1

A long cylinder has been cut into quarter cylinders that are insulated from each other. Alternate quarter cylinders are held at potentials +100 and −100. Find the electrostatic potential inside the cylinder.

We have to solve Laplace's equation with the following boundary conditions:

$$\vec{\nabla}^2 V(x,y) = 0\,, \qquad V(a,\theta) = \begin{cases} 100, & 0<\theta<\frac{\pi}{2},\ \pi<\theta<\frac{3\pi}{2}, \\ -100, & \frac{\pi}{2}<\theta<\pi,\ \frac{3\pi}{2}<\theta<2\pi. \end{cases} \tag{1}$$

Because we have a long cylinder, there will be no dependence on the z direction. The Laplacian in two dimensions is

$$\vec{\nabla}^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}. \tag{2}$$

We separate the equation in a radial and angular equation:

$$V(r,\Theta) = R(r)\Theta(\theta) \qquad \Longrightarrow \qquad \frac{r}{R}(rR')' = -\frac{\Theta''}{\Theta} = n^2, \tag{3}$$

$$\Theta(\theta) = e^{\pm in\theta}\,, \qquad r^2R'' + rR' - n^2R = 0. \tag{4}$$

n must be an integer so that the function is single valued, $\Theta(\theta+2\pi) = \Theta(\theta)$, while the radial equation is solved by $r^{\pm n}$. The most general solution is then

$$V = \sum_{n=0}^{\infty}\left[r^n(A_n\sin n\theta + B_n\cos n\theta) + r^{-n}(A'_n\sin n\theta + B'_n\cos n\theta)\right] \tag{5}$$

Now we have to apply the boundary conditions (1): because we are looking for a solution inside the cylinder, we will put $A' = B' = 0$ to avoid divergent behaviors: we are left with

$$V(a,\theta) = \sum_{n=0}^{\infty} a^n\left[A_n\sin n\theta + B_n\cos n\theta\right] \tag{6}$$

First, we see that the boundary conditions are odd around $\theta = 0$; then we will put $B_n = 0$ for all n because the cosine is even. Finally, we have

$$\begin{aligned} a^nA_n &= \frac{1}{\pi}\int_0^{2\pi} V(a,\theta)\sin n\theta\,d\theta \\ &= \frac{100}{\pi}\int_0^{\pi/2}\sin n\theta\,d\theta - \frac{100}{\pi}\int_{\pi/2}^{\pi}\sin n\theta\,d\theta + \frac{100}{\pi}\int_{\pi}^{3\pi/2}\sin n\theta\,d\theta - \frac{100}{\pi}\int_{\pi/2}^{2\pi}\sin n\theta\,d\theta \\ &= \frac{100}{n\pi}[2 - 2\cos(n\pi/2) + 2\cos n\pi - 2\cos(3\pi n/2)] = \frac{200}{n\pi}(1+\cos n\pi)[1-\cos(n\pi/2)] \end{aligned} \tag{7}$$

If we substitute $n = 2k+1$, the expression is zero. For $n = 2k$, we have $\frac{400}{2k\pi}(1-(-1)^k)$, so that only the odd k contribute. Finally, our potential is

$$V(r,\theta) = \frac{400}{\pi}\sum_{\text{odd } k}\frac{1}{k}\left(\frac{r}{a}\right)^{2k}\sin 2k\theta \tag{8}$$

2

Find the steady-state temperature in the region between two spheres with radii $r = 1$ and $r = 2$, respectively, if the surface of the outer sphere has its upper half held at 100° and its lower half at −100° and these temperatures are reversed for the inner sphere.

We have to solve Laplace's equation, $\vec{\boldsymbol{\nabla}}^2 u(r,\theta,\phi) = 0$, with the following boundary conditions:

$$u(2,\theta,\phi) = \begin{cases} 100, & 0 < \theta < \pi/2, \\ -100, & \pi/2 < \theta < \pi\,, \end{cases} \qquad u(1,\theta,\phi) = \begin{cases} -100, & 0 < \theta < \pi/2, \\ 100, & \pi/2 < \theta < \pi\,. \end{cases} \tag{9}$$

The separation of variables in Laplace's equation proceeds as always: once we take $u = R(r)\Theta(\theta)\Phi(\phi)$, we find

$$\frac{1}{R}(r^2 R')' + \frac{1}{\Theta}\frac{1}{\sin\theta}(\sin\theta\,\Theta')' + \frac{1}{\Phi}\frac{1}{\sin^2\theta}\Phi'' = 0 \tag{10}$$

which gives

$$u(r,\theta,\phi) = \left\{ \begin{array}{c} r^l \\ r^{-l-1} \end{array} \right\} \cdot P_l^m(\cos\theta) \cdot \left\{ \begin{array}{c} \sin m\phi \\ \cos m\phi \end{array} \right\} \tag{11}$$

Because the boundary conditions have no ϕ dependence, we will have no ϕ dependence in the solution, that is, $m = 0$ and the P_l^m reduce to P_l, the Legendre polynomials. Our solution will be given by

$$u(r,\theta) = \sum_{l=0}^{\infty} (a_l r^l + b_l r^{-l-1}) P_l(\cos\theta) \tag{12}$$

