INSTRUCTIONS: This is a take-home exam. The point value of each problem is indicated in brackets. The exam should be completed and returned to my office, 211 Kerr Hall, on Monday November 5 before 2 pm. You may also leave the completed exam in my mailbox in Kerr Hall as long as you do so within the allotted time.

The exam is based on the first eleven class lectures, which includes material from the first six chapters of Kenyon. While working on the exam, you are permitted to consult with your class notes, Kenyon and one other relativity textbook and/or mathematics book of your choosing. Please indicate on the exam any additional references that you consult. However, you should *not* collaborate with any other source or persons during the exam.

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1. [20] In special relativity, the $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ fields are combined into a second rank antisymmetric tensor, $F^{\mu\nu}$.

(a) The quantity $F^{\mu\nu}F_{\mu\nu}$ is a scalar quantity, and hence a relativistic invariant. Evaluate this invariant in terms of $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$.

(b) Can you construct a second independent invariant that is quadratic in the electromagnetic fields? [*HINT:* it involves the Levi-Cevita tensor, $\epsilon^{\mu\nu\alpha\beta}$]. Evaluate this second invariant in terms of $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$.

2. [20] Starting from $V_\mu = g_{\mu\nu}V^\nu$ and

$$V^\mu{}_{;\,\alpha} = \frac{\partial V^\mu}{\partial x^\alpha} + \Gamma^\mu{}_{\beta\alpha}V^\beta\,,$$

derive an expression for $V_{\mu\,;\,\alpha}$. Your final expression should be in terms of the covariant components of the vector V_μ.

HINT: Note that the Leibnitz rule (also called the product rule) is valid for the covariant differentiation of a product of two tensors.

3. [30] The line element of special relativity is given by $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$. Transform this line element from the usual $(ct; x, y, z)$ rectangular coordinates to new coordinates $(ct'; x', y', z')$ related by

$$\begin{aligned} t &= t' \,, \\ x &= x' \cos\omega t' - y' \sin\omega t' \,, \\ y &= x' \sin\omega t' + y' \cos\omega t' \,, \\ z &= z' \,. \end{aligned}$$

The new coordinates describe a rotating reference frame with angular velocity vector $\vec{\boldsymbol{\omega}} = (0, 0, \omega)$.

(a) Express ds^2 in terms of the new coordinates.

(b) In terms of the new coordinates [*i.e.*, using the invariant line element of part (a)], write down the geodesic equations.

(c) Using the results of part (b), identify the nonvanishing Christoffel symbols.

(d) Using the results of part (b), show that in the non-relativistic limit,

$$\frac{d^2\vec{\boldsymbol{r}}}{dt^2} = -\vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}) - 2\vec{\boldsymbol{\omega}} \times \frac{d\vec{\boldsymbol{r}}}{dt} \,,$$

where $\vec{\boldsymbol{r}} \equiv (x', y', z')$. What is the physical interpretation of the two terms on the right hand side of the above equation?

4. [30] Consider a spacetime described by the Schwarzschild metric:

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2 \,.$$

(a) A clock at fixed (r, θ, ϕ) measures an (infinitesimal) proper time interval, which we shall denote by dT, along its world line. Express dT (as a function of r) in terms of the coordinate time interval dt.

(b) A stationary observer at fixed (t, θ, ϕ) measures an (infinitesimal) radial distance, which we shall denote by dR. Express dR (as a function of r) in terms of the coordinate radial distance dr.

(c) Consider a particle falling radially into the center of the Schwarzschild metric (*i.e*, falling in radially towards $r = 0$). Assume that the particle initially starts from rest infinitely far away from $r = 0$. Since this is force-free motion, the particle follows a geodesic. Show that the geodesic equation for $dt/d\tau$ (where $s \equiv c\tau$) implies that the quantity

$$E = mc^2 \left(1 - \frac{2GM}{c^2 r}\right) \frac{dt}{d\tau}$$

is a constant. We can interpret E as the total conserved energy of the particle. Argue that at $r \to \infty$ (where the initial velocity of the particle is zero), we can set $t = \tau$ and therefore $E = mc^2$ at all points along the particle trajectory.

(d) Compute the particle's inward coordinate velocity, $v = dr/dt$, as a function of the coordinate radial distance r. Invert the equation, and integrate from $r = r_0$ to $r = r_s \equiv 2GM/c^2$ (the Schwarzschild radius), where r_0 is some finite coordinate distance, $r_0 > r_s$. Show that the elapsed coordinate time is infinite, *i.e.* it takes an infinite coordinte time to reach the Schwarzschild radius.

HINT: The easiest way to proceed is to note that the value of the Lagrangian used to obtain the geodesic equations is a constant. (What is the value of this constant?)

(e) Compute the velocity dR/dT as measured by a stationary observer at a coordinate radial distance r. Verify that $|dR/dT| \to c$ as $r \to r_s$ [where r_s is the Schwarzschild radius defined in part (d)].

HINT: Use the results for dT and dR obtained in parts (a) and (b).