y) = \int_0^\infty \frac{\sin k}{k \cosh k} \cos kx \cosh ky \, dk \tag{38}$$

7 Boas, p. 724, problem 15.1-4

A single card is drawn at random from a shuffled deck. What is the probability that it is red? That it is the ace of hearts? That it is either a 3 or a 5? That it is either an ace or red or both?

- the probability that one card is red is $26/52 = 1/2$ as there are 26 red cards (diamonds and hearts) out of 52.
- the probability that it is the ace of hearts is $1/52$.
- the probability that it is either a 3 or a 5 is $8/52 = 2/13$ as there are 8 favorable cards out of 52.
- the probability that it is an ace or a red or both is $28/52 = 7/13$, since there are 26 red cards and 2 black aces.

8 Boas, p. 724, problem 15.1-10

A shopping mall has four entrances, One on the North, one on the South and two on the East. If you enter at random, shop and exit at random, what is the probability that you enter and exit on the same side of the building?

There are three possible sides, each one having individual probabilities of being chosen given by

$$N:\ 1/4\,, \qquad S:\ 1/4\,, \qquad E:\ 1/2\,.$$

Let us list the individual probabilities:

- if you enter from N, you have a probability of 1/4 of exiting by the same side: $P = 1/4 * 1/4$
- if you enter by S, you have a probability of 1/4 of exiting by the same side: $P = 1/4 * 1/4$
- if you enter by E, you have a probability of 1/2 of exiting by the same side: $P = 1/2 * 1/2$

This is a set of mutually exclusive events so that the probability of entering and exiting on any same side is given by the sum of these probabilities and it is $6/16 = 3/8$.

9 Boas, p. 728, problem 15.2-13

A student claims in Problem 15.1-15 that if one child is a girl, the probability that both are girls is $\frac{1}{2}$. Use appropriate sample spaces to show what is wrong with the following argument: It doesn't matter whether the girl is the older child or the younger; in either case the probability is $\frac{1}{2}$ that the other child is a girl.

The sample space that describes the two children consists of four possible outcomes: BB, BG, GB, GG where the order indicates which one is the younger and which one is older and B/G stands for boy/girl. If one child is a girl, the reduced sample space consists only of three possibilities: BG, GB, GG. Without any further information, all three possibilities are equally likely. Hence, the probability of two girls is $P = \frac{1}{3}$.

The student who claims that the probability is $P = \frac{1}{2}$ is wrong as long as there is no further information. But suppose one is told that one of the girl's names is Mary. Now, if we denote the girl whose name is Mary by G^*, then the reduced sample space actually now contains four equally likely possibilities: BG^*, G^*B, G^*G, GG^*, in which case the probability of two girls is indeed $P = \frac{1}{2}$ as the student claimed. So, one must be extremely careful on how the problem is posed.

10 Boas, p. 728, problem 15.2-15

Use the sample spaces (2.4) and (2.5) in Boas, p. 727, 728 to answer the following questions about a toss of two dice.

(a) What is the probability that the sum is ≥ 4?

We will give the answer using both the uniform sample space (2.4) and the non uniform one (2.5):

- Only three events $(1,1)$, $(1,2)$ and $(2,1)$ give a sum < 4, which means that 33 events give a sum ≥ 4. Hence, the probability is
$$P = \frac{33}{36} = \frac{11}{12} \tag{39}$$
- The associated probabilities with the sums 2 and 3 are $\frac{1}{36}$and $\frac{2}{36}$, respectively. Adding up these probabilities yields $\frac{1}{12}$. Thus, the assoicated probabilities with the sums ≥ 4 must add up to:
$$P = 1 - \frac{1}{12} = \frac{11}{12} \tag{40}$$
as expected.

(b)What is the probability that the sum is even?

- This happens when both numbers are even or both are odd:
$$P = \frac{1}{36} \cdot 18 = \frac{1}{2} \tag{41}$$

- This happens for the points in the sample space where the sum is even:

$$P = \frac{1}{36} + \frac{3}{36} + \frac{5}{36} + \frac{5}{36} + \frac{3}{36} + \frac{2}{36} + \frac{1}{36} = \frac{1}{2} \tag{42}$$

(c)What is the probability that the sum is divisible by 3?

- This happens for 12 possible combinations:

$$P = \frac{12}{36} = \frac{1}{3} \tag{43}$$

- This happens when the sum is 3,6,9,12:

$$P = \frac{2}{36} + \frac{5}{36} + \frac{4}{36} + \frac{1}{36} = \frac{1}{3} \tag{44}$$

(d) If the sum is odd, what is the probability that it is equal to 7?

- Our sample space is given by half of the original sample space, and we here have 6 possible combinations

$$P = \frac{6}{18} = \frac{1}{3} \tag{45}$$

- The modified sample space is the one in (2.5) but with new probabilities:

$$P(3) = \frac{2}{18}\,,\; P(5) = \frac{4}{18}\,,\; P(7) = \frac{6}{18}\,,\; P(9) = \frac{4}{18}\,,\; P(11) = \frac{2}{18}\,.$$

  The probability for 7 is then

$$P(7) = \frac{6}{18} = \frac{1}{3} \tag{46}$$

(e) What is the probability that the product of the numbers on the two dice is 12?

- This happens for $(3,4),(4,3),(2,6),(6,2)$; the probability is then

$$P = \frac{4}{36} = \frac{1}{9} \tag{47}$$

- the sample space (2.5) is not useful in this case; we could introduce a sample space in which we associate a probability to the product of the numbers of the two dice. We would associate the probability $\frac{1}{9}$ to the point having product equal to 12.

11 Boas, p. 729, problem 15.2-18

Are the following correct nonuniform sample spaces for a throw of two dices? If so, find the probabilities of the given sample points. If not show what is wrong.

(a) First die shows an even number.
First die shows an odd number.

This is a set of all mutually exclusive outcomes, then it is a sample space. The probabilities are $\frac{1}{2}$, $\frac{1}{2}$.

(b) Sum of two numbers on dice is even.
First die is even and second odd.
First die is odd and second even.

This is also a sample space: the first event happens when both numbers are even or both are odd, while the other two events describe the two possible outcomes that give a odd number. The probabilities are $\frac{1}{2}, \frac{1}{4}, \frac{1}{4}$.

(c) First die shows a number ≤ 3.
At least one die shows a number > 3.

This is not a sample space since the two events are not mutually exclusive.

12 Boas, p. 729, problem 15.2-19

Consider the set of all permutations of the numbers $1, 2, 3$. If you select a permutation at random, what is the probability that the number 2 is in the middle position? In the first position? Do your answer suggest a simple way of answering the same questions for the set of all permutations of the numbers 1 to 7?

The number of permutations of $1, 2, 3$ is $3! = 6$. There are two permutations with 2 in the middle position, $(1, 2, 3), (3, 2, 1)$. The probability that the number 2 is in the middle position is then $1/3$ and is the same as the probability that the number 2 is in the first position. The same applies to the set of permutations of the numbers 1 to 7, for which the probability of having a given number in one given spot is $1/7$.

In general, for the permutation of the numbers 1 to n numbers, if we fix a particular number in a given place, the number of other arrangements in which that number is in the same place is equal to the number of permutations of the remaining $n - 1$ numbers, i.e. $(n - 1)!$. This should be compared to the total number of permutations $n!$. Hence, the relevant probability is

$$P = \frac{(n-1)!}{n!} = \frac{1}{n} \, .$$