Physics 171 Solution Set #4 Fall 2001

1. By definition,

$$V^\mu{}_{;\alpha} = V^\mu{}_{,\alpha} + \Gamma^\mu_{\rho\alpha} V^\rho$$

(a) Now, for a mixed second rank tensor,

$$T^\mu{}_{\alpha;\beta} = T^\mu{}_{\alpha,\beta} + \Gamma^\mu_{\sigma\beta} T^\sigma{}_\alpha - \Gamma^\sigma_{\alpha\beta} T^\mu{}_\sigma$$

Thus,

$$\begin{aligned} V^\mu{}_{;\alpha;\beta} &= V^\mu{}_{;\alpha,\beta} + \Gamma^\mu_{\sigma\beta} V^\sigma{}_{;\alpha} - \Gamma^\sigma_{\alpha\beta} V^\mu{}_{;\sigma} \\ &= V^\mu{}_{,\alpha,\beta} + \Gamma^\mu_{\rho\alpha,\beta} V^\rho + \Gamma^\mu_{\rho\alpha} V^\rho{}_{,\beta} \\ &\quad + \Gamma^\mu_{\sigma\beta}\left(V^\sigma{}_{,\alpha} + \Gamma^\sigma_{\tau\alpha} V^\tau\right) - \Gamma^\sigma_{\alpha\beta}\left(V^\mu{}_{,\sigma} + \Gamma^\mu_{\tau\sigma} V^\tau\right) \\ &= V^\mu{}_{,\alpha,\beta} + \Gamma^\mu_{\rho\alpha,\beta} V^\rho + \Gamma^\mu_{\rho\alpha} V^\rho{}_{,\beta} + \Gamma^\mu_{\sigma\beta} V^\sigma{}_{,\alpha} - \Gamma^\sigma_{\alpha\beta} V^\mu{}_{,\sigma} \\ &\quad + \Gamma^\mu_{\sigma\beta}\Gamma^\sigma_{\tau\alpha} V^\tau - \Gamma^\sigma_{\alpha\beta}\Gamma^\mu_{\tau\sigma} V^\tau \end{aligned}$$

Note that $V^\mu{}_{,\alpha,\beta} = V^\mu{}_{,\beta,\alpha}$

$\Gamma^\sigma_{\alpha\beta} = \Gamma^\sigma_{\beta\alpha}$

and $\Gamma^\mu_{\rho\alpha} V^\rho{}_{,\beta} + \Gamma^\mu_{\sigma\beta} V^\sigma{}_{,\alpha}$ is symmetric under interchange of α and β.

Thus,

$$\begin{aligned} V^\mu{}_{;\alpha;\beta} - V^\mu{}_{;\beta;\alpha} &= \left(\Gamma^\mu_{\rho\alpha,\beta} - \Gamma^\mu_{\rho\beta,\alpha}\right) V^\rho + \left(\Gamma^\mu_{\sigma\beta}\Gamma^\sigma_{\tau\alpha} - \Gamma^\mu_{\sigma\alpha}\Gamma^\sigma_{\tau\beta}\right) V^\tau \\ &= \left(\Gamma^\mu_{\rho\alpha,\beta} - \Gamma^\mu_{\rho\beta,\alpha} + \Gamma^\mu_{\sigma\beta}\Gamma^\sigma_{\rho\alpha} - \Gamma^\mu_{\sigma\alpha}\Gamma^\sigma_{\rho\beta}\right) V^\rho \end{aligned}$$

Recalling that

$$R^\mu{}_{\rho\alpha\beta} = \Gamma^\mu_{\rho\beta,\alpha} - \Gamma^\mu_{\rho\alpha,\beta} + \Gamma^\sigma_{\rho\beta}\Gamma^\mu_{\sigma\alpha} - \Gamma^\sigma_{\rho\alpha}\Gamma^\mu_{\sigma\beta}$$

we see that

$$V^\mu{}_{;\alpha;\beta} - V^\mu{}_{;\beta;\alpha} = -R^\mu{}_{\rho\alpha\beta} V^\rho$$

Note: This result can also be derived starting with the result derived in class:

$$V_{\mu;\alpha;\beta} - V_{\mu;\beta;\alpha} = R^\rho{}_{\mu\alpha\beta} V_\rho$$