Note that we have kept both the positive and the negative powers of r, since the range of r relevant for this problem is $1 \le r \le 2$ which excludes both $r = 0$ and $r = \infty$. We first apply the boundary conditions at $r = 1$:

$$u(1,\theta,\phi) \;\; = \;\; \sum_l (a_l + b_l) P_l(\cos\theta)\,. \tag{13}$$

We can solve for $a_l + b_l$ by multiplying both sides of this equation by $P_{l'}(\cos\theta)$ and integrating over $-1 \le \cos\theta \le 1$. Using the orthogonality relation of the Legendre polynomials, it follows that

$$\begin{aligned} a_l + b_l &= \frac{2l+1}{2}\int_{-1}^{1} u(1,\theta,\phi) P_l(\cos\theta)\, d\cos\theta = 100\frac{2l+1}{2}\left(-\int_0^1 P_l(x)dx + \int_{-1}^{0} P_l(x)dx\right) \\ &= \begin{cases} -100(2l+1)\int_0^1 P_l(x)dx\,, & \text{for odd } l, \\ 0\,, & \text{for even } l, \end{cases} \end{aligned} \tag{14}$$

where we have used the fact that P_l is an even function of x when l is even and an off function of x when l is odd. Making use of the result of problem 12-23.3 on p. 615 of Boas,

$$c_n \equiv \int_0^1 P_{2n+1}(x)dx = \frac{(-1)^n(2n-1)!!}{2^{n+1}(n+1)!}\,, \qquad \text{for } n = 0,1,2,3,\ldots, \tag{15}$$

so that

$$a_{2n+1} + b_{2n+1} = -100(4n+3)c_n\,, \tag{16}$$

$$a_{2n} + b_{2n} = 0\,. \tag{17}$$

Next, we apply the boundary conditions at $r = 2$:

$$u(2, \theta, \phi) = \sum_l (a_l 2^l + b_l 2^{-l-1}) P_l(\cos\theta) \,. \tag{18}$$

We solve for $a_l 2^l + b_l 2^{-l-1}$ following the same procedure as above.

$$\begin{aligned} a_l 2^l + b_l 2^{-l-1} &= \frac{2l+1}{2} \int_{-1}^{1} u(2, \theta, \phi) P_l(\cos\theta) d\cos\theta = 100 \frac{2l+1}{2} \left(\int_0^1 P_l(x) dx - \int_{-1}^{0} P_l(x) dx \right) \\ &= \begin{cases} 100(2l+1) \int_0^1 P_l(x) dx \,, & \text{for odd } l, \\ 0 \,, & \text{for even } l. \end{cases} \end{aligned} \tag{19}$$

Using (15),

$$2^{2n+1} a_{2n+1} + 2^{-2n-2} b_{2n+1} = 100(4n+3) c_n \,, \tag{20}$$

$$2^{2n} a_{2n} + 2^{-2n-1} b_{2n} = 0 \,. \tag{21}$$

Eqs. (17) and (21) imply that $a_{2n} = b_{2n} = 0$. Adding eqs. (16) and (20) yields

$$a_{2n+1} \left(1 + 2^{2n+1}\right) = -b_{2n+1} \left(1 + \frac{1}{2^{2n+2}}\right) \,. \tag{22}$$

which yields

$$a_1 = -\tfrac{5}{12} b_1 \,, \quad a_3 = -\tfrac{17}{144} b_3 \,, \quad \text{etc.} \tag{23}$$

We then use (16) to obtain a_{2n+1} and b_{2n+1} separately. For example, noting that $c_0 = \frac{1}{2}$ and $c_1 = -\frac{1}{8}$, it follows that:

$$a_1 = \tfrac{750}{7} \,, \quad b_1 = -\tfrac{1800}{7} \,, \quad a_3 = -\tfrac{2975}{254} \,, \quad b_3 = \tfrac{12600}{127} \,, \quad \text{etc.} \tag{24}$$

For general odd values of $\ell = 2n + 1 = 1, 3, 5, \ldots$,

$$a_l = -100(2l+1) c_n \frac{1 + 2^{l+1}}{1 - 2^{2l+1}} \,,$$

$$b_l = 100(2l+1) c_n \frac{2^{2l+1} + 2^{l+1}}{1 - 2^{2l+1}} \,,$$

where c_n is given by (15). For even values of l, $a_l = b_l = 0$. Inserting these values back in (12) yields the solution to the problem.

3

Using the Green function technique, solve Poisson's equation,

$$\vec{\boldsymbol{\nabla}}^{\mathbf{2}} \phi(\vec{\boldsymbol{r}}) = f(\vec{\boldsymbol{r}}) = e^{-r/a} \tag{25}$$

under the assumption that $\phi(\vec{\boldsymbol{r}}) \to 0$ as $r \to \infty$ The parameter a is a constant with units of length.

