You have three hours to complete this exam. The point value of each of the three problems is indicated in brackets, so use your time wisely.

While working on the exam, you are permitted to consult with your class notes, all class handouts (including problem set solutions) and the course textbook by Kenyon. You should *not* collaborate with anyone or any other source material during the exam.

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1. [25] Maxwell's equations in curved spacetime read:

$$F^{\mu\nu}{}_{;\,\mu} = \frac{4\pi J^\nu}{c}\,, \tag{1}$$

where $F^{\mu\nu}$ is the (antisymmetric) electromagnetic field strength tensor and the current J^ν satisfies the covariant conservation law

$$J^\nu{}_{;\,\nu} = 0\,. \tag{2}$$

Equations (1) and (2) are consistent only if

$$F^{\mu\nu}{}_{;\,\mu\,;\,\nu} = 0\,. \tag{3}$$

Prove that *any* antisymmetric tensor satisfies equation (3), and hence Maxwell's equations in curved spacetime are consistent.

HINT: Problem 1(b) on problem set 4 may be helpful.

2. [40] Consider a spacetime with a spacetime interval

$$ds^2 = e^{-2ax/c^2}\, c^2dt^2 - dx^2 - dy^2 - dz^2\,,$$

where a is a constant.*

(a) Find all the nonzero Christoffel symbols. Employ the method in which you compute the geodesic equations.

(b) Suppose that a particle has a zero instantaneous velocity at $t = 0$ (*i.e.*, $dx/dt = dy/dt = dz/dt = 0$ at $t = 0$). Show that the acceleration, $d^2x/d\tau^2$, (where τ is the *proper* time parameter) at $t = 0$ is a constant given by

$$\left.\frac{d^2x}{d\tau^2}\right|_{t=0} = a\,.$$

Because of this, it may be said that this spacetime describes a uniform gravitational field in the x direction.

HINT: Start with the geodesic equation for $\ddot{x}$. Simplify this expression at $t = 0$ by noting that $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 1$. Here the dot superscript refers to d/ds (where $ds = c\,d\tau$).

(c) Find the nonzero components of the Ricci tensor. Show that apart from R_{00} and R_{11}, all the other components of the Ricci tensor are zero.

(d) Using the Einstein equations, evaluate the components T_{00}, T_{11}, T_{22} and T_{33} of the energy-momentum tensor that generates this gravitational field. In fact, the resulting energy-momentum tensor does not represent a "reasonable" matter distribution (some would say it is not physically acceptable). Can you explain why this might be the case?

(e) As a final check that you have done your algebra correctly, show that the energy-momentum tensor found in part (d) satisfies $T^{\mu\nu}{}_{;\nu} = 0$.

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*Some helpful advice: First, do not confuse the variable x with the four-vector x^μ. As usual, $x^1 \equiv x$, $x^2 \equiv y$, $x^3 \equiv z$ and $x^0 \equiv ct$. Second, be very careful about minus signs. In particular, note that $g_{11} = g_{22} = g_{33} = -1$. Although the algebra is not difficult in this problem, work carefully to avoid careless errors. Part (e) is one way to check that you have not made a mistake.

3. [35] The universe today seems to be described by the following parameters: $\Omega_m = 0.3$, $\Omega_\Lambda = 0.7$, $\Omega_\gamma = 5 \times 10^{-5}$ and the inverse Hubble parameter today is $H_0^{-1} = 14$ Gyr. Here, $\Omega_x \equiv \rho_x/\rho_{c0}$ where $x = m$, Λ and γ refer respectively to the densities of matter, "vacuum", and photons that make up the cosmic microwave background radiation (the latter is measured to have a temperature $T_0 = 2.725°\text{K}$).

(a) Using the parameters given above, find the relative value of the cosmic scale parameter, $a(t) \equiv R(t)/R(t_0)$, when $\rho_m = \rho_\gamma$.

(b) What was the temperature of the universe (corresponding to the temperature of the blackbody cosmic photons) when $\rho_m = \rho_\gamma$?

(c) In class, we showed that the Hubble parameter at time t is given by:

$$H^2(t) = H_0^2 \left[\frac{\Omega_m}{a^3} + \frac{\Omega_r}{a^4} + \Omega_\Lambda + \frac{1 - \Omega_T}{a^2} \right] , \qquad (4)$$

where $\Omega_T \equiv \Omega_m + \Omega_r + \Omega_\Lambda$, and $a \equiv R(t)/R(t_0)$. This is in fact a differential equation, which can be used to solve for t as a function of a (or vice versa). Integrate this equation from $t = 0$ (the time of the big bang) to t_{eq}, the time at which $\rho_m = \rho_\gamma$. Explain why you may safely ignore the terms in equation (4) proportional to Ω_Λ and $1 - \Omega_T$ when $0 \leq t \leq t_{\text{eq}}$. Then, evaluate the integral and obtain an expression for t_{eq}, under the assumption that $\Omega_r = \Omega_\gamma$. Evaluate t_{eq} numerically (in years).

HINT: The upper limit of the a integration is obtained from the result of part (a). (What is the lower limit of the a integration, corresponding to $t = 0$?) The integral you need to evaluate is of the form:

$$\int \frac{x\,dx}{(a + bx)^{1/2}} = \frac{2}{b^2} \left[\tfrac{1}{3}(a + bx)^{3/2} - a(a + bx)^{1/2} \right] .$$

(d) [*OPTIONAL*] More accurately, $\Omega_r = \Omega_\gamma + \Omega_\nu$, where Ω_ν is the density of cosmic background neutrinos relative to the critical density. One can show that $\Omega_\nu \simeq 0.68\Omega_\gamma$. Thus, replace Ω_γ in part (c) with $\Omega_r = 1.68\Omega_\gamma$ and obtain an improved numerical result for the value of t_{eq} (which now corresponds to the cosmic time elapsed after the big bang at which $\rho_m = \rho_\gamma + \rho_\nu$). That is, t_{eq} corresponds to the age of the universe at "matter–radiation equality." Hence, for $t < t_{\text{eq}}$ the universe was radiation-dominated, while for $t > t_{\text{eq}}$ the universe is matter-dominated.