Physics 171 — Solution Set #3 — Fall 2001

① (a) The simplest way to verify that raising and lowering indices commutes with covariant differentiation is to note that

$$A_{\mu;\alpha} = (g_{\mu\nu}A^{\nu})_{;\alpha}$$
$$= g_{\mu\nu;\alpha}A^{\nu} + g_{\mu\nu}A^{\nu}{}_{;\alpha}$$

using Leibniz's rule for differentiation of a product. But

$$g_{\mu\nu;\alpha} = 0$$

Hence,
$$A_{\mu;\alpha} = g_{\mu\nu}A^{\nu}{}_{;\alpha}$$

The same argument works for

$$A^{\mu}{}_{;\alpha} = (g^{\mu\nu}A_{\nu})_{;\alpha} = g^{\mu\nu}A_{\nu;\alpha}$$

since $g^{\mu\nu}{}_{;\alpha}=0$. The argument is easily extended to raising or lowering the index of any tensor.

(b) Following the examples given in class,

$$A^{\alpha}{}_{\beta\rho;\sigma} = A^{\alpha}{}_{\beta\rho,\sigma} + \Gamma^{\alpha}_{\mu\sigma}A^{\mu}{}_{\beta\rho} - \Gamma^{\mu}_{\beta\sigma}A^{\alpha}{}_{\mu\rho} - \Gamma^{\mu}_{\rho\sigma}A^{\alpha}{}_{\beta\mu}$$

(c) By definition, the metric connection is

$$\Gamma^{\mu}_{\alpha\beta} = \tfrac{1}{2}g^{\mu\nu}\left[g_{\nu\beta,\alpha} + g_{\nu\alpha,\beta} - g_{\alpha\beta,\nu}\right]$$

If μ,α and β are all different, and if $g_{\alpha\beta}$ is diagonal then $g_{\alpha\beta}=0$. Moreover, in the implicit sum over ν, we must have $\mu=\nu$ otherwise $g^{\mu\nu}=0$. Note that $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$ and is therefore diagonal as well. Finally, since $\mu=\nu$ but ν differs from α and β, we see that $g_{\nu\beta}=g_{\nu\alpha}=0$. We conclude that if $g_{\mu\nu}$ is diagonal and μ,α, and β are distinct, then $\Gamma^{\mu}_{\alpha\beta}=0$.

② (a)
$$A_{\mu;\nu} = A_{\mu,\nu} - A_{\alpha}\Gamma^{\alpha}_{\mu\nu}$$
$$A_{\nu;\mu} = A_{\nu,\mu} - A_{\alpha}\Gamma^{\alpha}_{\nu\mu}$$
$$= A_{\nu,\mu} - A_{\alpha}\Gamma^{\alpha}_{\mu\nu}$$

since $\Gamma^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\nu\mu}$. Hence

$$A_{\mu;\nu} - A_{\nu;\mu} = A_{\mu,\nu} - A_{\nu,\mu}.$$

(b)
$$F_{\mu\nu;\rho} = F_{\mu\nu,\rho} - \Gamma^{\sigma}_{\mu\rho}F_{\sigma\nu} - \Gamma^{\sigma}_{\nu\rho}F_{\mu\sigma}$$
$$F_{\rho\mu;\nu} = F_{\rho\mu,\nu} - \Gamma^{\sigma}_{\rho\nu}F_{\sigma\mu} - \Gamma^{\sigma}_{\mu\nu}F_{\rho\sigma}$$
$$F_{\nu\rho;\mu} = F_{\nu\rho,\mu} - \Gamma^{\sigma}_{\nu\mu}F_{\sigma\rho} - \Gamma^{\sigma}_{\rho\mu}F_{\nu\sigma}$$

Adding the three equations and using the fact that

$$F_{\mu\sigma} = -F_{\sigma\mu}$$
$$\Gamma^{\sigma}_{\rho\nu} = \Gamma^{\sigma}_{\nu\rho}$$

we see that

$$F_{\mu\nu;\rho} + F_{\rho\mu;\nu} + F_{\nu\rho;\mu} = F_{\mu\nu,\rho} + F_{\rho\mu,\nu} + F_{\nu\rho,\mu}$$
$$- \Gamma^{\sigma}_{\mu\rho}(F_{\sigma\nu} + F_{\nu\sigma}) - \Gamma^{\sigma}_{\nu\rho}(F_{\mu\sigma} + F_{\sigma\mu}) - \Gamma^{\sigma}_{\mu\nu}(F_{\rho\sigma} + F_{\sigma\rho})$$

or

$$F_{\mu\nu;\rho} + F_{\rho\mu;\nu} + F_{\nu\rho;\mu} = F_{\mu\nu,\rho} + F_{\rho\mu,\nu} + F_{\nu\rho,\mu}$$

(c) In Minkowski space, Maxwell's equations are:

$$\partial_{\mu}F^{\mu\nu} = \frac{4\pi}{c}J^{\nu}$$
$$\partial^{\rho}F^{\mu\nu} + \partial^{\mu}F^{\nu\rho} + \partial^{\nu}F^{\rho\mu} = 0$$

where $F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}$

If we lower the indices, we can write

$$\partial_{\rho}F_{\mu\nu} + \partial_{\mu}F_{\nu\rho} + \partial_{\nu}F_{\rho\mu} = 0$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

Using the comma notation, we therefore have:

$$F^{\mu\nu}{}_{,\mu} = \frac{4\pi}{c} J^\nu$$

$$F_{\mu\nu,\rho} + F_{\nu\rho,\mu} + F_{\rho\mu,\nu} = 0$$

where $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$.