Given Poisson's equation, we can solve it by using its Green function,

$$\phi(\vec{\boldsymbol{r}}) = -\frac{1}{4\pi} \int \frac{f(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|} d^3 r' = -\frac{1}{4\pi} \int \frac{e^{-r'/a}}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|} d^3 r' \,. \tag{26}$$

We insert the expansion

$$\frac{1}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}=4\pi\sum_{\ell=0}^{\infty}\frac{1}{2\ell+1}\frac{r'^{\ell}}{r^{\ell+1}}\sum_{m=-\ell}^{\ell}Y_\ell^m(\theta,\phi)Y_\ell^m(\theta',\phi')^*\,,\qquad\qquad \text{for } r>r'\,, \tag{27}$$

and

$$\frac{1}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}=4\pi\sum_{\ell=0}^{\infty}\frac{1}{2\ell+1}\frac{r^{\ell}}{r'^{\ell+1}}\sum_{m=-\ell}^{\ell}Y_\ell^m(\theta,\phi)Y_\ell^m(\theta',\phi')^*\,,\qquad\qquad \text{for } r<r'\,, \tag{28}$$

into (26), which yields

$$\begin{aligned}\phi(\vec{\boldsymbol{r}})=&-\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{1}{2\ell+1}Y_\ell^m(\theta,\phi)\int_0^r r'^2dr'\frac{r'^{\ell}}{r^{\ell+1}}e^{-r'/a}\int Y_\ell^m(\theta',\phi')^*d\Omega'\\ &-\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{1}{2\ell+1}Y_\ell^m(\theta,\phi)\int_r^\infty r'^2dr'\frac{r^{\ell}}{r'^{\ell+1}}e^{-r'/a}\int Y_\ell^m(\theta',\phi')^*d\Omega'\,,\end{aligned} \tag{29}$$

after using $d^3r'=r'^2dr'd\Omega'$. The integral over solid angles is trivial using the orthonomality relation,

$$\int Y_\ell^m(\theta',\phi')Y_{\ell'}^{m'}(\theta',\phi')^*d\Omega'=\delta_{\ell\ell'}\delta_{mm'}\,. \tag{30}$$

If we set $\ell=m=0$ in the integral above, the $Y_{00}(\theta',\phi')=1/\sqrt{4\pi}$, and we conclude that

$$\int Y_\ell^m(\theta',\phi')^*d\Omega'=\sqrt{4\pi}\delta_{\ell 0}\delta_{m0}\,. \tag{31}$$

Hence, only the $\ell=m=0$ term in (29) survives, and we end up with

$$\phi(\vec{\boldsymbol{r}})=-\left\{\frac{1}{r}\int_0^r r'^2e^{-r'/a}dr'+\int_r^\infty r'e^{-r'/a}dr'\right\}\,. \tag{32}$$

The integrals are elementary, and the final result is:

$$\begin{aligned}\phi(\vec{\boldsymbol{r}})&=\frac{1}{r}\,e^{-r'/a}\left[ar'^2+2a^2r'+2a^3\right]\Bigg|_0^r+e^{-r'/a}\left[r'a+a^2\right]\Bigg|_r^\infty\\ &=\frac{1}{r}\,e^{-r/a}\left[ar^2+2a^2r+2a^3\right]-\frac{2a^3}{r}-e^{-r/a}\left[ra+a^2\right]\\ &=a^2e^{-r/a}-\frac{2a^3}{r}\left[1-e^{-r/a}\right]\,.\end{aligned} \tag{33}$$

One can check this result by verifying that

$$\vec{\boldsymbol{\nabla}}^{\mathbf{2}}\phi(\vec{\boldsymbol{r}})=\frac{d^2\phi}{dr^2}+\frac{2}{r}\,\frac{d\phi}{dr}=e^{-r/a}\,. \tag{34}$$

4

Consider a metal place covering the first quadrant. The edge along the y axis is insulated and the edge along the x axis has a fixed temperature profile given by:

$$u(x,0) = \begin{cases} 100(2-x), & \text{for } 0 < x < 2, \\ 0, & \text{for } x > 2. \end{cases} \tag{35}$$

Find the steady-state temperature distribution as a function of x and y. You may leave your final answer as an integral.

To find the steady-state temperature, we solve Laplace's equation with the given boundary condition (35) and the condition $\frac{\partial u}{\partial x}(0,y) = 0$. Solving Laplace's equation by separation of variables gives

$$u = X(x)Y(y) \qquad \Longrightarrow \qquad \frac{X''}{X} = -\frac{Y''}{Y} = -k^2 \Longrightarrow X = \begin{cases} \cos kx \\ \sin kx \end{cases}, \quad Y = \begin{cases} e^{-ky} \\ e^{ky} \end{cases}, \tag{36}$$

where $k \geq 0$ (since changing the sign of k does not produce any new solutions). We eliminate the e^{ky} solution because we are searching for a solution which does not blow up at large y; because $\frac{\partial u}{\partial x}(0,y) = 0$, we do not take the $\sin kx$ solution either. Our solution will be given by a linear combination of $\cos kx e^{-ky}$ for each possible value of k; not having any other condition to set k to a discrete set of values, we can take an integral sum:

$$u(x,y) = \int_0^\infty B(k) \cos kx \, e^{-ky} dk \tag{37}$$

To find B, we apply condition (35):

$$u(x,0) \quad = \quad \int_0^\infty B(k) \cos kx \, dk = \begin{cases} 100(2-x), & \text{for } 0 < x < 2, \\ 0, & \text{for } x > 2. \end{cases} \tag{38}$$

