Physics 171 — Solution Set #2 — Fall 2001

1 (a) Given

$$t = \left(\frac{c}{g} + \frac{x'}{c}\right)\sinh\left(\frac{gt'}{c}\right)$$
$$x = c\left(\frac{c}{g} + \frac{x'}{c}\right)\cosh\left(\frac{gt'}{c}\right) - \frac{c^2}{g}$$
$$y = y'$$
$$z = z'$$

we compute:

$$dt = \frac{\partial t}{\partial t'}dt' + \frac{\partial t}{\partial x'}dx'$$
$$= \frac{g}{c}\left(\frac{c}{g} + \frac{x'}{c}\right)\cosh\left(\frac{gt'}{c}\right)dt' + \frac{1}{c}\sinh\left(\frac{gt'}{c}\right)dx'$$

and

$$dx = \frac{\partial x}{\partial t'}dt' + \frac{\partial x}{\partial x'}dx'$$
$$= g\left(\frac{c}{g} + \frac{x'}{c}\right)\sinh\left(\frac{gt'}{c}\right)dt' + \cosh\left(\frac{gt'}{c}\right)dx'$$

Then,

$$c^2dt^2 - dx^2 = \left[g\left(\frac{c}{g} + \frac{x'}{c}\right)\cosh\left(\frac{gt'}{c}\right)dt' + \sinh\left(\frac{gt'}{c}\right)dx'\right]^2$$
$$- \left[g\left(\frac{c}{g} + \frac{x'}{c}\right)\sinh\left(\frac{gt'}{c}\right)dt' + \cosh\left(\frac{gt'}{c}\right)dx'\right]^2$$
$$= g^2\left(\frac{c}{g} + \frac{x'}{c}\right)^2 dt'^2 - dx'^2$$
$$= \left(1 + \frac{gx'}{c^2}\right)^2 c^2dt'^2 - dx'^2$$

where I have used $\cosh^2\left(\frac{gt'}{c}\right) - \sinh^2\left(\frac{gt'}{c}\right) = 1$.

Since $y = y'$ and $z = z'$, we conclude that the transformed metric is:

$$ds^2 = \left(1 + \frac{gx'}{c^2}\right)^2 c^2dt'^2 - dx'^2 - dy'^2 - dz'^2$$

(b) For $\frac{gt'}{c} \ll 1$, we may use the expansions

$$\sinh\left(\frac{gt'}{c}\right) \simeq \frac{gt'}{c}$$
$$\cosh\left(\frac{gt'}{c}\right) \simeq 1 + \frac{1}{2}\left(\frac{gt'}{c}\right)^2$$

Then,

$$t = \left(\frac{c}{g} + \frac{x'}{c}\right)\left(\frac{gt'}{c}\right)$$
$$= t'\left(1 + \frac{gx'}{c^2}\right)$$

and

$$x = c\left(\frac{c}{g} + \frac{x'}{c}\right)\left(1 + \frac{1}{2}\left(\frac{gt'}{c}\right)^2\right) - \frac{c^2}{g}$$
$$= x' + \frac{1}{2}gt'^2 + \frac{1}{2}x'\frac{g^2t'^2}{c^2}$$
$$= x' + \frac{1}{2}gt'^2\left(1 + \frac{gx'}{c^2}\right)$$

If we assume that $\frac{gx'}{c^2} \ll 1$, then we are assured of nonrelativistic motion, i.e. $v \ll c$, over the time interval of interest. Thus,

$$t = t'$$
$$x = x' + \frac{1}{2}gt'^2$$

which is simply the relationship between a stationary frame and a uniformly accelerating frame in Newtonian mechanics.

(c) Compare the rate of clocks at a fixed position. That is, we take $dx^2+dy^2+dz^2=0$ and $dx'^2=dy'^2=dz'^2=0$. Noting that $ds^2=c^2d\tau^2$ where $d\tau$ is the proper time interval, we see that

$$d\tau = \left(1+\frac{gx'}{c^2}\right)dt'$$

That is, an at-rest clock in the accelerated frame measures time intervals

$$dt' = \frac{d\tau}{1+\frac{gx'}{c^2}}$$

at position x'. Thus

$$\frac{dt'(x'=h)}{dt'(x'=0)} = \frac{1}{1+\frac{gx'}{c^2}}$$

so $dt'(x'=h) < dt'(x'=0)$ which means that the at-rest clock runs fast (shorter time intervals) than the clock at rest at $x'=0$ by a factor of $1+\frac{gx'}{c^2}$.

This is related to the equivalence principle as follows. We can replace the uniform acceleration with a constant gravitational field with potential

$$\phi(x') = gx'$$

Thus,

$$d\tau = \left(1+\frac{\phi(x')}{c^2}\right)dt'$$

which is precisely the result obtained in class that yields the gravitational red shift.