To extend Maxwell's equations to curved spacetime, we make the above equations generally covariant. To accomplish this, we replace ordinary derivatives with covariant derivatives. For example, we now define

$$F_{\mu\nu} = A_{\nu;\mu} - A_{\mu;\nu}$$

Note that based on part (a), we can simply write this as

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$$

so there is no change in the definition of $F_{\mu\nu}$. But now,

$$F^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta}$$

where $g^{\mu\alpha}$ is the inverse of the metric tensor of curved spacetime. The Maxwell equations in covariant form are now:

$$F^{\mu\nu}{}_{;\mu} = \frac{4\pi}{c} J^\nu$$

$$F_{\mu\nu;\rho} + F_{\nu\rho;\mu} + F_{\rho\mu;\nu} = 0$$

Based on the results of part (b), the second equation can be rewritten as

$$F_{\mu\nu,\rho} + F_{\nu\rho,\mu} + F_{\rho\mu,\nu} = 0$$

so again, there is no change from the Minkowski space version.

Finally, current conservation in Minkowski space is $\partial_\mu J^\mu = 0$, which we can write as $J^\mu{}_{,\mu} = 0$. In curved spacetime, we again replace the ordinary derivative with a covariant derivative. Thus, in curved spacetime,

$$J^\mu{}_{;\mu} = 0.$$

<u>Remark</u>: If one takes the covariant derivative of $F^{\mu\nu}{}_{;\mu} = \frac{4\pi}{c} J^\nu$ with respect to x^ν, one can verify after much algebra that

$$(F^{\mu\nu}{}_{;\mu})_{;\nu} = 0$$

In deriving this result, one must use the fact that $F^{\mu\nu} = -F^{\nu\mu}$. Thus, Maxwell's equation is consistent with the requirement that $J^\mu{}_{;\mu} = 0$.

③ (a) Start with the result:

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\rho} \left[g_{\rho\nu,\mu} + g_{\rho\mu,\nu} - g_{\mu\nu,\rho} \right]$$

Setting $\alpha = \mu$ and summing over μ,

$$\Gamma^\mu_{\mu\nu} = \frac{1}{2} g^{\mu\rho} \left[g_{\rho\nu,\mu} + g_{\rho\mu,\nu} - g_{\mu\nu,\rho} \right]$$

Note that $g^{\mu\rho}[g_{\rho\nu,\mu} - g_{\mu\nu,\rho}] = 0$ since $g^{\mu\rho} = g^{\rho\mu}$ while $g_{\rho\nu,\mu} - g_{\mu\nu,\rho}$ is antisymmetric under the interchange of μ and ρ.

[Equivalently, one can simply relabel the second term $g^{\mu\rho} g_{\mu\nu,\rho}$ by writing ρ for μ and μ for ρ. This is allowed since ρ and μ are dummy indices; that is they are summed indices.]

Hence,

$$\Gamma^\mu_{\mu\nu} = \frac{1}{2} g^{\mu\rho} g_{\rho\mu,\nu}$$

$$= \frac{1}{2} g^{\mu\rho} \frac{\partial g_{\rho\mu}}{\partial x^\nu}$$

To prove that this is equivalent to

$$\Gamma^\mu_{\mu\nu} = \frac{1}{2g} \frac{\partial g}{\partial x^\nu}, \qquad g \equiv \det g_{\mu\nu}$$

Let us consider an arbitrary matrix A. By definition of the differential,

$$d(\det A) = \det(A + dA) - \det A$$

We can perform the following manipulation:

$$\det(A + dA) = \det(A[1 + A^{-1} dA])$$

$$= \det A \det(1 + A^{-1} dA)$$

where 1 is the identity matrix (with matrix elements δ_{ij}) and we have used the fact that $\det AB = \det A \det B$.

I can evaluate $\det(1+\delta A)$ as follows:

$$\det(1+\delta A) = \det\begin{pmatrix} 1+\delta A_{11} & \delta A_{12} & \cdots & \delta A_{1n} \\ \delta A_{21} & 1+\delta A_{22} & \cdots & \delta A_{2n} \\ \vdots & & & \\ \delta A_{n1} & \delta A_{n2} & \cdots & 1+\delta A_{nn} \end{pmatrix}$$

$$= (1+\delta A_{11})(1+\delta A_{22})\cdots(1+\delta A_{nn}) + O((\delta A_{ij})^2)$$

$$= 1 + \delta A_{11} + \delta A_{22} + \ldots + \delta A_{nn} + O((\delta A_{ij})^2)$$

$$= 1 + \mathrm{Tr}(\delta A)$$

where I have dropped all higher order corrections in δA_{ij}. Note that Tr stands for trace of the matrix, which is equal to the sum of the diagonal elements.