Hence,

$$\begin{aligned} B(k) &= \frac{2}{\pi} \int_0^\infty u(x,0) \cos kx \, dx = \frac{200}{\pi} \int_0^2 (2-x) \cos kx \, dx \\ &= \frac{200}{\pi} \left[(2-x) \frac{\sin kx}{k} \Big|_0^2 + \frac{1}{k} \int_0^2 \sin kx \right] = \frac{200}{\pi} \frac{1}{k^2} (1 - \cos 2k), \end{aligned} \tag{39}$$

The solution is thus given by

$$u(x,y) = \frac{200}{\pi} \int_0^\infty \frac{1 - \cos 2k}{k^2} \cos kx \, e^{-ky} dk \, . \tag{40}$$

<u>*REMARK*</u>: In problems where the parameter k takes on only discrete values, one had to treat the $k = 0$ separately. There is no need to do that in this problem. In fact, one can easily check that (39) yields

$$\lim_{k \to 0} B(k) = \frac{400}{\pi} \, . \tag{41}$$

The $k = 0$ term is automatically included in the integral expression given in (40).

5

Consider the motion of a semi-infinite string with an external time-dependent force acting on it given by

$$F(t) = \cos \omega t, \qquad t \geq 0$$

where ω is a constant. One end of the string is kept fixed while the other end is allowed to move freely in the vertical direction. Assume that at $t = 0$, the string is initially at rest in its equilibrium position, i.e.,

$$y(x, 0) = 0, \qquad \frac{\partial y}{\partial t}(x, 0) = 0 \tag{42}$$

The displacement of the string $y(x, t)$ is governed by the inhomogeneous wave equation,

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2} + F(t) \tag{43}$$

(a) What are the relevant boundary conditions for this problem at $x = 0$ and $x = \infty$?

The boundary conditions in $x = 0$ are those given by the string being fixed $y(0, t) = 0$ for all values of $t \geq 0$. At any given time $t \geq 0$, the slope of the string must vanish at infinity, where the sting is unconstrained (this is the analog of the insulated end for heat flow). That is,

$$\lim_{x \to \infty} \frac{\partial y}{\partial x}(x, t) = 0. \tag{44}$$

(b) Solve this differential equation using method of Laplace transforms. Show that the Laplace transformation of the inhomogeneous wave equation yields an ordinary differential equation. Transform the boundary conditions and then solve the resulting differential equation. Finally, apply the relevant inverse Laplace transforms to obtain the final result.

Consider the Laplace transform with respect to the variable t,

$$L(y) = Y(x, p) = \int_0^\infty y(x, t) e^{-pt} dt \tag{45}$$

The properties of the Laplace transformation give

$$L\left(\frac{\partial y}{\partial t}\right) = pY - y(x, t = 0)\,, \qquad L\left(\frac{\partial^2 y}{\partial t^2}\right) = p^2 Y - p\left(\frac{\partial y}{\partial t}\right)_{t=0} - y(x, t = 0) = p^2 Y, \tag{46}$$

$$L\left(\frac{\partial^2 y}{\partial x^2}\right) = \frac{\partial^2 Y}{\partial x^2}\,, \qquad L(\cos \omega t) = \frac{p}{p^2 + \omega^2}, \tag{47}$$

where we have used the initial conditions given in (37). Thus, the wave equation (43) is transformed into

$$p^2 Y = v^2 \frac{d^2 Y}{dx^2} + \frac{p}{p^2 + \omega^2} \tag{48}$$

This is a simple ordinary differential equation in x (where p is a constant parameter). The most general solution can be determined by inspection,

$$Y(x, p) = Ae^{px/v} + Be^{-px/v} + \frac{1}{p(p^2 + \omega^2)}\,, \tag{49}$$

where A and B are constants that will be fixed by the boundary conditions.

The Laplace-transformed boundary conditions are:

$$Y(0,p) = 0, \qquad \lim_{x\to\infty} \frac{dY}{dx} = 0\,. \tag{50}$$

The solution for Y is therefore given by

$$Y(x,p) = \frac{1 - e^{-px/v}}{p(p^2+\omega^2)}\,. \tag{51}$$

We now have to perform an inverse Laplace transformation to obtain the desired solution $y(x,t)$. The first step is to rewrite the denominator of (51) by using the method of partial fractions,

$$\frac{1}{p(p^2+\omega^2)} = \frac{1}{\omega^2}\left[\frac{1}{p} - \frac{p}{p^2+\omega^2}\right]\,. \tag{52}$$

Using entries L1 and L3 of the Table of Laplace Transforms given on p. 469 of Boas, we deduce the following inverse Laplace transform,

$$L^{-1}\left(\frac{1}{p} - \frac{p}{p^2+\omega^2}\right) = 1 - \cos\omega t = 2\sin^2(\tfrac{1}{2}\omega t)\,. \tag{53}$$

Next, entry L24 of the Table of Laplace Transforms given on p. 470 of Boas yields[1]

$$L^{-1}\left(\frac{e^{-x/v}}{p}\right) = \Theta(vt - x) = \begin{cases} 1\,, & \text{if } x < vt\,, \\ 0\,, & \text{if } x > vt\,. \end{cases} \tag{54}$$

