Physics 171 FINAL EXAM SOLUTIONS Fall 2001

(1) We wish to prove that for any antisymmetric tensor, $F^{\mu\nu} = -F^{\nu\mu}$,

$$F^{\mu\nu}{}_{;\nu;\mu} = 0.$$

First, we use the results of problem 1(b) on problem set #4 to obtain:

$$F^{\mu\nu}{}_{;\nu;\mu} - F^{\mu\nu}{}_{;\mu;\nu} = -R^{\mu}{}_{\rho\nu\mu} F^{\rho\nu} - R^{\nu}{}_{\rho\nu\mu} F^{\mu\rho}$$
$$= R_{\rho\nu} F^{\rho\nu} - R_{\rho\mu} F^{\mu\rho}$$
$$= 0.$$

Above, we have used

$$R^{\nu}{}_{\rho\nu\mu} = -R^{\nu}{}_{\rho\mu\nu} = R_{\rho\nu}$$

which defines the Ricci tensor. Furthermore, $R_{\rho\nu} = R_{\nu\rho}$ and $F^{\rho\nu} = -F^{\nu\rho}$ implies that

$$R_{\rho\nu} F^{\rho\nu} = 0.$$

We conclude that

$$F^{\mu\nu}{}_{;\nu;\mu} = F^{\mu\nu}{}_{;\mu;\nu}$$
$$= -F^{\nu\mu}{}_{;\mu;\nu} \qquad \text{using } F^{\mu\nu} = -F^{\nu\mu}$$
$$= -F^{\mu\nu}{}_{;\nu;\mu} \qquad \text{after relabeling of indices } (\mu\to\nu,\ \nu\to\mu).$$

Hence,

$$2F^{\mu\nu}{}_{;\nu;\mu} = 0$$

or

$$F^{\mu\nu}{}_{;\nu;\mu} = 0.$$

Since $F^{\mu\nu} = -F^{\nu\mu}$, this is also equivalent to $F^{\mu\nu}{}_{;\mu;\nu} = 0$.

(2) Consider

$$ds^2 = e^{-2ax/c^2} c^2 dt^2 - dx^2 - dy^2 - dz^2$$

(a) To compute the Christoffel symbols, we use the Lagrangian

$$L = e^{-2ax/c^2} c^2 \dot{t}^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2$$

and compute the equations of motion using the Lagrange equations.

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{t}}\right) = \frac{\partial L}{\partial t} \implies \frac{d}{ds}\left(e^{-2ax/c^2}\dot{t}\right) = 0$$

$$e^{-2ax/c^2}\ddot{t} - \frac{2a}{c^2} e^{-2ax/c^2}\dot{x}\dot{t} = 0$$

$$\ddot{t} - \frac{2a}{c^2}\dot{x}\dot{t} = 0$$

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} \implies \frac{d}{ds}(-2\dot{x}) = -\frac{2a}{c^2} e^{-2ax/c^2} c^2 \dot{t}^2$$

$$\ddot{x} - a e^{-2ax/c^2}\dot{t}^2 = 0$$

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{y}}\right) = \frac{\partial L}{\partial y} \implies \ddot{y} = 0$$

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{z}}\right) = \frac{\partial L}{\partial z} \implies \ddot{z} = 0$$

Thus, the non-zero Christoffel symbols are

$$\Gamma^1_{00} = \frac{-a}{c^2} e^{-2ax/c^2}$$

$$\Gamma^0_{01} = \Gamma^0_{10} = \frac{-a}{c^2}$$

where we have put $x^0 = ct$ and compared the above results with

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0$$

(b) Suppose that at $t=0$ we have $\frac{dx}{dt}=\frac{dy}{dt}=\frac{dz}{dt}=0$. Then from part (a),

$$\frac{d}{ds}\left(e^{-2ax/c^2}\dot{t}\right)=0$$

implies that

$$\frac{dt}{d\tau}=Ke^{2ax/c^2}$$

for some constant K. It follows that at $t=0$, $\frac{dx}{d\tau}=\frac{dx}{dt}\frac{dt}{d\tau}=0$, etc.
Thus, $\dot{x}=\dot{y}=\dot{z}=0$ at $t=0$.
Because $L=g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}=1$, we have

$$e^{-2ax/c^2}c^2\dot{t}^2-\dot{x}^2-\dot{y}^2-\dot{z}^2=1.$$

It follows that

$$e^{-2ax/c^2}c^2\dot{t}^2=1 \qquad \text{at } t=0.$$

In part (a), we found that

$$\ddot{x}=ae^{-2ax/c^2}\dot{t}^2$$

Thus, at $t=0$, we can combine the last two equations to obtain:

$$\ddot{x}=\frac{d^2x}{ds^2}=\frac{a}{c^2} \qquad \text{at } t=0.$$

Since $s=c\tau$, we conclude that

$$\frac{d^2x}{d\tau^2}=a \qquad \text{at } t=0.$$

(c) We use:

$$R_{\mu\nu}=\Gamma^\alpha_{\mu\nu,\alpha}-\Gamma^\alpha_{\mu\alpha,\nu}+\Gamma^\rho_{\mu\nu}\Gamma^\sigma_{\rho\sigma}-\Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\rho\nu}$$

to compute $R_{\mu\nu}$. In part (a), we saw that the only non-zero Christoffel symbols are Γ^1_{00}, Γ^0_{01} and Γ^0_{10}.

Thus,

$$\begin{aligned}R_{00}&=\Gamma^1_{00,1}+\Gamma^1_{00}\Gamma^0_{10}-\Gamma^1_{00}\Gamma^0_{10}-\Gamma^0_{01}\Gamma^1_{00}\\&=-\frac{a}{c^2}\frac{d}{dx}e^{-2ax/c^2}-\frac{a^2}{c^4}e^{-2ax/c^2}\\&=\frac{a^2}{c^4}e^{-2ax/c^2}\end{aligned}$$

$$\begin{aligned}R_{11}&=-\Gamma^0_{10,1}-\Gamma^0_{10}\Gamma^0_{01}\\&=-\frac{a^2}{c^4}\end{aligned}$$

$$R_{01}=\Gamma^0_{01,0}=0$$

All other components of $R_{\mu\nu}$ are trivially zero.

(d) The Einstein equations are

$$G_{\mu\nu}=\frac{8\pi G}{c^4}T_{\mu\nu}$$

where $G_{\mu\nu}=R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R$.

We must now evaluate $G_{\mu\nu}$. First,

$$\begin{aligned}R=g^{\mu\nu}R_{\mu\nu}&=g^{00}R_{00}+g^{11}R_{11}\\&=e^{2ax/c^2}R_{00}-R_{11} &&\text{since } g^{00}=\frac{1}{g_{00}}=e^{2ax/c^2}\\&=\frac{2a^2}{c^4} &&\text{and } g^{11}=\frac{1}{g_{11}}=-1.\end{aligned}$$

Thus,

$$\begin{aligned}G_{00}&=R_{00}-\frac{a^2}{c^4}g_{00}=0\\G_{11}&=R_{11}-\frac{a^2}{c^4}g_{11}=0\\G_{22}&=R_{22}-\frac{a^2}{c^4}g_{22}=\frac{a^2}{c^4} &&\text{since } R_{22}=R_{33}=0\\G_{33}&=R_{33}-\frac{a^2}{c^4}g_{33}=\frac{a^2}{c^4} &&\text{and } g_{22}=g_{33}=-1\end{aligned}$$

All other components of $G_{\mu\nu}$ are zero.

It follows that

$$T_{\mu\nu} = \frac{c^4}{8\pi G} G_{\mu\nu}$$

Hence, the only non-zero components of $T_{\mu\nu}$ are:

$$T_{22} = T_{33} = \frac{a^2}{8\pi G}$$

This does not seem to correspond to any "reasonable" matter distribution. For example, we saw that for a perfect fluid,

$$T^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) u^\mu u^\nu - g^{\mu\nu} p$$

In the non-relativistic limit where $u^\mu \simeq (c;0,0,0)$, we have

$$T^{00} \simeq c^2 \rho$$

But in our example, $T_{00} = 0$ (which implies $T^{00} = (g^{00})^2 T_{00} = 0$). That is, $\rho = 0$. In "reasonable" matter distributions, one finds $\rho \geq |p|$.

(e) As a check on our algebra, we should verify that $T^{\mu\nu}{}_{;\nu} = 0$. Use the fact that

$$T^{\mu\nu}{}_{;\nu} = T^{\mu\nu}{}_{,\nu} + \Gamma^\mu_{\sigma\nu} T^{\sigma\nu} + \Gamma^\nu_{\sigma\nu} T^{\mu\sigma}$$

From part (d), after raising indices we have $T^{22} = T^{33} = \dfrac{a^2}{8\pi G}$.