Taking $\delta A = A^{-1} dA$

it follows that to leading order in dA,

$$\det(A+dA) = \det A\left[1 + \mathrm{Tr}(A^{-1}dA)\right]$$

$$= \det A + \det A \,\mathrm{Tr}(A^{-1}dA)$$

Hence,

$$\boxed{d(\det A) = \det A \,\mathrm{Tr}(A^{-1}dA)}$$

which is true for any matrix A. Applying this result to $A = g_{\mu\nu}$, and noting that by definition, $A^{-1} = g^{\mu\nu}$, it follows that

$$\mathrm{Tr}(A^{-1}dA) = g^{\nu\mu} dg_{\mu\nu}$$

$$= g^{\mu\nu} dg_{\mu\nu}$$

since $g^{\mu\nu} = g^{\nu\mu}$. Moreover, $\det A = g$. Hence

$$\boxed{dg = g\, g^{\mu\nu} dg_{\mu\nu}}$$

Finally, using the chain rule,

$$dg = \frac{\partial g}{\partial x^\beta} dx^\beta$$

$$dg_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial x^\beta} dx^\beta$$

Hence,

$$\frac{1}{g}\frac{\partial g}{\partial x^\beta} dx^\beta = g^{\mu\nu}\frac{\partial g_{\mu\nu}}{\partial x^\beta} dx^\beta$$

This is true for an arbitrary infinitesimal dx^β, so we conclude that:

$$\boxed{\frac{1}{g}\frac{\partial g}{\partial x^\beta} = g^{\mu\nu}\frac{\partial g_{\mu\nu}}{\partial x^\beta}}$$

Comparing with the previous result, $\Gamma^\mu{}_{\mu\nu} = \frac{1}{2} g^{\mu\rho}\dfrac{\partial g_{\rho\mu}}{\partial x^\nu}$

(and using $g^{\mu\rho} = g^{\rho\mu}$), we see that

$$\Gamma^\mu{}_{\mu\nu} = \frac{1}{2g}\frac{\partial g}{\partial x^\nu}$$

Finally, we note that

$$\frac{\partial\sqrt{-g}}{\partial x^\nu} = \frac{-1}{2\sqrt{-g}}\frac{\partial g}{\partial x^\nu}$$

Hence,

$$\frac{1}{\sqrt{-g}}\frac{\partial\sqrt{-g}}{\partial x^\nu} = \frac{1}{2g}\frac{\partial g}{\partial x^\nu}$$

so that

$$\Gamma^\mu{}_{\mu\nu} = \frac{1}{\sqrt{-g}}\frac{\partial\sqrt{-g}}{\partial x^\nu}.$$

Remark: Another proof of $dg = g\, g^{\mu\nu} dg_{\mu\nu}$

From the definition of the determinant, we have

$$g_{\mu\rho}\, g_{\nu\sigma}\, g_{\alpha\delta}\, g_{\beta\tau}\, \varepsilon^{\rho\sigma\delta\tau} = -g\, \varepsilon_{\mu\nu\alpha\beta}$$

Note the minus sign, since by convention $\varepsilon^{0123} = -\varepsilon_{0123} = 1$.
Take the differential of this equation to obtain:

$$-dg\, \varepsilon_{\mu\nu\alpha\beta} = \varepsilon^{\rho\sigma\delta\tau}\Big[dg_{\mu\rho}\, g_{\nu\sigma}\, g_{\alpha\delta}\, g_{\beta\tau} + g_{\mu\rho}\, dg_{\nu\sigma}\, g_{\alpha\delta}\, g_{\beta\tau} + g_{\mu\rho}\, g_{\nu\sigma}\, dg_{\alpha\delta}\, g_{\beta\tau} + g_{\mu\rho}\, g_{\nu\sigma}\, g_{\alpha\delta}\, dg_{\beta\tau}\Big]$$

We also note that

$$g_{\nu\sigma}\, g_{\alpha\delta}\, g_{\beta\tau}\, \varepsilon^{\rho\sigma\delta\tau} = -g\, g^{\gamma\rho}\, \varepsilon_{\gamma\nu\alpha\beta}$$

since $g^{\mu\rho}$ is the inverse of $g_{\mu\rho}$. Likewise

$$g_{\mu\rho}\, g_{\alpha\delta}\, g_{\beta\tau}\, \varepsilon^{\rho\sigma\delta\tau} = -g\, g^{\gamma\sigma}\, \varepsilon_{\mu\gamma\alpha\beta}$$
$$g_{\mu\rho}\, g_{\nu\sigma}\, g_{\beta\tau}\, \varepsilon^{\rho\sigma\delta\tau} = -g\, g^{\gamma\delta}\, \varepsilon_{\mu\nu\gamma\beta}$$
$$g_{\mu\rho}\, g_{\nu\sigma}\, g_{\alpha\delta}\, \varepsilon^{\rho\sigma\delta\tau} = -g\, g^{\gamma\tau}\, \varepsilon_{\mu\nu\alpha\gamma}$$