Finally, using entry L28 of the Table of Laplace Transforms with $g(t) = \cos\omega t$ yields,

$$L^{-1}\left(\frac{pe^{-x/v}}{p^2+\omega^2}\right) = \cos\omega(t - x/v)\Theta(vt - x) = \begin{cases} \cos\omega(t - x/v)\,, & \text{if } x < vt\,, \\ 0\,, & \text{if } x > vt\,. \end{cases} \tag{55}$$

Putting it all together,

$$y(x,t) = L^{-1}[Y(x,p)] = \frac{1}{\omega^2}\Big\{2\sin^2(\tfrac{1}{2}\omega t) - \Theta(vt-x)\,[1 - \cos\omega(t - x/v)]\Big\}\,, \tag{56}$$

which we can rewrite as

$$y(x,t) = \frac{2}{\omega^2}\Big\{\sin^2(\tfrac{1}{2}\omega t) - \Theta(vt - x)\sin^2[\tfrac{1}{2}\omega(t - x/v)]\Big\}\,. \tag{57}$$

Explicitly, the above result corresponds to

$$y(x,t) = \begin{cases} \frac{2}{\omega^2}\Big\{\sin^2(\tfrac{1}{2}\omega t) - \sin^2[\tfrac{1}{2}\omega(t - x/v)]\Big\}\,, & \text{if } x < vt\,, \\ \frac{2}{\omega^2}\sin^2(\tfrac{1}{2}\omega t)\,, & \text{if } x > vt\,. \end{cases} \tag{58}$$

Note that the two cases agree on the boundary when $x = vt$. The form of the above solution demonstrates that a signal induced by the external harmonic force propagates along the string with velocity v.

[1] I prefer to use Θ for the Heaviside step function rather than u, which is used by Boas in entry L24 of the table.

6

Suppose you have two quarters and a dime in your left pocket and two dimes and three quarters in your right pocket. You select a pocket at random and from it a coin at random.

(a) What is the probability that the coin you selected is a dime?

There is a probability $\frac{1}{2}$ to get each of the pockets, and then a probability $\frac{1}{3}$ of getting any coin in the left pocket and a probability of $\frac{1}{5}$ of getting any coin from the right pocket. The probability that we get a dime is

$$P(D) = \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{2}{5} = \frac{11}{30} \tag{59}$$

(b) Let x be the amount of money you selected. What is the expectation value, $E(x)$?

We compute the expected value in cents. Let A be the event where a quarter is selected and D be the event where a dime is selected. Then,

$$E(x) = 25 \cdot P(Q) + 10 \cdot P(D) = 25 \cdot \frac{19}{30} + 10 \cdot \frac{11}{30} = 19.5 \tag{60}$$

(c) Suppose you selected a dime in part (a). What is the probability that it came from your right pocket?

This is a conditional probability. Let R be the event where the coin comes from the right pocket, and let D be the event that the coin drawn is a dime. Then, the probability that the coin came from your right pocket is $P(R|D)$ which is given by

$$P(R|D) = \frac{P(R \cap D)}{P(D)}\,, \tag{61}$$

using Bayes' formula. Using (59), the probability for events R *and* D is given by

$$P(R \cap D) = \frac{1}{2} \cdot \frac{2}{5} = \frac{1}{5}\,. \tag{62}$$

Hence,

$$P(R|D) = \frac{1/5}{11/30} = \frac{6}{11}\,. \tag{63}$$

The same result can be obtained by denoting by D_L the event that a dime is drawn from the left pocket and by D_R the event that the dime is drawn from the right pocket. Then, the calculation of part (a) shows that

$$P(D_L) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6} \quad \text{and} \quad P(D_R) = \frac{1}{2} \cdot \frac{2}{5} = \frac{1}{5}\,. \tag{64}$$

As expected, $P(D) = P(D_L) + P(D_R) = 11/30$. Then, if you select a dime, the probability that it came from your right pocket is simply

$$\frac{P(D_R)}{P(D_L) + P(D_R)} = \frac{1/6}{11/30} = \frac{6}{11}\,, \tag{65}$$

in agreement with (63).

(d) Suppose you do not replace the dime, but select another coin which is also a dime. What is the probability that this second coin came from your right pocket?

Let us denote by D_R the event where a dime is drawn from the right pocket at by D_L the event where the dime is drawn from the left pocket. Then,

$$P(DD) = P(D_L D_R) + P(D_R D_L) + P(D_R D_R)\,. \tag{66}$$

Note that $P(D_L D_L) = 0$ since the left pocket contains only one dime. The corresponding probabilities are straightforward:

$$P(D_L D_R) = \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{2}\cdot\frac{2}{5} = \frac{1}{30}\,, \tag{67}$$
$$P(D_R D_L) = \frac{1}{2}\cdot\frac{2}{5}\cdot\frac{1}{2}\cdot\frac{1}{3} = \frac{1}{30}\,, \tag{68}$$
$$P(D_R D_R) = \frac{1}{2}\cdot\frac{2}{5}\cdot\frac{1}{2}\cdot\frac{1}{4} = \frac{1}{40}\,. \tag{69}$$

