INSTRUCTIONS: This is a one-hour and 45 minute exam. During the exam, you may refer to the textbook, the class handouts (including solution sets to homework and practice problems) or your own personal notes. Collaboration with your neighbor is strictly forbidden.

In answering the questions, it is not sufficient to simply give a final result. You must provide the intermediate steps needed to arrive at the final solution in order to get full credit. *However, if you are employing any result already obtained in class or in the textbook, you may simply cite the source. You do not need to re-derive it.*

1. Solve the differential equation:

$$xy'' + (\tfrac{1}{2} - x)y' - \tfrac{1}{2}y = 0\,, \tag{1}$$

by expanding the solution in a generalized power series using the method of Frobenius.

(a) Determine the generalized power series for the two linearly independent solutions of eq. (1). Show that one of the two series solutions, denoted by $y_1(x)$, is an elementary function by explicitly summing the series.

(b) In class, I showed that once an explicit form for $y_1(x)$ is known, the second solution $y_2(x)$ can be determined from:

$$y_2(x) = y_1(x)v(x)\,, \qquad \text{where} \quad v(x) \equiv \int w(x)\,dx\,,$$

and $w(x)$ satisfies a first order linear differential equation. Inserting $y = y_1(x)v(x)$ into eq. (1) [and using the fact that $y_1(x)$ satisfies eq. (1)], find the equation that $w(x)$ satisfies, and solve it.

(c) *EXTRA CREDIT:* Evaluate the indefinite integral of $w(x)$ in terms of one of the special functions that we studied in Physics 116A.

2. The spherical modified Bessel functions are defined by:

$$i_n(x) \equiv \sqrt{\frac{\pi}{2x}}\, I_{n+\frac{1}{2}}(x)\,,$$

$$k_n(x) \equiv \sqrt{\frac{2}{\pi x}}\, K_{n+\frac{1}{2}}(x)\,,$$

where n is an integer. $I_{n+\frac{1}{2}}(x)$ and $K_{n+\frac{1}{2}}(x)$ are modified Bessel functions of order $n+\frac{1}{2}$.

(a) Express $i_0(x)$ and $k_0(x)$ explicitly in terms of elementary functions.* Cite any pertinent formulae that you take from Boas.

(b) Compute the Wronskian of $k_0(x)$ and $i_0(x)$. Are these two functions linearly independent?

3. In quantum mechanics, the Schrodinger equation for a free particle in one dimension is†

$$\frac{\partial^2\psi(x,t)}{\partial x^2} = -i\frac{\partial\psi(x,t)}{\partial t}\,, \tag{2}$$

where the (complex) wave function $\psi(x,t)$ provides information on the probability that the particle is located in the vicinity of x at time t.

Using the separation of variables technique, solve eq. (2) subject to the boundary conditions $\psi(0,t)=\psi(1,t)=0$ and the initial condition $\psi(x,0)=1$ $(0<x<1)$. These conditions correspond to the physical situation of a particle trapped in a "box" (more precisely, an interval) of length 1, whose location at time $t=0$ is equally probable anywhere in the interval $0<x<1$.

HINT: Your final solution for $\psi(x,t)$ should take the form of an infinite sum over solutions to eq. (2) that satisfy the boundary conditions. Make sure that you impose the initial condition to determine all remaining unknown coefficients.

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*Elementary functions include any function that can be expressed in terms of polynomials, powers, exponentials, logarithms, trigonometric functions, hyperbolic trigonometric functions and their inverses.

†To make the algebra easier, I have taken the mass of the particle equal to $\frac{1}{2}$ and have chosen units where Planck's constant is equal to 2π.