Thus,

$$dg\, \varepsilon_{\mu\nu\alpha\beta} = g\Big[g^{\gamma\rho} dg_{\mu\rho}\, \varepsilon_{\gamma\nu\alpha\beta} + g^{\gamma\sigma} dg_{\nu\sigma}\, \varepsilon_{\mu\gamma\alpha\beta} + g^{\gamma\delta} dg_{\alpha\delta}\, \varepsilon_{\mu\nu\gamma\beta} + g^{\gamma\tau} dg_{\beta\tau}\, \varepsilon_{\mu\nu\alpha\gamma}\Big]$$

Multiply by $\varepsilon^{\mu\nu\alpha\beta}$ and sum over repeated indices.
Using:

$$\varepsilon^{\mu\nu\alpha\beta}\, \varepsilon_{\mu\nu\alpha\beta} = -24$$
$$\varepsilon^{\mu\nu\alpha\beta}\, \varepsilon_{\gamma\nu\alpha\beta} = -6\, \delta^{\mu}_{\gamma}$$

etc., we obtain:

$$dg = \frac{g}{24}\, 6\Big[g^{\mu\rho} dg_{\mu\rho} + g^{\nu\sigma} dg_{\nu\sigma} + g^{\alpha\delta} dg_{\alpha\delta} + g^{\beta\tau} dg_{\beta\tau}\Big]$$

All four terms are identical after relabeling. Thus, we end up with

$$dg = g\, g^{\mu\nu} dg_{\mu\nu}$$

as before. We now turn to (a) and (b).

(a) Start with $A^{\mu}{}_{;\nu} = A^{\mu}{}_{,\nu} + \Gamma^{\mu}_{\beta\nu}$. Set $\nu=\mu$ and sum over μ. Using $\Gamma^{\mu}_{\beta\nu} = \Gamma^{\mu}_{\nu\beta}$, we can write:

$$A^{\mu}{}_{;\mu} = A^{\mu}{}_{,\mu} + \Gamma^{\mu}_{\mu\beta} A^{\beta} = A^{\mu}{}_{,\mu} + \frac{1}{\sqrt{-g}} \frac{\partial \sqrt{-g}}{\partial x^{\beta}} A^{\beta}.$$

This should be compared with:

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^{\mu}} \left(\sqrt{-g}\, A^{\mu}\right) = \frac{\partial}{\partial x^{\mu}} A^{\mu} + A^{\mu} \frac{1}{\sqrt{-g}} \frac{\partial \sqrt{-g}}{\partial x^{\mu}}$$

and we see that indeed

$$A^{\mu}{}_{;\mu} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^{\mu}} \left(\sqrt{-g}\, A^{\mu}\right).$$

(b) Start with $F^{\mu\nu}{}_{;\alpha} = F^{\mu\nu}{}_{,\alpha} + \Gamma^{\mu}_{\beta\alpha} F^{\beta\nu} + \Gamma^{\nu}_{\beta\alpha} F^{\mu\beta}$. Set $\alpha=\mu$ and sum over μ. It follows that:

$$F^{\mu\nu}{}_{;\mu} = F^{\mu\nu}{}_{,\mu} + \Gamma^{\mu}_{\mu\beta} F^{\beta\nu} + \Gamma^{\nu}_{\mu\beta} F^{\mu\beta}$$

By assumption, $F^{\mu\nu}$ is an antisymmetric tensor. Therefore, $\Gamma^{\nu}_{\mu\beta} F^{\mu\beta} = 0$ since $F^{\mu\beta}$ is antisymmetric whereas $\Gamma^{\nu}_{\mu\beta}$ is symmetric under the interchange of μ and β. Hence,

$$F^{\mu\nu}{}_{;\mu} = F^{\mu\nu}{}_{,\mu} + \frac{1}{\sqrt{-g}} \frac{\partial \sqrt{-g}}{\partial x^{\beta}} F^{\beta\nu}$$

The ν index goes along for the ride. Otherwise, the result is similar to that of part (a), so that

$$F^{\mu\nu}{}_{;\mu} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^{\mu}} \left(\sqrt{-g}\, F^{\mu\nu}\right) \qquad \text{if } F^{\mu\nu} = -F^{\nu\mu}.$$

(4) $$ds^2 = \left(1-\frac{2GM}{c^2r}\right)c^2dt^2 - \left(1-\frac{2GM}{c^2r}\right)^{-1}dr^2 - r^2d\theta^2 - r^2\sin^2\theta\, d\phi^2$$