First, recall that

$$R_{\tau\mu\alpha\beta} = g_{\tau\rho} R^\rho{}_{\mu\alpha\beta}.$$

Thus,

$$\begin{aligned} R^\rho{}_{\mu\alpha\beta} V_\rho &= R^\rho{}_{\mu\alpha\beta}\, g_{\tau\rho} V^\tau \\ &= R_{\tau\mu\alpha\beta} V^\tau \\ &= -R_{\mu\tau\alpha\beta} V^\tau \end{aligned}$$

since $R_{\tau\mu\alpha\beta}$ is antisymmetric under interchange of the first two indices (or the last two indices). Hence,

$$V_{\mu;\alpha;\beta} - V_{\mu;\beta;\alpha} = -R_{\mu\tau\alpha\beta} V^\tau$$

Finally, we note that $g^{\rho\mu} R_{\mu\tau\alpha\beta} = R^\rho{}_{\tau\alpha\beta}$. So, if we multiply the above equation by $g^{\rho\lambda}$ and note, e.g. that

$$g^{\rho\mu} V_{\mu;\alpha;\beta} = V^\rho{}_{;\alpha;\beta}$$

since $g^{\rho\lambda}{}_{;\alpha} = 0$ so that raising indices commutes with covariant differentiation, we end up with

$$V^\rho{}_{;\alpha;\beta} - V^\rho{}_{;\beta;\alpha} = -R^\rho{}_{\tau\alpha\beta} V^\tau$$

which when relabeled is the desired result.

(b) Suppose V_ν is a covariant vector. Then $V_\nu T^{\mu\nu}$ is a contravariant vector. Using the results of part (a),

$$(V_\nu T^{\mu\nu})_{;\alpha;\beta} - (V_\nu T^{\mu\nu})_{;\beta;\alpha} = -R^\mu{}_{\rho\alpha\beta} V_\nu T^{\rho\nu}$$

Next, evaluate the left-hand side above using the Leibniz (product rule) for covariant differentiation.

$$\begin{aligned} (V_\nu T^{\mu\nu})_{;\alpha;\beta} &= \left(V_{\nu;\alpha} T^{\mu\nu} + V_\nu T^{\mu\nu}{}_{;\alpha}\right)_{;\beta} \\ &= V_{\nu;\alpha;\beta} T^{\mu\nu} + V_{\nu;\alpha} T^{\mu\nu}{}_{;\beta} + V_{\nu;\beta} T^{\mu\nu}{}_{;\alpha} + V_\nu T^{\mu\nu}{}_{;\alpha;\beta} \end{aligned}$$

Noting that $V_{\nu;\alpha} T^{\mu\nu}{}_{;\beta} + V_{\nu;\beta} T^{\mu\nu}{}_{;\alpha}$ is symmetric under the interchange of α and β,

$$(V_\nu T^{\mu\nu})_{;\alpha;\beta} - (V_\nu T^{\mu\nu})_{;\beta;\alpha}$$

$$= (V_{\nu;\alpha;\beta} - V_{\nu;\beta;\alpha}) T^{\mu\nu} + V_\nu (T^{\mu\nu}{}_{;\alpha;\beta} - T^{\mu\nu}{}_{;\beta;\alpha})$$

Finally, using $V_{\nu;\alpha;\beta} - V_{\nu;\beta;\alpha} = R^\rho{}_{\nu\alpha\beta} V_\rho$, we get

$$(V_\nu T^{\mu\nu})_{;\alpha;\beta} - (V_\nu T^{\mu\nu})_{;\beta;\alpha} = R^\rho{}_{\nu\alpha\beta} V_\rho T^{\mu\nu} + V_\nu (T^{\mu\nu}{}_{;\alpha;\beta} - T^{\mu\nu}{}_{;\beta;\alpha})$$

Since we already showed that

$$(V_\nu T^{\mu\nu})_{;\alpha;\beta} - (V_\nu T^{\mu\nu})_{;\beta;\alpha} = -R^\mu{}_{\rho\alpha\beta} V_\nu T^{\rho\nu}$$

it follows that:

$$V_\nu (T^{\mu\nu}{}_{;\alpha;\beta} - T^{\mu\nu}{}_{;\beta;\alpha}) = -R^\mu{}_{\rho\alpha\beta} V_\nu T^{\rho\nu} - R^\rho{}_{\nu\alpha\beta} V_\rho T^{\mu\nu}$$