2 (a) In class, we showed that

$$t(\phi) = t(0)\left[1+\frac{\phi}{c^2}\right]$$

Thus,

$$t(\phi_1)-t(\phi_2) = t(0)\left[\frac{\phi_1-\phi_2}{c^2}\right]$$

For the gravitational field of the earth

$$\phi = -\frac{GM}{r}$$

If ϕ_1 is the gravitational potential at the bottom of the Empire State Building (where $r=R_E$ is the radius of the earth), and ϕ_2 is the gravitational potential at the top of the Empire State Building (where $r=R_E+h$), where $h=380$ m, then

$$\phi_1-\phi_2 = -GM\left(\frac{1}{R_E}-\frac{1}{R_E+h}\right)$$

$$= \frac{-GMh}{R_E(R_E+h)}$$

$$\simeq \frac{-GMh}{R_E^2} \qquad \text{since } h \ll R_E$$

From Appendix H of Kenyon,

$$G = 6.6726\times10^{-11}\ kg^{-1}m^3s^{-2}$$

$$M = 5.98\times10^{24}\ kg$$

$$R_E = 6.37\times10^6\ m$$

Thus,

$$\Delta\phi \equiv \phi_1-\phi_2 = \frac{-(6.6726\times10^{-11})(5.98\times10^{24})(380)}{(6.37\times10^6)^2}$$

$$= -3.737\times10^3\ m^2s^{-2}$$

$$\frac{\Delta\phi}{c^2} = \frac{-3.737\times10^3\ m^2s^{-2}}{(2.99792\times10^8)^2 m^2s^{-2}} = -4.16\times10^{-14}$$

Now, clocks run slower in a region of lower gravitational potential. Since $\phi<0$, this means that clocks run slower where $-\phi$ is larger by a factor of $1+\phi/c^2$ (relative to the region of zero gravitational potential). Thus, the clock at the bottom of the Empire State Building runs slower than one at the top of the Empire State Building by a factor of

$$\frac{1+\phi_1/c^2}{1+\phi_2/c^2} \simeq 1+\frac{\phi_1-\phi_2}{c^2}$$

$$\simeq 1+\frac{\Delta\phi}{c^2}$$

(b) In part (a), we saw that the fractional change in the two clocks was 4.16×10^{-14}. Thus, the two clocks will differ by $1\,ns = 10^{-9}\,s$ in a time

$$\frac{10^{-9}s}{4.16\times10^{-14}} = 2.4\times10^{4}\,s$$

$$= 6.68 \text{ hours}$$

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③ We shall make use of the principle of equivalence. Note that the centrifugal acceleration is

$$a=\frac{v^2}{R} = \omega^2 R$$

where $\omega=\frac{v}{R}$ is the angular velocity. Thus, the equivalent gravitational field is one in which the gravitational force points radially outward

$$\vec{F} = m\omega^2 R\,\hat{r}$$

The corresponding gravitational potential is defined by:

$$\vec{F} = -m\vec{\nabla}\phi$$

$$= -m\hat{r}\frac{\partial\phi}{\partial r}$$

for a radial field. Thus, taking $\vec{F} = m\omega^2 r\,\hat{r}$

$$\frac{\partial\phi}{\partial r} = -\omega^2 r$$

$$\phi = -\frac{1}{2}\omega^2 r^2$$

(a) Since the potential is lower at $r=R$ (on the rim of the centrifuge) as compared to the axis of the centrifuge at $r=0$, we conclude that clocks run slower on the rim.

(b) The fractional difference in the rates of the two clocks is

$$\left|\frac{\Delta\phi}{c^2}\right| = \frac{\omega^2R^2}{2c^2} = \frac{v^2}{2c^2}$$

Note that this is valid for $v\ll c$. For arbitrary v, one would have to obtain a transformation between the rest frame and the rotating frame (a result which would then permit the calculation of $ds^2=c\,d\tau^2$ as in problem 1).

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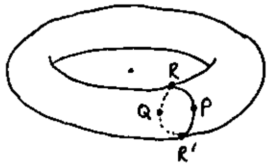
(4) Consider the torus:

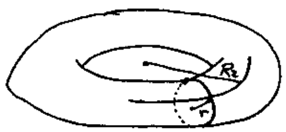


Point P lies on the outer edge of the torus and point Q lies on the inner edge.

The inner radius is $R_1 = 2\,cm$.
The outer radius is $R_2 = 5\,cm$

Thus, the radius of the circle shown in the figure above (with P and Q on the circumference) is $\frac{1}{2}(R_2 - R_1) = 1.5\,cm$.

(a) At the point P, the two directions of principle curvature are:



$$r \equiv \tfrac{1}{2}(R_2 - R_1)$$

so that
$$K = \frac{1}{R_2 r} = \frac{1}{(5\,cm)(1.5\,cm)} = 0.133\ cm^{-2}$$

(b) At the point Q, the two principle curvatures are:
$$K_1 = -\frac{1}{R_1} \quad \text{and} \quad K_2 = \frac{1}{r}$$

Note that $K_1 < 0$ so that
$$K = -\frac{1}{R_1 r} = \frac{-1}{(2\,cm)(1.5\,cm)} = -0.33\ cm^{-2}$$

(c) By continuity, K must vanish somewhere between (a) and (b). An explicit computation shows that K vanishes precisely at the top and bottom of the torus (points R and R' above)

An explicit computation:

A torus can be constructed by rotating a circle about an axis that lies outside the circle. Thus, consider the vector

$$\vec{x}(\xi_1, \xi_2) = \big((r_0 + r\cos\xi_2)\cos\xi_1\,,\ (r_0 + r\cos\xi_2)\sin\xi_1\,,\ r\sin\xi_2\big)$$

As the angles ξ_1 and ξ_2 vary between 0 and 2π, we map out a torus of inner radius $r_0 - r$ and outer radius $r_0 + r$.