(a) Consider the Lagrangian

$$L = \left(1-\frac{2GM}{c^2r}\right)(\dot{x}^0)^2 - \left(1-\frac{2GM}{c^2r}\right)^{-1}\dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\,\dot{\phi}^2$$

Using the Lagrange equations:

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{x}^0}\right) = \frac{\partial L}{\partial x^0} \implies \frac{d}{ds}\left[2\dot{x}^0\left(1-\frac{2GM}{c^2r}\right)\right] = 0$$

or $$\ddot{x}^0\left(1-\frac{2GM}{c^2r}\right) + \frac{2GM}{c^2r^2}\dot{r}\dot{x}^0 = 0$$

which we rewrite as

$$\boxed{\ddot{x}^0 + \frac{2GM}{c^2r^2}\left(1-\frac{2GM}{c^2r}\right)^{-1}\dot{r}\dot{x}^0 = 0}$$

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r} \implies$$

$$\frac{d}{ds}\left[-2\dot{r}\left(1-\frac{2GM}{c^2r}\right)^{-1}\right] = \frac{2GM}{c^2r^2}(\dot{x}^0)^2 + \left(1-\frac{2GM}{c^2r}\right)^{-2}\left(\frac{2GM}{c^2r^2}\right)\dot{r}^2 - 2r\dot{\theta}^2 - 2r\sin^2\theta\,\dot{\phi}^2$$

Hence,

$$\ddot{r}\left(1-\frac{2GM}{c^2r}\right)^{-1} - \frac{GM}{c^2r^2}\left(1-\frac{2GM}{c^2r}\right)^{-2}\dot{r}^2 + \frac{GM}{c^2r^2}(\dot{x}^0)^2 - r\dot{\theta}^2 - r\sin^2\theta\,\dot{\phi}^2 = 0$$

Multiply by $1-\frac{2GM}{c^2r}$ to obtain:

$$\boxed{\ddot{r} - \frac{GM}{c^2r^2}\left(1-\frac{2GM}{c^2r}\right)^{-1}\dot{r}^2 + \frac{GM}{c^2r^2}\left(1-\frac{2GM}{c^2r}\right)(\dot{x}^0)^2 - r\left(1-\frac{2GM}{c^2r}\right)\dot{\theta}^2 - r\sin^2\theta\left(1-\frac{2GM}{c^2r}\right)\dot{\phi}^2 = 0}$$

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta} \implies \frac{d}{ds}\left[-2r^2\dot{\theta}\right] = -2r^2\sin\theta\cos\theta\,\dot{\phi}^2$$

That is,

$$\boxed{\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta\cos\theta\,\dot{\phi}^2 = 0}$$

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi} \implies \frac{d}{ds}\left[-2r^2\sin^2\theta\,\dot{\phi}\right] = 0$$

or $$\sin^2\theta\,\ddot{\phi} + \frac{2\dot{r}\dot{\phi}}{r}\sin^2\theta + 2\sin\theta\cos\theta\,\dot{\theta}\dot{\phi} = 0$$

Divide by $\sin^2\theta$ to obtain

$$\boxed{\ddot{\phi} + \frac{2\dot{r}\dot{\phi}}{r} + 2\cot\theta\,\dot{\theta}\dot{\phi} = 0}$$

We now compare with

$$\ddot{x}^\mu + \Gamma^\mu_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 0$$

and read off all the non-zero $\Gamma^\mu_{\alpha\beta}$. Keep in mind that $\Gamma^\mu_{\alpha\beta} = \Gamma^\mu_{\beta\alpha}$. Thus, for $\alpha \neq \beta$,

$$\Gamma^\mu_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = \frac{1}{2}\Gamma^\mu_{\alpha\beta}(\dot{x}^\alpha\dot{x}^\beta + \dot{x}^\beta\dot{x}^\alpha)$$

The result:

$$\Gamma^0_{0r} = \Gamma^0_{r0} = \frac{GM}{c^2r^2}\left(1-\frac{2GM}{c^2r}\right)^{-1}$$

$$\Gamma^r_{rr} = -\frac{GM}{c^2r^2}\left(1-\frac{2GM}{c^2r}\right)^{-1}$$

$$\Gamma^r_{00} = \frac{GM}{c^2r^2}\left(1-\frac{2GM}{c^2r}\right)$$

$$\Gamma^r_{\theta\theta} = -r\left(1-\frac{2GM}{c^2r}\right)$$

$$\Gamma^r_{\phi\phi} = -r\sin^2\theta\left(1-\frac{2GM}{c^2r}\right)$$

$$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}$$

$$\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$$

$$\Gamma^\phi_{\phi r} = \Gamma^\phi_{r\phi} = \frac{1}{r}$$

$$\Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta$$

These are the thirteen non-zero Christoffel symbols.