In the last computation, we noted that given that the first dime came from the right pocket, the second pocket is then left with three quarters and a dime, meaning that there is only a one in four chance that a coin drawn from the right pocket would be a dime. Hence,

$$P(DD) = \frac{1}{30} + \frac{1}{30} + \frac{1}{40} = \frac{11}{120}\,. \tag{70}$$

Let R_2 be the event that the second coin is drawn from the right pocket. Then if two dimes are drawn without replacement, then the probability that the second dime comes from your right pocket is $P(R_2|DD)$ which is given by

$$P(R_2|DD) = \frac{P(R_2 \cap DD)}{P(DD)}\,, \tag{71}$$

using Bayes' formula. The probability for the event R_2 *and* DD is given by

$$P(R_2 \cap DD) = P(D_L D_R) + P(D_R D_R) = \frac{1}{30} + \frac{1}{40} = \frac{7}{120}\,, \tag{72}$$

since in both cases the second dime came from the right pocket. Hence,

$$P(R_2|DD) = \frac{7/120}{11/120} = \frac{7}{11}\,. \tag{73}$$

One can also use an argument analogous to the one presented below (63). Namely, if two dimes are drawn without replacement, then the probability that the second dime comes from your right pocket is given by

$$\frac{P(D_L D_R) + P(D_R D_R)}{P(D_L D_R) + P(D_R D_L) + P(D_R D_R)} = \frac{7/120}{11/120} = \frac{7}{11}\,. \tag{74}$$

7

If four letters are put at random into four envelopes, what is the probability that at least one letter gets into a correct envelope?

Let us count the number of possibilities; there are $4! = 24$ ways in which the envelopes can be assorted.

- to have only 1 correct envelope: there are 4 ways to chose the right envelope, then the other three letters must go in the wrong envelopes, and there are two ways of doing this. We then have $4 \cdot 8$ possibilities.
- to have 2 correct envelopes, they could be any 2 out of the 4, that is, we have $C(4,2) = 6$ ways of choosing the right ones; then the other two must be chosen in the wrong way, and there is only one way to do this. We have $6 \cdot 1$ possibilities.
- if we put 3 correct envelopes, the fourth is also going to be right. The there is 1 case for both 3 and 4 correct envelopes.

We then have 15 possibilities out of 24 to put at least one letter inside the right envelope. Hence,

$$p = \frac{15}{24} = \frac{5}{8}\,. \tag{75}$$

Another way to do this problem is to compute the probability that no letter gets into a correct envelope. This is equivalent to asking the following question. Starting with the numbers 1234 in order, how many different permutations are there in which either 1 does not occur in the first position, 2 does not occur in the second position, 3 does not occur in the third position, or 4 does not in the fourth position? Such permutations, in which none of the numbers end up in their original positions, are called *derangements*. The probability that no letter gets into a correct envelope is then equal to $D_4/4!$, where D_4 is the number of derangements of four objects. It is an interesting exercise to compute D_n for arbitrary n. In the case of $n = 4$, one can simply enumerate all nine possible derangements explicitly:

$$2143\,,\ 2341\,,\ 2413\,,\ 3142\,,\ 3412\,,\ 3421\,,\ 4123\,,\ 4312\,,\ 4321\,. \tag{76}$$

Hence the probability of a derangement is $9/24 = 3/8$. Therefore, the probability that with four letters and four envelopes, at least one letter gets into a correct envelope is

$$p = 1 - \frac{3}{8} = \frac{5}{8}\,. \tag{77}$$

<u>*REMARK*</u>: The general formula for D_n is not so hard to derive. It is just an exercise in counting. The key step is to generalize the result of problem 15–3.8 on p. 734 of Boas to the probability of the union of n events. Then, if we define A_i be a permutation where the position of the ith number is unchanged, one can compute the probability of the event $A_1 \cup A_2 \cup \cdots \cup A_n$ (where "union" means "or"). Subtracting this probability from 1 yields the probability of a derangement,[2]

$$\frac{D_n}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!}\,. \tag{78}$$

Let us test this out for $n = 4$. The above formula then yields,

$$24\left[1 - 1 + \tfrac{1}{2} - \tfrac{1}{6} + \tfrac{1}{24}\right] = 24 - 24 + 12 - 4 + 1 = 9\,, \tag{79}$$

in agreement with our explicit computation above. An interesting fact about derangements is that the probability that a given permutation is a derangement in the limit of $n \to \infty$ is

$$\lim_{n\to\infty} \frac{D_n}{n!} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = e^{-1} \simeq 0.367\,. \tag{80}$$

Given this result, you should now be able to answer problem 15–11.5 of on p. 776 of Boas.

[2]See, e.g. pp. 104–106 of Charles M. Grinstead and J. Laurie Snell, *Introduction to Probability* (American Mathematical Society, Providence, RI, 1997). A link to this textbook can be found on the class webpage.

8

A bit (i.e., a binary digit) is 0 or 1. An ordered array of eight bits (such as 01101001) is a byte.

(a) How many different bytes are there?

There are $2^8 = 256$ different bytes.

(b) If you select a byte at random, what is the probability that you select a byte containing three 1's and five 0's?