But this is a constant, so $T^{\mu\nu}{}_{,\nu} = 0$. Moreover, since Γ^1_{00}, Γ^0_{01} and Γ^0_{10} are the only non-vanishing Christoffel symbols, it is clear that

$$T^{\mu\nu}{}_{;\nu} = 0.$$

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③ (a) For matter ($p=0$),

$$\rho_m R^3 = \text{constant}$$

For radiation ($p = \frac{1}{3}\rho c^2$),

$$\rho_r R^4 = \text{constant}$$

Thus, we can write

$$\rho_m R^3 = \rho_{m0} R_0^3 \qquad \text{where } R_0 \equiv R(t_0)$$

$$\rho_r R^4 = \rho_{r0} R_0^4$$

where the subscript zero refers to today. Dividing the two equations yields:

$$\frac{\rho_r}{\rho_m} R = \frac{\rho_{r0}}{\rho_{m0}} R_0$$

If we assume that $\rho_r = \rho_m$, then

$$\frac{R}{R_0} = \frac{\rho_{r0}}{\rho_{m0}} = \frac{\rho_{r0}/\rho_{c0}}{\rho_{m0}/\rho_{c0}} = \frac{\Omega_r}{\Omega_m}$$

Using $\Omega_r = \Omega_\gamma = 5 \times 10^{-5}$ [we neglect the neutrinos, here]

$\Omega_m = 0.3$

we obtain

$$\boxed{\frac{R}{R_0} = 1.67 \times 10^{-4}}$$

(b) In class [see Kenyon p. 152], we showed that:

$$\frac{T}{T_0} = \frac{R_0}{R}$$

Using $T_0 = 2.725\,°K$ and the results of part (a),

$$T = 16{,}350\,°K$$

at the epoch at which $\rho_\gamma = \rho_m$.

(c) In class, we derived the result:

$$H^2 = H_0^2\left[\frac{\Omega_m}{a^3} + \frac{\Omega_r}{a^4} + \Omega_\Lambda + \frac{1-\Omega_T}{a^2}\right]$$

where $a(t) \equiv \frac{R(t)}{R_0}$. Noting that $H \equiv \frac{1}{R}\frac{dR}{dt} = \frac{1}{a}\frac{da}{dt}$,

$$\frac{da}{dt} = H_0\left[\frac{\Omega_m}{a} + \frac{\Omega_r}{a^2} + \Omega_\Lambda a^2 + 1 - \Omega_T\right]^{1/2}$$

In the time interval $0 \le t \le t_{eq}$, where t_{eq} corresponds to $\rho_m = \rho_r$, we may neglect Ω_Λ and $1-\Omega_T$ above, since according to part (a), the scale factor varies in the range $0 \le a \le a_{eq}$ with $a_{eq} = 1.67\times10^{-4}$. Thus,

$$\frac{da}{dt} = \frac{H_0}{a}(\Omega_r + \Omega_m a)^{1/2}$$

Integrating,

$$H_0\int_0^{t_{eq}} dt = \int_0^{a_{eq}} \frac{a\,da}{(\Omega_r + \Omega_m a)^{1/2}}$$

From part (a), $a_{eq} = \frac{R(t_{eq})}{R_0} = \frac{\Omega_r}{\Omega_m}$. Thus,

$$H_0 t_{eq} = \frac{1}{\Omega_m^{1/2}}\int_0^{a_{eq}} \frac{a\,da}{(a_{eq}+a)^{1/2}}$$

$$= \frac{1}{\Omega_m^{1/2}}\left[\frac{2}{3}(a_{eq}+a)^{3/2} - 2a_{eq}(a_{eq}+a)^{1/2}\right]\Bigg|_0^{a_{eq}}$$

$$= \frac{1}{\Omega_m^{1/2}}\left[\frac{2}{3}(2a_{eq})^{3/2} - (2a_{eq})^{3/2} - \frac{2}{3}a_{eq}^{3/2} + 2a_{eq}^{3/2}\right]$$

$$= \frac{a_{eq}^{3/2}}{\Omega_m^{1/2}}\left(\frac{4}{3} - \frac{2}{3}\sqrt{2}\right)$$

Putting $a_{eq} = \Omega_r/\Omega_m$, we end up with

$$\boxed{t_{eq} = H_0^{-1}\frac{\Omega_r^{3/2}}{\Omega_m^2}\left(\frac{4}{3} - \frac{2}{3}\sqrt{2}\right)}$$

Putting in the numbers: $H_0^{-1} = 14\,\text{Gyr}$

$\Omega_r = 5\times10^{-5}$

$\Omega_m = 0.3$

we obtain

$$t_{eq} = 2.15\times10^4 \text{ years}$$

Remark: A more accurate calculation would include the effect of the neutrinos. One can show that three approximately massless neutrino types contribute roughly $0.68\,\Omega_\gamma$ to Ω_r. Thus, a better estimate for Ω_r is:

$$\Omega_r = 1.68\,\Omega_\gamma \simeq 8.5\times10^{-5}.$$

We then obtain

$$t_{eq} = 4.76\times10^4 \text{ years}.$$