DUE: TUESDAY OCTOBER 2, 2001

1. Suppose $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{b}}$ are proper three-vectors, while $\vec{\boldsymbol{c}}$ and $\vec{\boldsymbol{d}}$ are improper three-vectors (also called pseudovectors) . Show that

(a) $\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{b}}$ and $\vec{\boldsymbol{c}}\cdot\vec{\boldsymbol{d}}$ are both scalar quantities, whereas $\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{c}}$ is a pseudoscalar.

(b) $\vec{\boldsymbol{a}}\times\vec{\boldsymbol{b}}$ and $\vec{\boldsymbol{c}}\times\vec{\boldsymbol{d}}$ are pseudovectors, whereas $\vec{\boldsymbol{a}}\times\vec{\boldsymbol{c}}$ is a proper vector.

2. Consider a tensor T_{ij} in three dimensional Euclidean space. Under an arbitrary rotation of the three-dimensional coordinate space, the tensor is transformed. Show that:

(a) if T_{ij} is symmetric and traceless, then the transformed tensor is traceless and symmetric.

(b) if T_{ij} is antisymmetric, then the transformed tensor is antisymmetric.

State the analogous result for a tensor $T^{\mu\nu}$ in four-dimensional Minkowski space. Define carefully what you mean by a traceless tensor in this case.

3. Under a Lorentz transformation, a tensor transforms as follows:

$$F'^{\mu\nu} = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}F^{\alpha\beta}\,,$$

where Λ is the Lorentz transformation matrix. Consider an inertial frame K at rest, and a second inertial frame K' moving with velocity v along the x-direction with respect to K. Using the explicit result for Λ corresponding to the transformation between K and K', determine the electric and magnetic fields in frame K' in terms of the corresponding fields in frame K.

4. In electrodynamics, one can introduce a scalar potential Φ and a vector potential $\vec{\boldsymbol{A}}$ as follows (using CGS units):

$$\vec{\boldsymbol{E}} = -\vec{\boldsymbol{\nabla}}\Phi - \frac{1}{c}\frac{\partial\vec{\boldsymbol{A}}}{\partial t}\,, \qquad \vec{\boldsymbol{B}} = \vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{A}}\,.$$

Define the four-vector potential $A^\mu = (\Phi\,;\,\vec{\boldsymbol{A}})$.

(a) Verify that the electromagnetic field strength tensor $F^{\mu\nu}$ can be written in terms of the four-vector potential as

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu\,.$$

(b) Two of the four Maxwell's equations can be expressed in relativistic notation as:

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu\,. \tag{1}$$

Noting that $F^{\mu\nu}$ is an antisymmetric tensor, show that current conservation, $\partial_\nu J^\nu = 0$, is automatically satisfied.

(c) Define the dual electromagnetic field strength tensor by:

$$\widetilde{F}^{\mu\nu} \equiv \tfrac{1}{2}\epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}\,.$$

Explicitly work out the components of $\widetilde{F}^{\mu\nu}$. A *duality* transformation is an abstract transformation that changes $F^{\mu\nu}$ into $\widetilde{F}^{\mu\nu}$. How do the $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ fields change under this transformation?

(d) Using the result of part (a), show that

$$\partial_\mu \widetilde{F}^{\mu\nu} = 0\,. \tag{2}$$

By explicitly working out the components of this equation, show that one obtains the other two Maxwell equations.

REMARK: If $J^\nu = 0$ (no external currents), note the symmetry of Maxwell's equations [eqs. (1) and (2)] under the duality transformation.

5. Supernova 1987A was observed to occur at a distance of 170,000 light years from earth. Neutrinos and photons emitted from the supernova were observed by detectors on earth. The neutrinos (on average) had an energy of 10 MeV. Assume that the neutrinos have a rest mass-energy of $mc^2 = 1$ eV (whereas photons are massless).

(a) What is the elapsed time for one of the neutrinos to travel from the supernova to earth (as measured in the neutrino's frame)?

(b) Suppose the neutrino and photon were initially emitted from the supernova at the same time. How much later than the photon will the neutrino arrive at earth?