We must count the number of bytes with three 1's and five 0's: these is equivalent to the number of taking 3 indistinguishable objects out of a set of 8. This is $C(8,3)$:

$$C(8,3) = \frac{8!}{5!3!} = 8 \cdot 7 \qquad \Longrightarrow \qquad P = \frac{C(8,3)}{2^8} = \frac{7}{32} \tag{81}$$

9

A true coin is tossed 10,000 times.

(a) Find the probability of getting exactly 5000 heads.

This is given by the binomial distribution:

$$P^B_{5000} = \frac{10000!}{5000!5000!} 2^{-10000} = 0.00797865 \tag{82}$$

where we used Mathematica to do the calculation. For the normal approximation, with $\mu = np = 5000$, $\sigma^2 = npq = 2500 = 50^2$,

$$P^N_{5000} = \frac{1}{\sqrt{2\pi}\sigma} = 0.00797885 \tag{83}$$

the difference is as small as $2 \cdot 10^{-7}$.

(b) Find the probability of getting between 4900 and 5075 heads.

Using the binomial distribution, we have to sum between 4900 and 5075:

$$P^B = \sum_{n=4900}^{5075} \frac{10000!}{n!(10000-n)!} 2^{-10000} = 0.912267 \tag{84}$$

With the normal distribution, this is an integral

$$P^N = \frac{1}{\sqrt{2\pi}\sigma} \int_{4900}^{5075} e^{-(x-\mu)^2/2\sigma^2} dx = \frac{1}{2}\mathrm{erf}(1.5/\sqrt{2}) + \frac{1}{2}\mathrm{erf}(2/\sqrt{2}) = 0.910443 \tag{85}$$

In class, I argued that the fractional error of the normal approximation should be of $\mathcal{O}(1/\sqrt{n})$. In this problem $n = 10000$ so we should expect an error of the order of 1%. This is consistent with the fact that P^B and P^N differ in the third decimal place.

10

Suppose a 200-page book has, on average, one misprint every ten pages. On about how many pages would you expect to find two misprints?

There are 20 errors in the whole book. The average number of errors per page is 0.1, and we want to know the probability to have two errors on a single page. This is given by the Poisson distribution:

$$P_2 = \frac{(0.1)^2}{2!} e^{-0.1} = 0.0045 \tag{86}$$

The average number of pages with two errors will be $(0.0045)(200) = 0.9$. That is, we expect to find two misprints on at most one page in the book.

11

Let $x_1,\ x_2,\ \ldots, x_n$ be independent random variables, each with probability density function $f(x)$, mean μ and variance σ^2. Define the sample mean by

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \tag{87}$$

Compute the expectation value $E(\bar{x})$ and the variance $\mathrm{Var}(\bar{x})$.

The expectation value is

$$E(\bar{x}) = \frac{1}{n}\sum_i E(x_i) = \mu, \tag{88}$$

where we have used the fact that $E(x_i) = \mu$ independently of the index i.

The variance is

$$\mathrm{Var}(\bar{x}) = \frac{1}{n^2}\sum_i \mathrm{Var}(x_i)\,, \tag{89}$$

since x_i and x_j are independent events for $i \neq j$. We have also used the fact that $\mathrm{Var}(cx) = c^2\,\mathrm{Var}(x)$, when c is a constant and x is a random variable. Hence, using $\mathrm{Var}(x_i) = \sigma^2$ independently of the index i, it follows that

$$\mathrm{Var}(\bar{x}) = \frac{1}{n^2}\cdot n\sigma^2 = \frac{\sigma^2}{n}\,. \tag{90}$$

Hence, the expectation value of the mean is μ and the standard deviation of the mean is $\sigma/\sqrt{n}$.

<u>*REMARK:*</u>

One can also derive a formula for $\mathrm{Var}(\bar{x})$ starting from the definition of the variance. Recall that $\mathrm{Var}(x) = E[(x-\mu)^2] = E(x^2) - [E(x)]^2$. Then,

$$\begin{aligned}
\mathrm{Var}(\bar{x}) &= E(\bar{x}^2) - [E(\bar{x})]^2 = E\left(\left[\frac{1}{n}\sum_i x_i\right]^2\right) - \mu^2 = \frac{1}{n^2}E\left(\sum_i x_i^2 + \sum_{i\neq j} x_i x_j\right) - \mu^2 \\
&= \frac{1}{n^2}\sum_i E(x_i^2) + \frac{1}{n^2}\sum_{i\neq j} E(x_i x_j) - \mu^2 = \frac{1}{n^2}\left(n\sigma^2 + n\mu^2 + \sum_{i\neq j} E(x_i x_j)\right) - \mu^2\,,
\end{aligned} \tag{91}$$

where we have used $\sigma^2 = E(x_i^2) - [E(x_i)]^2 = E(x_i^2) - \mu^2$, independently of the index i. The expectation value of the product of two *independent* variables is $E(x_i x_j) = E(x_i)E(x_j) = \mu^2$ for $i \neq j$. Hence, noting that there are $n(n-1)$ terms in the sum $\sum_{i\neq j}$,

$$\mathrm{Var}(\bar{x}) = \frac{1}{n^2}\left[n\sigma^2 + n\mu^2 + n(n-1)\mu^2\right] - \mu^2 = \frac{\sigma^2}{n}\,, \tag{92}$$

which reproduces the result of (90).