(b) Use $\quad \Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}\left[g_{\nu\beta,\alpha} + g_{\nu\alpha,\beta} - g_{\alpha\beta,\nu}\right]$

Thus, $\Gamma^0_{0r} = \frac{1}{2}g^{00}\left[\frac{\partial g_{0r}}{\partial x^0} + \frac{\partial g_{00}}{\partial r} - \frac{\partial g_{0r}}{\partial x^0}\right]$

$$= \frac{1}{2}g^{00}\frac{\partial g_{00}}{\partial r}$$

Since $g_{\mu\nu}$ is diagonal, we have $g^{\mu\mu} = \frac{1}{g_{\mu\mu}}$ (no sum over μ).
Thus, $g^{00} = \left(1-\frac{2GM}{c^2r}\right)^{-1}$. Next,

$$\frac{\partial g_{00}}{\partial r} = \frac{2GM}{c^2r^2}$$

Thus, $\quad \Gamma^0_{0r} = \Gamma^0_{r0} = \frac{GM}{c^2r^2}\left(1-\frac{2GM}{c^2r}\right)^{-1}$

The other non-zero Christoffel symbols are also easy to obtain:

$$\Gamma^r_{00} = -\frac{1}{2}g^{rr}\frac{\partial g_{00}}{\partial r} = \frac{GM}{c^2r^2}\left(1-\frac{2GM}{c^2r}\right)$$

$$\Gamma^r_{rr} = \frac{1}{2}g^{rr}\frac{\partial g_{rr}}{\partial r} = -\frac{GM}{c^2r^2}\left(1-\frac{2GM}{c^2r}\right)^{-1}$$

$$\Gamma^r_{\theta\theta} = -\frac{1}{2}g^{rr}\frac{\partial g_{\theta\theta}}{\partial r} = -r\left(1-\frac{2GM}{c^2r}\right)$$

$$\Gamma^r_{\phi\phi} = -\frac{1}{2}g^{rr}\frac{\partial g_{\phi\phi}}{\partial r} = -r\sin^2\theta\left(1-\frac{2GM}{c^2r}\right)$$

$$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{2}g^{\theta\theta}\frac{\partial g_{\theta\theta}}{\partial r} = \frac{1}{2}\left(-\frac{1}{r^2}\right)(-2r) = \frac{1}{r}$$

$$\Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{1}{2}g^{\phi\phi}\frac{\partial g_{\phi\phi}}{\partial r} = \frac{1}{2}\left(-\frac{1}{r^2\sin^2\theta}\right)(-2r\sin^2\theta) = \frac{1}{r}$$

$$\Gamma^\theta_{\phi\phi} = -\frac{1}{2}g^{\theta\theta}\frac{\partial g_{\phi\phi}}{\partial\theta} = -\frac{1}{2r^2}2r^2\sin\theta\cos\theta = -\sin\theta\cos\theta$$

$$\Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \frac{1}{2}g^{\phi\phi}\frac{\partial g_{\phi\phi}}{\partial\theta} = \frac{1}{2r^2\sin^2\theta}2r^2\sin\theta\cos\theta$$

$$= \cot\theta$$

All other Γ's vanish.

(5) The metric for the surface of the sphere of radius one is

$$ds^2 = d\theta^2 + \sin^2\theta \, d\phi^2 .$$

We obtain the geodesic equations from $L = \dot\theta^2 + \sin^2\theta\,\dot\phi^2$.

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot\theta}\right) = \frac{\partial L}{\partial \theta} \implies \frac{d}{ds}(2\dot\theta) = 2\sin\theta\cos\theta\,\dot\phi^2$$

or

$$\boxed{\ddot\theta - \sin\theta\cos\theta\,\dot\phi^2 = 0}$$

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot\phi}\right) = \frac{\partial L}{\partial \phi} \implies \frac{d}{ds}(2\dot\phi\sin^2\theta) = 0$$

or

$$\ddot\phi\sin^2\theta + 2\sin\theta\cos\theta\,\dot\theta\dot\phi = 0$$

$$\boxed{\ddot\phi + 2\cot\theta\,\dot\theta\dot\phi = 0}$$

Lines of longitude on the sphere correspond to

$$\phi = \text{constant}$$
$$\theta = s$$

where s is the arclength measured along the geodesic. It is trivial to check that this satisfies the two geodesic equations given above.

<u>Remark</u>: It is not much more difficult to obtain all geodesics. From the equation

$$\frac{d}{ds}(\dot\phi\sin^2\theta) = 0$$

we learn that $\dot\phi\sin^2\theta = J$ where J is a constant. Insert this result into the first geodesic equation and eliminate $\dot\phi$. But, there is an easier method. Noting that

$$1 = \left(\frac{d\theta}{ds}\right)^2 + \sin^2\theta\left(\frac{d\phi}{ds}\right)^2$$

we see that the value of L on the geodesic is $L = 1$. Thus,

$$\dot\theta^2 + \sin^2\theta\,\dot\phi^2 = 1$$

Putting $\dot\phi\sin^2\theta = J$, we have

$$\dot\theta^2 = 1 - \frac{J^2}{\sin^2\theta}$$

$$\frac{d\theta}{ds} = \frac{\sqrt{\sin^2\theta - J^2}}{\sin\theta}$$

Define $J \equiv \sin\theta_0$. Then, we can integrate this to obtain

$$\cos\theta = \cos\theta_0 \sin(s - s_0)$$

where s_0 is the constant of integration. For simplicity, set $s_0 = 0$ which simply defines the origin of the arclength variable.

