DUE: TUESDAY OCTOBER 30, 2001

MIDTERM ALERT: The midterm exam will be a take-home exam. The exam will be handed out in class on Thursday November 1, and should be handed back to me in my office on Monday November 5 at 2 pm. The exam will be based on the material covered on the first three problem sets, and on the material in the first six chapters of Kenyon. While working on the exam, you are permitted to consult with your class notes, Kenyon and one other relativity textbook and/or mathematics book of your choosing. However, you should *not* collaborate with anyone else during the exam.

1. (a) Show that raising and lowering of indices commutes with covariant differentiation; *e.g.*, $A_{\mu\,;\alpha} = g_{\mu\nu} A^{\nu}{}_{;\alpha}$.

(b) Write the expression for $A^{\alpha}{}_{\beta\rho\,;\sigma}$ in terms of $A^{\alpha}{}_{\beta\rho\,,\sigma}$ and expressions involving the metric connection coefficients $\Gamma^{\mu}{}_{\alpha\beta}$.

(c) Show that if $g_{\mu\nu}$ is diagonal, then $\Gamma^{\mu}{}_{\alpha\beta} = 0$ if μ, α and β are all different.

2. Suppose that A_μ is a covariant vector and $F_{\mu\nu}$ is an antisymmetric tensor. Prove that:

(a) $A_{\mu\,;\nu} - A_{\nu\,;\mu} = A_{\mu\,,\nu} - A_{\nu\,,\mu}$

(b) $F_{\mu\nu\,;\rho} + F_{\rho\mu\,;\nu} + F_{\nu\rho\,;\mu} = F_{\mu\nu\,,\rho} + F_{\rho\mu\,,\nu} + F_{\nu\rho\,,\mu}\,.$

(c) Using the strong equivalence principle, find the appropriate generalization of Maxwell's equations in curved spacetime. How should the continuity equation, which describes the conservation of charge, be generalized?

HINT: Make sure that your equations are generally covariant. The results of parts (a) and (b) should be useful.

3. Prove that

$$\Gamma^{\mu}{}_{\mu\nu} = \frac{1}{2g}\frac{\partial g}{\partial x^{\nu}} = \frac{1}{\sqrt{-g}}\frac{\partial\sqrt{-g}}{\partial x^{\nu}}\,,$$

where $g \equiv \det\, g_{\mu\nu}$. Note the implicit sum over the index μ. Using this result, show that if A^{μ} is a contravariant vector and $F^{\mu\nu}$ is an antisymmetric tensor, then

(a) $A^{\mu}{}_{;\,\mu} = \dfrac{1}{\sqrt{-g}}\,\partial_{\mu}(\sqrt{-g}\,A^{\mu})\,,$

(b) $F^{\mu\nu}{}_{;\,\mu} = \dfrac{1}{\sqrt{-g}}\,\partial_{\mu}(\sqrt{-g}\,F^{\mu\nu})\,,$

where $\partial_{\mu} \equiv \partial/\partial x^{\mu}$.

4. The Schwarzschild metric is given by:

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right)c^2 dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2\,.$$

(a) From the corresponding Lagrangian $L = g_{\mu\nu}q^{\mu}q^{\nu}$, where $q^{\mu} \equiv dx^{\mu}/ds$, write down the Euler-Lagrange equations which (as shown in class) are equivalent to the geodesic equations

$$\frac{dq^{\mu}}{ds} + \Gamma^{\mu}{}_{\alpha\beta}q^{\alpha}q^{\beta} = 0\,.$$

Use this result to work out the thirteen nonvanishing metric connection coefficients for the Schwarzschild metric.

(b) Check a few of these metric connection coefficients by calculating them directly from the formula for $\Gamma^{\mu}{}_{\alpha\beta}$ in terms of the derivatives of the metric tensor.

5. Using the geodesic equation, show that all lines of longitude, corresponding to constant azimuthal angle ϕ on the (two-dimensional) surface of a sphere, are geodesics.