In the second term above, both ρ and ν are dummy labels. Thus, we can relabel by swapping them. Then, we get

$$V_\nu (T^{\mu\nu}{}_{;\alpha;\beta} - T^{\mu\nu}{}_{;\beta;\alpha}) = -V_\nu (R^\mu{}_{\rho\alpha\beta} T^{\rho\nu} + R^\nu{}_{\rho\alpha\beta} T^{\mu\rho})$$

Since V_ν is arbitrary, we conclude that

$$\boxed{T^{\mu\nu}{}_{;\alpha;\beta} - T^{\mu\nu}{}_{;\beta;\alpha} = -R^\mu{}_{\rho\alpha\beta} T^{\rho\nu} - R^\nu{}_{\rho\alpha\beta} T^{\mu\rho}}$$

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② $T^{\mu\nu}{}_{;\nu} = 0$ implies that

$$(\rho u^\mu u^\nu)_{;\nu} = 0$$

$$u^\mu (\rho u^\nu)_{;\nu} + \rho u^\nu u^\mu{}_{;\nu} = 0$$

Multiply by $g_{\alpha\mu} u^\alpha$ and use $g_{\alpha\nu} u^\alpha u^\nu = c^2$ to obtain

$$c^2 (\rho u^\nu)_{;\nu} + \rho u^\nu g_{\alpha\mu} u^\alpha u^\mu{}_{;\nu} = 0$$

I claim that the second term above vanishes. To see this, take the covariant derivative of the equation $g_{\alpha\mu} u^\alpha u^\mu = c^2$

$$g_{\alpha\mu} (u^\alpha u^\mu{}_{;\nu} + u^\alpha{}_{;\nu} u^\mu) = 0$$

Since $g_{\alpha\mu} = g_{\mu\alpha}$, this equation is equivalent to

$$g_{\alpha\mu} u^\alpha u^\mu{}_{;\nu} = 0$$

after dividing by two. We conclude that

$$(\rho u^\nu)_{;\nu} = 0$$

since c^2 is a constant. Returning to the result

$$u^\mu (\rho u^\nu)_{;\nu} + \rho u^\nu u^\mu{}_{;\nu} = 0$$

since the first term is now known to vanish, it follows that

$$u^\nu u^\mu{}_{;\nu} = 0$$

or

$$u^\nu (u^\mu{}_{,\nu} + \Gamma^\mu_{\rho\nu} u^\rho) = 0$$

Finally, by using the chain rule, we note that

$$u^\nu u^\mu{}_{,\nu} = \frac{dx^\nu}{d\tau} \frac{\partial u^\mu}{\partial x^\nu} = \frac{du^\mu}{d\tau}$$

Thus,

$$\frac{du^\mu}{d\tau} + \Gamma^\mu_{\rho\nu} u^\rho u^\nu = 0$$

Putting $u^\mu = \frac{dx^\mu}{d\tau}$, we obtain the geodesic equation

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\nu} \frac{dx^\rho}{d\tau} \frac{dx^\nu}{d\tau} = 0 .$$

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3 (a) Using the results of Kenyon, Appendix D, the metric takes the form

$$ds^2 = A(r)c^2dt^2 - B(r)dr^2 - r^2d\Omega^2$$

and

$$R_{00} = \frac{A''}{2B} - \frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{A'}{rB}$$

$$R_{11} = \frac{-A''}{2A} + \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{B'}{rB}$$

$$R_{22} = 1 - \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right) - \frac{1}{B}$$

$$R_{33} = R_{22}\sin^2\theta$$

We wish to solve for $A(r)$ and $B(r)$ given $R_{\mu\nu} = -\Lambda g_{\mu\nu}$

First we note that

$$\frac{R_{00}}{A} + \frac{R_{11}}{B} = \frac{A'/A + B'/B}{rB} = -\Lambda\left(\frac{g_{00}}{A} + \frac{g_{11}}{B}\right)$$

But $g_{00} = A$ and $g_{11} = -B$, so we find

$$\frac{A'}{A} + \frac{B'}{B} = 0$$

$$\frac{1}{AB}\frac{d}{dr}(AB) = 0$$

Hence, $AB =$ constant. At $r \to \infty$, the metric must reduce to the Minkowski metric where $A = B = 1$. Hence,

$$AB = 1$$

or $B = \frac{1}{A}$. Substitute [redacted] this result into the equation for R_{22}.