12

Suppose that x and y are discrete random variables, not necessarily independent.

(a) Prove that

$$E(xy) = E(x)E(y) + \mathrm{Cov}(x,y)\,, \tag{93}$$

where $\mathrm{Cov}(x,y)$ is covariance of x and y.

Denote the joint probability distribution by p_{ij}. Since the total probability must be equal to one, it follows that

$$\sum_i \sum_j p_{ij} = 1\,. \tag{94}$$

Then, we have

$$E(xy) = \sum_i \sum_j x_i y_j\, p_{ij}\,, \tag{95}$$

$$E(x) = \sum_i \sum_j x_i\, p_{ij}\,, \tag{96}$$

$$E(y) = \sum_i \sum_j y_j\, p_{ij}\,, \tag{97}$$

$$\mathrm{Cov}(x,y) = \sum_i \sum_j [x_i - E(x)][y_j - E(y)]p_{ij}\,. \tag{98}$$

It then follows that:

$$\begin{aligned}
\mathrm{Cov}(x,y) &= \sum_i \sum_j x_i y_j\, p_{ij} - E(x)\sum_i \sum_j y_j\, p_{ij} - E(y)\sum_i \sum_j x_i\, p_{ij} + E(x)E(y)\sum_i \sum_j p_{ij} \\
&= E(xy) - E(x)E(y) - E(y)E(x) + E(x)E(y) \\
&= E(xy) - E(x)E(y)\,. \qquad (99)
\end{aligned}$$

after making use of eqs. (94)–(98). It then follows that:

$$E(xy) = E(x)E(y) + \mathrm{Cov}(x,y)\,, \tag{100}$$

as required.

(b) A set of n measurements are made (called the "sample"), and the resulting data are (x_1, y_1), $(x_2, y_2), \ldots, (x_n, y_n)$. You may assume that each measurement is independent, which implies that (x_i, y_i) and (x_j, y_j) are independent for $i \neq j$. But you cannot assume that x_i and y_i are independent. We wish to estimate the population covariance. Consider

$$V_n(x, y) = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y}), \tag{101}$$

where $\bar{x} = \frac{1}{n}\sum_i x_i$ and $\bar{y} = \frac{1}{n}\sum_i y_i$ are the corresponding sample means. Evaluate the expectation value of $V_n(x, y)$ and prove that $E(V_n(x, y)) = \mathrm{Cov}(x, y)$.

First, we note that that we can rewrite $V_n(x, y)$ by multiplying out the terms,

$$V_n(x, y) = \frac{1}{n-1}\left\{\sum_{n=1}^{n} x_i y_i - n\bar{x}\bar{y} - n\bar{x}\bar{y} + n\bar{x}\bar{y}\right\} = \frac{1}{n-1}\left\{\sum_{n=1}^{n} x_i y_i - n\bar{x}\bar{y}\right\}. \tag{102}$$

Then,

$$\begin{aligned}
E(V_n(x, y)) &= \frac{1}{n-1}\left\{\sum_{n=1}^{n} x_i y_i - nE(\bar{x}\bar{y})\right\} \\
&= \frac{1}{n-1}\left\{\sum_{n=1}^{n} x_i y_i - \frac{1}{n}E\left(\sum_{i=1}^{n} x_i \sum_{j=1}^{n} y_j\right)\right\} \\
&= \frac{1}{n-1}\left\{\sum_{n=1}^{n} x_i y_i - \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} E(x_i y_i)\right\} \\
&= \frac{1}{n-1}\left\{\left(1 - \frac{1}{n}\right)\sum_{n=1}^{n} x_i y_i - \frac{1}{n}\sum_{\substack{i=1 \\ \text{for } i\neq j}}^{n}\sum_{j=1}^{n} E(x_i y_j)\right\}.
\end{aligned} \tag{103}$$

Since x_i and y_j are independent when $i \neq j$, it follows that

$$E(x_i y_j) = E(x_i)E(y_j) = \mu_x \mu_y, \qquad \text{for } i \neq j, \tag{104}$$

where $E(x_i) = \mu_x$ independently of the index i and $E(y_j) = \mu_y$ independently of the index j. Since x_i and y_i may not be independent, it follows from part (a) that

$$E(x_i y_i) = E(x_i)E(y_i) + \mathrm{Cov}(x_i, y_i) = \mu_x \mu_y + \mathrm{Cov}(x, y), \tag{105}$$

since $\mathrm{Cov}(x_i, y_i) = \mathrm{Cov}(x, y)$ independently of the index i. Inserting these results into (103), and noting that there are $n(n-1)$ terms in the double sum $\sum_{i\neq j}$, we end up with:

$$E(V_n(x, y)) = \frac{1}{n-1}\left\{\left(\frac{n-1}{n}\right) n\left[\mu_x \mu_y + \mathrm{Cov}(x, y)\right] - \frac{n(n-1)}{n}\mu_x \mu_y\right\} = \mathrm{Cov}(x, y), \tag{106}$$

as required.