Here is a collection of practice problems suitable for the midterm exam.

1. Consider the differential equation:

$$y' - 2xy = 1\,, \qquad y(0) = 0\,. \tag{1}$$

(a) Solve eq. (1) using the series solution technique. Writing the solution in the form $y = \sum_n a_n x^n$, obtain a general expression for a_n for arbitrary n. If possible, express a_n in terms of a double factorial.

(b) Solve eq. (1) by another technique and show that the solution can be written in terms of a special function studied in Physics 116A.

2. Evaluate $P^1_{2n+1}(0)$, where $P^m_n(x)$ is the associated Legendre function and n is a non-negative integer.

3. Consider the differential equation,

$$xy'' + 2y' - y = 0\,. \tag{2}$$

(a) Find a generalized power series solution to eq. (2) using the method of Frobenius.

(b) Show that the second linearly independent solution to eq. (2) cannot be expressed as a generalized power series due to the presence of a logarithm. Determine the second linearly independent solution to be of the form:

$$y(x) = \ln x \sum_{n=0}^{\infty} a_n x^{n+r_1} + \sum_{n=0}^{\infty} b_n x^{n+r_2}\,.$$

Obtain explicitly the values of r_1 and r_2 and the general expressions for a_n and b_n for arbitrary n.

4. Find the linearly independent solutions to the differential equation,

$$y'' + e^{2x} y = 0\,.$$

Show that the solutions can be expressed in terms of Bessel functions.

HINT: First try a change of variables, $z = e^x$.

5. Using either the recursion relations or the generating function for the Hermite polynomials $H_n(x)$, obtain a general expression for $H_n(0)$, where n is a non-negative integer.

6. Define $T_n(\cos\theta) = \cos n\theta$. One can show that $\cos n\theta$ can be expressed as a polynomial in $\cos\theta$ of degree n.* That is, one can write $T_n(x)$ in the form:

$$T_n(x) = \sum_n a_n \cos^n \theta \,,$$

where the a_n can be obtained, for example, by the method outlined in the footnote below. For convenience, denote $x = \cos\theta$. The $T_n(x)$ are called Chebyshev polynomials.

(a) Evaluate $T_0(x)$, $T_1(x)$ and $T_2(x)$.

(b) Note that $y(\theta) = T_n(\cos\theta) = \cos n\theta$ satisfies

$$\frac{d^2y}{d\theta^2} + n^2 y = 0 \,.$$

Introduce a new variable $x = \cos\theta$ and let $y(x) = T_n(x)$. By appropriate use of the chain rule, find the differential equation that $y(x)$ satisfies.

(c) Show explicitly that the differential equation for $T_n(x)$ obtained in part (b) can be written in the form of a Sturm-Liouville boundary value problem over the interval $-1 \leq x \leq 1$. (Make sure you verify that the Sturm-Liouville boundary condition is satisfied.)

(d) Write down the orthogonality relation satisfied by the T_n.

REMARK: The Chebyshev polynomials play a central role in the mathematical theory of the approximation of functions.

7. Consider the Sturm-Liouville problem,

$$\frac{d^2y}{dx^2} + k^2 y = 0 \,,$$

for $-\ell \leq x \leq \ell$, with periodic boundary conditions,

$$y(\ell) = y(-\ell) \,.$$

(a) Show that the Sturm-Liouville boundary condition is satisfied.

*You need not offer a proof here, but in case you are curious, the proof makes use of the relations $\cos n\theta = \text{Re } e^{in\theta} = \text{Re } (\cos\theta + i\sin\theta)^n$, and $\sin^2\theta = 1 - \cos^2\theta$.

(b) Show that the eigenfunctions of the Sturm-Liouville problem can be chosen to be $\{e^{ikx}\}$, where k can only take on discrete values. Determine the allowed values of k (noting that both positive and negative values of k are allowed).

(c) Write down orthonormality relations satisfied by the eigenfunctions obtained in part (b).

(d) The eigenfunctions obtained in part (b) form a complete set. Expand the function $f(x) = x^2$ (where $-\ell \leq x \leq \ell$) in terms of the eigenfunctions obtained in part (b).

(e) Set $x = \ell$ in the result obtained in part (d). Show that the resulting equation provides a method for computing the infinite sum,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} .$$

8. Consider a rectangular plate covering the area $0 < x < 1$ and $0 < y < 2$. The temperature for $y = 2$ is $T = 0$ and the temperature for $y = 0$ is $T = 1 - x$.

(a) Find the steady state temperature distribution if $T = 0$ for $x = 0$ and $x = 1$.

(b) Find the steady state temperature distribution if the sides $x = 0$ and $x = 1$ are insulated.

9. A bar of length 2 is initially at 0°. For time $t \geq 0$, the $x = 0$ end of the bar is insulated and the $x = 2$ end is held at 100°. Find the time-dependent temperature distribution of the bar.

10. A string of length ℓ has a zero initial velocity and a displacement given by:

$$y_0(x) = \begin{cases} \dfrac{4hx}{\ell}\,, & \text{for } 0 \leq x \leq \tfrac{1}{4}\ell\,, \\ 2h - \dfrac{4h}{\ell}x\,, & \text{for } \tfrac{1}{4}\ell \leq x \leq \tfrac{1}{2}\ell\,, \\ 0\,, & \text{for } \tfrac{1}{2}\ell \leq x \leq \ell\,. \end{cases}$$

Assuming that the string is pinned at $x = 0$ and free at $x = \ell$, find the displacement of the string as a function of x and t.

11. A square membrane of side ℓ is distorted into the shape,

$$f(x, y) = xy(\ell - x)(\ell - y)\,,$$

and released. Express its shape at subsequent times as an infinite series.