Using $B' = \frac{d}{dr}\left(\frac{1}{A}\right) = \frac{-A'}{A^2}$, we find:

$$1 - rA' - A = -\Lambda g_{22}$$

Using $g_{22} = -r^2$, we obtain

$$1 - rA' - A = \Lambda r^2$$

or equivalently,

$$\frac{d}{dr}[r(1-A)] = \Lambda r^2$$

Thus,

$$r(1-A) = C_1 + \frac{1}{3}\Lambda r^3$$

where C_1 is a constant of integration. Solving for A yields:

$$A = 1 - \frac{C_1}{r} - \frac{1}{3}\Lambda r^2$$

We identify the constant by noting that when $\Lambda = 0$, $C_1 = \frac{2GM}{c^2}$. Thus,

$$A(r) = B^{-1}(r) = 1 - \frac{2GM}{rc^2} - \frac{1}{3}\Lambda r^2$$

Thus, the new metric when $\Lambda \neq 0$ takes the form:

$$ds^2 = \left(1 - \frac{2GM}{rc^2} - \frac{1}{3}\Lambda r^2\right)c^2dt^2 - \left(1 - \frac{2GM}{rc^2} - \frac{1}{3}\Lambda r^2\right)^{-1}dr^2 - r^2d\Omega^2$$

(b) To obtain the orbit equation, we follow the derivation given in class for the Schwarzschild metric. First, we work out the geodesic equations.

$$L = \left(1 - \frac{2GM}{rc^2} - \frac{1}{3}\Lambda r^2\right)c^2\dot{t}^2 - \left(1 - \frac{2GM}{rc^2} - \frac{1}{3}\Lambda r^2\right)^{-1}\dot{r}^2 - r^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2)$$

We take the orbit to lie in the equatorial plane and set $\theta = \frac{\pi}{2}$. Then,

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{t}}\right) = \frac{\partial L}{\partial t} \implies \frac{d}{d\tau}\left[\left(1 - \frac{2GM}{rc^2} - \frac{1}{3}\Lambda r^2\right)\dot{t}\right] = 0$$

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Then,
$$E = mc^2\left(1-\frac{2GM}{rc^2}-\frac{1}{3}\Lambda r^2\right)\dot{t}$$
is a constant. Next,
$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot\phi}\right)=\frac{\partial L}{\partial \phi}$$
For $\theta=\frac{\pi}{2}$, we have $r^2\dot\phi = J$ is a constant. Finally, we set
$$L = g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}=\frac{(c\,d\tau)^2}{(d\tau)^2}=c^2$$
Thus, for $\theta=\frac{\pi}{2}$,
$$\left(1-\frac{2GM}{rc^2}-\frac{1}{3}\Lambda r^2\right)c^2\dot t^2-\left(1-\frac{2GM}{rc^2}-\frac{1}{3}\Lambda r^2\right)^{-1}\dot r^2-r^2\dot\phi^2=c^2$$
Insert $\dot t = \frac{E}{mc^2}\left(1-\frac{2GM}{rc^2}-\frac{1}{3}\Lambda r^2\right)^{-1}$ and $\dot\phi=\frac{J}{r^2}$

to obtain
$$\frac{1}{2}\left(\frac{E^2}{mc^2}-mc^2\right)=\frac{1}{2}m\dot r^2+\frac{mJ^2}{2r^2}\left(1-\frac{2GM}{rc^2}-\frac{1}{3}\Lambda r^2\right)-\frac{GMm}{r}-\frac{1}{6}\Lambda mc^2r^2$$
Define $u=\frac{1}{r}$. We can write:
$$\dot r=\frac{dr}{d\tau}=\frac{dr}{d\phi}\frac{d\phi}{d\tau}=\frac{dr}{d\phi}\dot\phi=\frac{J}{r^2}\frac{dr}{d\phi}=-J\frac{du}{d\phi}$$
Hence,
$$\frac{1}{2}\left(\frac{E^2}{mc^2}-mc^2\right)=\frac{1}{2}mJ^2\left(\frac{du}{d\phi}\right)^2+\frac{m}{2}J^2u^2\left(1-\frac{2GMu}{c^2}-\frac{\Lambda}{3u^2}\right)-GMmu-\frac{\Lambda mc^2}{6u^2}$$
Taking the derivative with respect to ϕ, and dividing by $\frac{du}{d\phi}$, we obtain:

$$J^2\frac{d^2u}{d\phi^2}+J^2u\left(1-\frac{2GMu}{c^2}-\frac{\Lambda}{3u^2}\right)-\frac{1}{2}J^2u^2\left(\frac{2GM}{c^2}-\frac{2\Lambda}{3u^3}\right)-GM+\frac{\Lambda c^2}{3u^3}=0$$
Simplifying the above expression yields:
$$\boxed{\frac{d^2u}{d\phi^2}+u-\frac{GM}{J^2}=\frac{3GMu^2}{c^2}-\frac{\Lambda c^2}{3J^2u^3}}$$