$$\boxed{\cos\theta = \cos\theta_0 \sin s}$$

In this case, $s = 0$ corresponds to $\theta = \frac{\pi}{2}$ (the equator). Finally, from

$$\dot\phi\sin^2\theta = \sin\theta_0$$

Thus,

$$\frac{d\phi}{ds} = \frac{\sin\theta_0}{1 - \cos^2\theta_0 \sin^2 s}$$

Integration gives:

$$\boxed{\tan(\phi - \phi_0) = \sin\theta_0 \tan s}$$

where ϕ_0 is the constant of integration. The two boxed equations provide a parametric representation of any geodesic, and depend on the constants θ_0 and ϕ_0. For example, if $\theta_0 = 0$, then we have

$$\phi = \phi_0$$
$$\theta = \frac{\pi}{2} - s$$

which describes one of the longitudes. If $\theta_0 = \frac{\pi}{2}$, then we obtain

$$\theta = \frac{\pi}{2}$$
$$\phi = \phi_0 + s$$

which is the equator.

Addendum to Problem 3

The key result

$$\frac{1}{g}\frac{\partial g}{\partial x^\beta} = g^{\mu\nu}\frac{\partial g_{\mu\nu}}{\partial x^\beta}$$

can be proved in other ways. Here are two alternate methods:

Method 1: For any matrix A,

$$\det(\exp A) = \exp(\operatorname{tr} A)$$

where

$$\exp A \equiv \sum_{n=0}^{\infty}\frac{A^n}{n!}$$

is defined by the usual power series of the exponential (which converges for all matrices). Take $\frac{\partial}{\partial x^\beta}$ of both sides

$$\frac{\partial}{\partial x^\beta}\det(\exp A) = \frac{\partial}{\partial x^\beta}\exp(\operatorname{tr} A)$$

assuming that the matrix elements of A depend on x^β. Now,

$$\frac{\partial}{\partial x^\beta}\exp(\operatorname{tr} A) = \exp(\operatorname{tr} A)\frac{\partial}{\partial x^\beta}\operatorname{tr} A$$

$$= \exp(\operatorname{tr} A)\operatorname{tr}\left(\frac{\partial A}{\partial x^\beta}\right)$$

where we have used the fact that the trace depends linearly on the matrix elements (since it is the sum of the diagonal elements), and the derivative is a linear operator. At this point, note that

$$\frac{\partial}{\partial x^\beta}\exp A = \exp A\,\frac{\partial A}{\partial x^\beta}$$

(which can be proved using the chain rule and the power series definition of $\exp A$). Thus,

$$\frac{\partial A}{\partial x^\beta} = (\exp A)^{-1}\frac{\partial}{\partial x^\beta}\exp A$$

Thus, if $B = \exp A$, we have proven that

$$\frac{\partial}{\partial x^\beta}\det B = \det B\,\operatorname{tr}\left(B^{-1}\frac{\partial B}{\partial x^\beta}\right)$$

Putting $B = g_{\mu\nu}$, $B^{-1} = g^{\mu\nu}$ and $\det B = g$ yields the desired result.

Remarks: If $B = \exp A$, then we can write $A = \ln B$. In this case, the above proof can be rewritten as follows:

$$\ln(\det B) = \operatorname{tr}(\ln B)$$

$$\frac{\partial}{\partial x^\beta}\ln(\det B) = \frac{1}{\det B}\frac{\partial}{\partial x^\beta}\det B$$

$$\frac{\partial}{\partial x^\beta}\operatorname{tr}(\ln B) = \operatorname{tr}\left(\frac{\partial}{\partial x^\beta}\ln B\right) = \operatorname{tr}\left(B^{-1}\frac{\partial B}{\partial x^\beta}\right)$$

from which it follows that

$$\frac{1}{\det B}\frac{\partial}{\partial x^\beta}\det B = \operatorname{tr}\left(B^{-1}\frac{\partial B}{\partial x^\beta}\right)$$

[Note that the proof of

$$\frac{\partial}{\partial x^\beta}\ln B = B^{-1}\frac{\partial B}{\partial x^\beta}$$

is easy if $B = \exp A$, since $\ln(\exp A) = A$ by definition.]

The only remaining question is whether we can apply these results to $g_{\mu\nu}$. Note the implicit assumption that there exists an A such that $B = \exp A$ (or equivalently that $\ln B$ exists).

Since $g_{\mu\nu}$ is a real symmetric matrix (and by assumption it is non-singular), it can be diagonalized by an orthogonal transformation. Let G be the matrix $g_{\mu\nu}$. Then

$$\mathcal{O}^T G\,\mathcal{O} = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$$

where diag is a matrix with diagonal elements given by the eigenvalues.