Now,
$$ds^2 = g_{ij}\, d\xi_i\, d\xi_j$$
on the surface of the torus. Since
$$ds^2 = d\vec{x}\cdot d\vec{x} = \frac{\partial\vec{x}}{\partial\xi_i}\cdot\frac{\partial\vec{x}}{\partial\xi_j}\, d\xi_i\, d\xi_j$$
we identify:
$$g_{ij} = \frac{\partial\vec{x}}{\partial\xi_i}\cdot\frac{\partial\vec{x}}{\partial\xi_j}$$

Now,
$$\frac{\partial\vec{x}}{\partial\xi_1} = \big(-(r_0 + r\cos\xi_2)\sin\xi_1\,,\ (r_0 + r\cos\xi_2)\cos\xi_1\,,\ 0\big)$$
$$\frac{\partial\vec{x}}{\partial\xi_2} = \big(-r\sin\xi_2\cos\xi_1\,,\ -r\sin\xi_2\sin\xi_1\,,\ r\cos\xi_2\big)$$

Thus,
$$g_{11} = (r_0 + r\cos\xi_2)^2$$
$$g_{12} = 0$$
$$g_{22} = r^2$$

In class, we showed that for $g_{12}=0$,

$$K = \frac{1}{2g_{11}g_{22}}\left\{-\frac{\partial^2 g_{11}}{\partial \xi_2^2} - \frac{\partial^2 g_{22}}{\partial \xi_1^2} + \frac{1}{2g_{11}}\left[\frac{\partial g_{11}}{\partial \xi_1}\frac{\partial g_{22}}{\partial \xi_1} + \left(\frac{\partial g_{11}}{\partial \xi_2}\right)^2\right] + \frac{1}{2g_{22}}\left[\frac{\partial g_{11}}{\partial \xi_2}\frac{\partial g_{22}}{\partial \xi_2} + \left(\frac{\partial g_{22}}{\partial \xi_1}\right)^2\right]\right\}$$

Using $g_{11} = (r_0 + r\cos\xi_2)^2$ and $g_{22} = r^2$,

$$\frac{\partial g_{11}}{\partial \xi_2} = -2(r_0 + r\cos\xi_2) r \sin\xi_2$$

$$\frac{\partial^2 g_{11}}{\partial \xi_2^2} = -2(r_0 + r\cos\xi_2) r\cos\xi_2 + 2r^2\sin^2\xi_2$$

Thus,

$$K = \frac{1}{2r^2(r_0 + r\cos\xi_2)^2}\left[2r(r_0 + r\cos\xi_2)\cos\xi_2 - 2r^2\sin^2\xi_2 + \frac{1}{2(r_0 + r\cos\xi_2)^2}\left[4(r_0+r\cos\xi_2)^2 r^2 \sin^2\xi_2\right]\right]$$

which simplifies to

$$\boxed{K = \frac{\cos\xi_2}{r(r_0 + r\cos\xi_2)}}$$

Part (a) corresponds to $\xi_2 = 0$

Part (b) corresponds to $\xi_2 = \pi$

Part (c) corresponds to $\xi_2 = \frac{\pi}{2}, \frac{3\pi}{2}$.

5. (a) The coordinate velocity in the radial direction is $\frac{dr}{dt}$.

Light travels along a path characterized by $ds^2 = 0$. Setting $d\theta = d\phi = 0$ for the radial path,

$$0 = \left(1 - \frac{2GM}{c^2 r}\right)c^2 dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2$$

Thus,

$$\left(\frac{dr}{dt}\right)^2 = c^2\left(1 - \frac{2GM}{c^2 r}\right)^2$$

or

$$\frac{dr}{dt} = c\left(1 - \frac{2GM}{c^2 r}\right)$$

(b) In the transverse direction, set $dr = d\phi = 0$. Then,

$$0 = \left(1 - \frac{2GM}{c^2 r}\right)c^2 dt^2 - r^2 d\theta$$

Thus,

$$r\frac{d\theta}{dt} = c\left(1 - \frac{2GM}{c^2 r}\right)^{1/2}$$

(c) The physical consequences of these results is the slowing of light as it passes a massive body. You can read about the experimental verification of the time delay of light in C.M. Will, gr-qc/010303 pp 39–41. It was first observed by Shapiro in 1964 in radar ranging data. Subsequent experiments using artificial sattelites (including the recent Viking Mars landers and orbiters) have significantly improved the tests of general relativity. See also the discussion in Kenyon, section 8.3 (pp. 95–97).