(c) We shall now solve the new orbit equation following the same strategy used in class.

First, we note the solution to
$$\frac{d^2u}{d\phi^2}+u-\frac{GM}{J^2}=0$$
is
$$u=\frac{GM}{J^2}(1+e\cos\phi)$$
Hence,
$$\frac{d^2u}{d\phi^2}+u-\frac{GM}{J^2}=\frac{3GM}{c^2}\left(\frac{GM}{J^2}\right)^2(1+e\cos\phi)^2-\frac{\Lambda c^2}{3J^2}\left(\frac{J^2}{GM}\right)^3\frac{1}{(1+e\cos\phi)^3}$$
We now assume that the orbit is nearly circular, i.e. $|e|\ll 1$. Then, we can expand the last term above:
$$\frac{1}{(1+e\cos\phi)^3}\simeq 1-3e\cos\phi+6e^2\cos^2\phi$$

So, we must solve

$$\frac{d^2u}{d\phi^2} + u - \frac{GM}{J^2} = k_1 + k_2 \cos\phi + k_3 \cos^2\phi$$

where
$$k_1 = \frac{3G^3M^3}{c^2J^4} - \frac{\Lambda c^2 J^4}{3G^3M^3}$$

$$k_2 = \left(\frac{6G^3M^3}{c^2J^4} + \frac{\Lambda c^2J^4}{G^3M^3}\right) e$$

$$k_3 = \left(\frac{3G^3M^3}{c^2J^4} - \frac{2\Lambda c^2J^4}{G^3M^3}\right) e^2$$

Following the analysis given in class, the solution is

$$u = \frac{GM}{J^2}(1 + e\cos\phi) + k_1 + \tfrac{1}{2}k_2\phi\sin\phi + \tfrac{1}{2}k_3\left(1 - \tfrac{1}{3}\cos 2\phi\right)$$

It is the $\phi\sin\phi$ term that contributes significantly to the advance of the perihelion, since this term can become significant after many revolutions. Hence,

$$u = \frac{GM}{J^2}\left(1 + e\cos\phi + e\alpha\phi\sin\phi\right)$$

where $\frac{2GM}{J^2}e\alpha \equiv k_2$. Since $|\alpha| \ll 1$, we can approximate

$$\cos\phi + \alpha\phi\sin\phi \simeq \cos[(1-\alpha)\phi]$$

so that
$$u = \frac{GM}{J^2}\left[1 + e\cos[(1-\alpha)\phi]\right]$$

At the perihelion, $(1-\alpha)\phi = 2\pi n$, or

$$\phi \simeq 2\pi n(1+\alpha)$$

which implies an advance of the perihelion per rotation of

$$\Delta\phi = 2\pi\alpha = \frac{\pi J^2 k_2}{GMe}$$

The parameter a is defined as half the distance between perihelion and aphelion. In the Newtonian approximation, the orbit equation is

$$u = \frac{1}{r} = \frac{1 + e\cos\phi}{a(1-e^2)}$$

so that at $\phi = \pi$, $r = a(1+e)$ [aphelion] and at $\phi = 0$, $r = a(1-e)$ (perihelion). Thus, we can identify

$$\frac{J^2}{GM} = a(1-e^2)$$

Hence, to first order in the relativistic correction, using the explicit form for k_2,

$$\Delta\phi = \frac{\pi J^2}{GM}\left(\frac{6G^3M^3}{c^2J^4} + \frac{\Lambda c^2J^4}{G^3M^3}\right)$$

Putting $J^2 = GMa(1-e^2)$, we arrive at the final result:

$$\boxed{\Delta\phi = \frac{6\pi GM}{c^2a(1-e^2)} + \frac{\pi\Lambda c^2a^3(1-e^2)^3}{GM}}$$

The first term on the right hand side corresponds to the prediction for $\Lambda = 0$ obtained in class.