Then,

$$\mathcal{O}^T \ln G\, \mathcal{O} = \operatorname{diag}(\ln\lambda_1, \ln\lambda_2, \ln\lambda_3, \ln\lambda_4)$$

and

$$\ln G = \mathcal{O}\operatorname{diag}(\ln\lambda_1, \ln\lambda_2, \ln\lambda_3, \ln\lambda_4)\,\mathcal{O}^T$$

If $g_{\mu\nu}$ is positive definite, then all the $\lambda_i > 0$ and $\ln G$ is well defined

However, in general relativity, $g_{\mu\nu}$ has one positive eigenvalue and three negative eigenvalues. We can still define $\ln G$, but we must be careful since the log of a negative number is multivalued.

Nevertheless, the end result,

$$\frac{1}{\det B}\frac{\partial}{\partial x^\beta}\det B = \operatorname{tr}\left(B^{-1}\frac{\partial B}{\partial x^\beta}\right)$$

is true for all matrices B, and does not depend on ambiguities in the meaning of $\ln B$.

By the way, in the solution to problem 3, I derived

$$\frac{1}{\det A}\, d(\det A) = \operatorname{Tr}(A^{-1} dA)$$

If I integrate this equation, then I obtain

$$\int \frac{1}{\det A}\, d(\det A) = \ln(\det A)$$

and

$$\int \operatorname{Tr}(A^{-1}dA) = \operatorname{Tr}\int A^{-1}dA = \operatorname{Tr}(\ln A)$$

In the last line, I used the linear dependence of the trace on the matrix elements and the linearity of the integral. The final equality is formal but suggestive, and it follows that

$$\ln(\det A) = \operatorname{Tr}(\ln A)$$

There is no integration constant, since this result is trivially satisfied when $A = 1$ (the identity matrix). A more rigorous proof would replace A above with $\exp A$. It is easy to see how to obtain:

$$\det(\exp A) = \exp(\operatorname{tr} A)$$

Method 2: In this method, we compute $\frac{\partial}{\partial x^\beta}\det B$ more directly.

First, let us review some matrix algebra. For any nonsingular matrix B, its inverse is given by

$$B^{-1} = \frac{1}{\det B}\, B^{\rm adj}$$

where $B^{\rm adj}$ is the "adjoint" of B (not to be confused with hermitian adjoint $B^\dagger$) which is defined by:

$$B^{\rm adj}_{ji} = \operatorname{cof}(B_{ij})$$

(note the order of indices), where the cofactor of B_{ij} is given by

$$\operatorname{cof}(B_{ij}) \equiv (-1)^{i+j}\det\begin{pmatrix} B_{11} & \cdots & B_{1,j-1} & B_{1,j+1} & \cdots & B_{1n} \\ \vdots & & \vdots & \vdots & & \\ B_{i-1,1} & \cdots & B_{i-1,j-1} & B_{i-1,j+1} & \cdots & B_{i-1,n} \\ B_{i+1,1} & \cdots & B_{i+1,j-1} & B_{i+1,j+1} & \cdots & B_{i+1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ B_{n1} & \cdots & B_{n,j-1} & B_{n,j+1} & \cdots & B_{nn}\end{pmatrix}$$

That is, the cofactor of B_{ij} is obtained by deleting the i^{th} row and j^{th} column from B, taking the determinant of the resulting $(n-1)\times(n-1)$ matrix and multiplying by a sign $(-1)^{i+j}$.

One can compute $\det B$ using the so-called cofactor expansion (or expansion by minors). One has the following formula:

$$\det B = \sum_{j=1}^{n} B_{ij}\operatorname{cof}(B_{ij})$$

for fixed i (any choice of i gives the same result).

If B_{ij} depends on x, then

$$\frac{\partial}{\partial B_{ij}}\det B = \operatorname{cof}(B_{ij})$$

since $\operatorname{cof}(B_{ij})$ does not depend on B_{ij} since this element has been deleted from the matrix in computing the determinant (see above).

By the chain rule,

$$\frac{\partial}{\partial x^\beta} \det B = \frac{\partial B_{ij}}{\partial x^\beta} \frac{\partial}{\partial B_{ij}} \det B$$

$$= \frac{\partial B_{ij}}{\partial x^\beta} \operatorname{cof}(B_{ij})$$

$$= \frac{\partial B_{ij}}{\partial x^\beta} B^{adj}_{ji}$$

$$= \operatorname{Tr}\left(B^{adj} \frac{\partial B}{\partial x^\beta} \right)$$

$$= \det B \operatorname{Tr}\left(B^{-1} \frac{\partial B}{\partial x^\beta} \right)$$

where we have used $B^{-1} = \frac{1}{\det B} B^{adj}$. The factor $\det B$ is just a number which we can pull out of the trace. Thus,

$$\frac{1}{\det B} \frac{\partial}{\partial x^\beta} \det B = \operatorname{Tr}\left(B^{-1} \frac{\partial B}{\partial x^\beta} \right)$$

for any non-singular matrix B.