The observed $\Delta\phi$ for mercury is 43.11 ± 0.45 arcseconds per century, and the prediction of general relativity, for $\Lambda = 0$ is 43.03 in the same units. Thus, we can be fairly confident that the effect of the term proportional to Λ cannot be as large as about 1 arcsecond per century. We will use this to set a bound on Λ.

Mercury makes 415 revolutions per century, has an eccentricity of $e = 0.2056$ and a semi-major axis of $a = 5.791 \times 10^{10}$ m. Thus,

$$\frac{\pi c^2a^3(1-e^2)^3}{GM} = \frac{\pi(3\times10^8\,\mathrm{ms^{-1}})^2(5.791\times10^{10}\,\mathrm{m})^3[1-(0.2056)^2]^3}{(6.6726\times10^{-11}\,\mathrm{kg^{-1}m^3s^{-2}})(1.99\times10^{30}\,\mathrm{kg})}$$

$$= 3.6327\times10^{29}\ \mathrm{m^2}$$

To obtain arcseconds per century, we must multiply by 415 (corresponding to the number of revolutions mercury makes around the sun per century) and convert radians to arcseconds by multiplying by

$$(3600)\left(\frac{180}{\pi}\right)$$

since there are $180/\pi$ degrees in one radian, 60 arcminutes in $1°$ and 60 arcseconds in one arcminute.

Thus, the term proportional to Λ contributes

$$\left(3.6327\times10^{29}\,m^2\right)(415)(3600)\left(\frac{180}{\pi}\right)\Lambda$$

$$= \left(3.11\times10^{37}\,m^2\right)\Lambda \quad \text{arcseconds per century}$$

We demand that

$$\left(3.11\times10^{37}\,m^2\right)\Lambda \lesssim 1$$

or

$$\boxed{\Lambda \lesssim 3.2\times10^{-38}\,m^{-2}}$$

This should be compared with $\Lambda \sim 10^{-52}\,m^{-2}$, which is the value of the cosmological constant deduced from astrophysics data.

Remark: as a numerical check, note that:

$$\frac{6\pi GM}{c^2 a(1-e^2)} = \frac{6\pi(6.6726\times10^{-11}kg^{-1}m^3s^{-2})(1.99\times10^{30}kg)}{(2.99792\times10^8 ms^{-1})^2(5.791\times10^{10}m)\left[1-(0.2056)^2\right]}$$

$$= 5.02\times10^{-7} \text{ radians per revolution}$$

Multiply by $(415)(3600)\left(\frac{180}{\pi}\right)$ to obtain 43 arcseconds per century.

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(4) The Lagrangian corresponding to the Schwarzschild metric is:

$$L = \left(1-\frac{2GM}{rc^2}\right)c^2\dot{t}^2 - \left(1-\frac{2GM}{rc^2}\right)^{-1}\dot{r}^2 - r^2\left(\dot{\theta}^2+\sin^2\theta\,\dot{\phi}^2\right)$$

(a) From

$$\frac{d}{ds}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = \frac{\partial L}{\partial\theta}, \quad \text{we obtain}$$

$$2\frac{d}{ds}\left(r^2\dot{\theta}\right) = 2r^2\sin\theta\cos\theta\,\dot{\phi}^2$$

$$r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} = r^2\sin\theta\cos\theta\,\dot{\phi}^2$$

$$\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta\cos\theta\,\dot{\phi}^2 = 0$$

(b) At $\tau=0$, we may always choose the axes such that the particle lies in the x-y plane, corresponding to $\theta=\frac{\pi}{2}$.

Moreover, if $\dot{\theta}\neq0$, simply choose coordinates rotating with respect to the original coordinates such that in the new coordinate system, $\dot{\theta}=0$.

Thus, without loss of generality, we can assume that $\theta=\frac{\pi}{2}$ and $\dot{\theta}=0$ at $\tau=0$. Then, from the result of part (a), it follows that $\ddot{\theta}=0$ at $\tau=0$. Moreover, if we differentiate the result of part (a) with respect to τ and use the fact that

$$\frac{d}{d\tau}f(\theta) = \dot{\theta}\frac{df}{d\theta}$$

for any function of θ, it follows that $\dddot{\theta}=0$ at $\tau=0$. Repeated differentiation with respect to τ yields the conclusion that:

$$\frac{d^n\theta}{d\tau^n} = 0 \quad \text{at } \tau=0 \quad \text{for all } n=1,2,3,\ldots$$

By considering the Taylor series of $\theta(\tau)$ about $\tau=0$, it then follows that $\theta(\tau)=\frac{\pi}{2}$ for all τ. That is, the orbital motion is planar.

⑤ (a) Start from the Lagrangian

$$L = g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = c^2\left(1-\frac{2GM}{rc^2}\right)\dot{t}^2 - \left(1-\frac{2GM}{rc^2}\right)^{-1}\dot{r}^2 - r^2\dot{\phi}^2$$

where we have set $\theta = \frac{\pi}{2}$ = constant [so that $\dot{\theta}=0$]. The parameter λ is an affine parameter.

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial\dot{\phi}}\right) = \frac{\partial L}{\partial\phi} \quad\Longrightarrow\quad r^2\dot{\phi} = \bar{J}$$

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial\dot{t}}\right) = \frac{\partial L}{\partial t} \quad\Longrightarrow\quad \frac{d}{d\lambda}\left[\left(1-\frac{2GM}{rc^2}\right)\dot{t}\right] = 0$$

so that $\left(1-\frac{2GM}{rc^2}\right)\dot{t} = \bar{E}$

where $\bar{J}$ and $\bar{E}$ are constants of the motion.

Finally, we note that for photon orbits, $g_{\mu\nu}dx^\mu dx^\nu = 0$. Hence

$$c^2\left(1-\frac{2GM}{rc^2}\right)\dot{t}^2 - \left(1-\frac{2GM}{rc^2}\right)^{-1}\dot{r}^2 - r^2\dot{\phi}^2 = 0$$

Inserting $\dot{t} = \bar{E}\left(1-\frac{2GM}{rc^2}\right)^{-1}$ and $\dot{\phi} = \bar{J}/r^2$ above yields:

$$\bar{E}^2c^2\left(1-\frac{2GM}{rc^2}\right)^{-1} - \left(1-\frac{2GM}{rc^2}\right)^{-1}\dot{r}^2 - \frac{\bar{J}^2}{r^2} = 0$$

Solving for $\bar{E}$ gives:

$$\bar{E}^2 = \frac{1}{c^2}\dot{r}^2 + \frac{\bar{J}^2}{c^2r^2}\left(1-\frac{2GM}{rc^2}\right)$$

(b) Define the effective potential:

$$\bar{E}^2 = \frac{1}{c^2}\dot{r}^2 + V_{eff}, \qquad \text{where} \quad V_{eff}(r) = \frac{\bar{J}^2}{c^2r^2}\left(1-\frac{2GM}{rc^2}\right)$$

First, let us compute the extremum of the potential, $V_{eff}(r)$:

$$\frac{dV_{eff}}{dr} = 0 \quad\Longrightarrow\quad \frac{-2\bar{J}^2}{c^2r^3}\left(1-\frac{2GM}{rc^2}\right) + \frac{\bar{J}^2}{c^2r^2}\frac{2GM}{c^2r^2} = 0$$

This simplifies to:

$$r\left(1-\frac{2GM}{rc^2}\right) = \frac{GM}{c^2}$$

$$r = \frac{3GM}{c^2} \equiv \frac{3}{2}r_s.$$

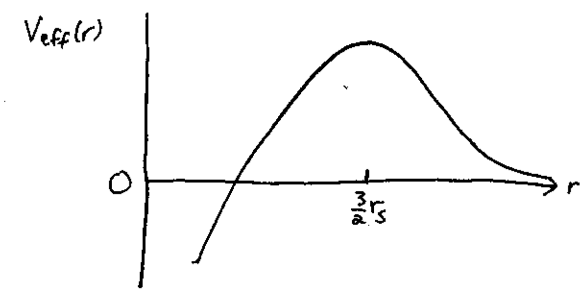
Compute the second derivative:

$$\frac{d^2V_{eff}}{dr^2} = \frac{6\bar{J}^2}{c^2r^4}\left(1-\frac{2GM}{rc^2}\right) - \frac{2\bar{J}^2}{c^2r^3}\frac{2GM}{r^2c^2} - \frac{8\bar{J}^2GM}{c^4r^5}$$

$$= \frac{6\bar{J}^2}{c^2r^4}\left(1-\frac{3r_s}{r}\right) \qquad \text{where } r_s \equiv \frac{2GM}{c^2}$$

At $r = \frac{3}{2}r_s$, $\frac{d^2V_{eff}}{dr^2} = -\frac{6\bar{J}^2}{c^2r^4} < 0$.

Thus, $r = \frac{3}{2}r_s$ is a maximum of $V_{eff}(r)$:



That is, the point $r=\frac{3}{2}r_s$ corresponds to an unstable circular orbit. That is, since $V_{eff}(r) = \frac{\bar{J}^2}{3c^2r_s^2}$ at $r = \frac{3}{2}r_s$, we see that if $\bar{E}^2 = \frac{4\bar{J}^2}{27c^2r_s^2}$, then at $r = \frac{3}{2}r_s$ we have $\dot{r}=0$ corresponding to circular motion.

(c) Start from the result

$$c^2\left(1-\frac{2GM}{rc^2}\right)\dot{t}^2-\left(1-\frac{2GM}{rc^2}\right)^{-1}\dot{r}^2-r^2\dot{\phi}^2=0$$

For circular motion, $\dot{r}=0$. Thus,

$$\dot{\phi}^2=\frac{c^2}{r^2}\left(1-\frac{2GM}{rc^2}\right)\dot{t}^2$$

For $r=\frac{3}{2}r_s$, this reduces to

$$\dot{\phi}^2=\frac{4}{27}\frac{c^2}{r_s^2}\dot{t}^2$$

Now, $$\frac{d\phi}{dt}=\frac{d\phi/d\lambda}{dt/d\lambda}=\frac{\dot{\phi}}{\dot{t}}=\frac{2}{3\sqrt{3}}\frac{c}{r_s}$$

Thus, $$t=\frac{3\sqrt{3}\,r_s}{2c}\int d\phi$$

Integrating over one revolution yields a period of

$$\Delta t=\frac{3\sqrt{3}\,\pi r_s}{c}$$

On problem 4 of the midterm, we noted that an observer fixed at a point (r,θ,ϕ) in a Schwarzschild geometry measures proper time equal to

$$\Delta T=\left(1-\frac{2GM}{c^2r}\right)^{1/2}\Delta t=\left(1-\frac{r}{r_s}\right)^{1/2}\Delta t$$

At $r=\frac{3}{2}r_s$, the observer measures the time to complete one revolution

$$\Delta T=\frac{1}{\sqrt{3}}\Delta t=\frac{3\pi r_s}{c}$$

(d) The very distant observer measures Schwarzschild time, so the orbital period measured is

$$\Delta t=\frac{3\sqrt{3}\,\pi r_s}{c}$$

(e) The orbit equation for the photon is:

$$\frac{d^2u}{d\phi^2}+u=\frac{3GM}{c^2}u^2$$

where $u=\frac{1}{r}$. In terms of r,

$$\frac{du}{dr}=\frac{d}{dr}\left(\frac{1}{r}\right)=-\frac{1}{r^2}\frac{dr}{d\phi}$$

$$\frac{d^2u}{dr^2}=-\frac{1}{r^2}\frac{d^2r}{d\phi^2}+\frac{2}{r^3}\left(\frac{dr}{d\phi}\right)^2$$

Thus,

$$\frac{d^2r}{d\phi^2}-\frac{2}{r}\left(\frac{dr}{d\phi}\right)^2=r-\frac{3GM}{c^2}$$

$$=r-\frac{3}{2}r_s$$

Note that $r=\frac{3}{2}r_s$ is a solution to this equation.
If we substitute $r=\frac{3}{2}r_s+\eta$, we obtain

$$\frac{d^2\eta}{d\phi^2}-\frac{2}{\frac{3}{2}r_s+\eta}\left(\frac{d\eta}{d\phi}\right)^2=\eta$$

If $|\eta/r_s|\ll 1$, then we can neglect the term proportional to $(d\eta/d\phi)^2$ which is quadratic in η. Then, we are left with:

$$\frac{d^2\eta}{d\phi^2}=\eta$$

The solution to this equation is $\eta=Ae^{\phi}+Be^{-\phi}$ which exhibits exponential growth in ϕ. Thus, the size of the perturbation grows without bound [rather than oscillating as in the case of a stable orbit]. Hence, the circular orbit at $r=\frac{3}{2}r_s$ is unstable.