Physics 171 Solution Set #5 Fall 2001

① The Friedmann-LeMaître equations are:

$$\frac{\dot{R}^2}{R^2} + \frac{kc^2}{R^2} - \frac{1}{3}c^2\Lambda = \frac{8\pi G\rho}{3}$$

$$\frac{2\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{kc^2}{R^2} - c^2\Lambda = -\frac{8\pi G p}{c^2}$$

(a) Multiply the first equation above by $R^3$ and then differentiate with respect to time.

$$\frac{d}{dt}\left(\dot{R}^2R + kc^2R - \tfrac{1}{3}c^2\Lambda R^3\right) = \frac{8\pi G}{3}\frac{d}{dt}(R^3\rho)$$

$$\dot{R}^3 + 2R\dot{R}\ddot{R} + kc^2\dot{R} - c^2\Lambda R^2\dot{R} = \frac{8\pi G}{3}\frac{d}{dt}(R^3\rho)$$

$$\dot{R}R^2\left[\frac{\dot{R}^2}{R^2} + \frac{2\ddot{R}}{R} + \frac{kc^2}{R^2} - c^2\Lambda\right] = \frac{8\pi G}{3}\frac{d}{dt}(R^3\rho)$$

Using the second equation above, we recognize the expression in brackets as $\frac{-8\pi G p}{c^2}$. Hence,

$$-\frac{8\pi G p}{c^2}\dot{R}R^2 = \frac{8\pi G}{3}\frac{d}{dt}(R^3\rho)$$

or

$$\frac{d}{dt}(\rho R^3) = -\frac{3p}{c^2}R^2\dot{R}$$

On the left hand side, use the chain rule to write $\frac{d}{dt} = \frac{dR}{dt}\frac{d}{dR} = \dot{R}\frac{d}{dR}$. Canceling the common factor of $\dot{R}$, we obtain:

$$\frac{d}{dR}(\rho R^3) = -\frac{3pR^2}{c^2}$$



(b) For radiation, the appropriate equation of state is

$$p = \tfrac{1}{3}\rho c^2$$

Inserting the result for $p$ into the result of part (a),

$$\frac{d}{dR}(\rho R^3) = -\rho R^2$$

$$R^3\frac{d\rho}{dR} + 3R^2\rho = -\rho R^2$$

$$\frac{d\rho}{dR} = -\frac{4\rho}{R}$$

$$\frac{d\rho}{\rho} = -4\frac{dR}{R}$$

Integrating this equation yields

$$\rho \propto \frac{1}{R^4}$$

2 (a) In class, we showed that the distance to our particle horizon is

$$d_H = cR(t_0)\int_0^{t_0} \frac{dt}{R(t)}$$

This was derived by noting that for radial motion in a Robertson-Walker metric,

$$ds^2 = c^2 dt^2 - R^2(t)\frac{d\sigma^2}{1-k\sigma^2}$$

For light travel, $ds^2 = 0$ so

$$\frac{d\sigma}{dt} = \pm \frac{c\sqrt{1-k\sigma^2}}{R(t)}$$

A light signal emitted from a radial coordinate position $\sigma_H$ at time $t=0$ reaches us today at radial coordinate position $\sigma=0$ at time $t=t_0$. Then,

$$\int_0^{t_0} \frac{c\,dt}{R(t)} = \int_0^{\sigma_H} \frac{d\sigma}{\sqrt{1-k\sigma^2}}$$

The proper distance from the source of the light signal to our present position is

$$d_H = \int \sqrt{g_{\sigma\sigma}}\, d\sigma = R(t_0)\int_0^{\sigma_H} \frac{d\sigma}{\sqrt{1-k\sigma^2}}$$

Using the result of the previous equation, the result stated at the top of the page follows.

Let us now change the variable of integration from $t$ to $R$. Using

$$dt = \frac{dt}{dR}dR = \frac{dR}{\dot{R}} = \frac{dR}{HR}$$

where the last step follows from the definition $H \equiv \dot{R}/R$, it follows that

$$d_H = cR(t_0)\int_0^{R(t_0)} \frac{dR}{R^2 H}$$



where we have used the fact that the scale factor vanishes at $t=0$, i.e. $R(0)=0$.

Finally, define $a(t) \equiv \dfrac{R(t)}{R(t_0)}$. Changing variables from $R$ to $a$ in the integral, $dR = R(t_0)\,da$, and we end up with

$$d_H = c\int_0^1 \frac{da}{a^2 H}$$

(b) In class, we showed that

$$H^2 = H_0^2\left[\frac{\rho_m + \rho_\Lambda}{\rho_{c0}} + (1-\Omega_T)\frac{R^2(t_0)}{R^2(t)}\right]$$

where $\rho_{c0} \equiv \dfrac{3H_0^2}{8\pi G}$ is the critical density today.

In an Einstein-de Sitter cosmology, $\Omega_T = 1$ and $\Omega_\Lambda = 0$ (that is, $\rho_\Lambda = 0$). Hence,

$$H^2 = \frac{H_0^2 \rho_m}{\rho_{c0}}$$

where $\rho_m$ is the mass density of the matter at time $t$. For a matter-dominated universe, we may set the pressure to zero. Then,

$$\frac{d}{dt}(\rho_m R^3) = 0$$

[see problem 1, part (a)], so that $\rho_m R^3$ is constant.

Today, at time $t_0$, $\rho_m(t_0) = \rho_{c0}$. Thus,

$$\rho_m(t) R^3(t) = \rho_{c0} R^3(t_0)$$

or

$$\frac{\rho_m}{\rho_{c0}} = \frac{R^3(t_0)}{R^3(t)} = \frac{1}{a^3(t)}$$

Thus,

$$H^2 = \frac{H_0^2}{a^3(t)} .$$

Inserting this into the result of part (a),

$$d_H = \frac{c}{H_0}\int_0^1 \frac{da}{a^{1/2}} = \frac{2c}{H_0}$$

If we denote $t_H = H_0^{-1}$, then

$$d_H = 2ct_H .$$

(c) If $\Omega_m = \Omega_\Lambda = 0$, and $\Omega_T = \Omega_r = 1$ (radiation-dominated universe), then,

$$H^2 = \frac{H_0^2 \rho_r}{\rho_{c0}}$$

where $\rho_r c^2$ is the energy density of the radiation at time $t$. Using the result of problem 1, part (b),

$$\rho_r R^4 = \text{constant}.$$

Since $\Omega_T = \Omega_r = 1$, $\rho_r(t_0) = \rho_{c0}$ [this also follows by setting $t = t_0$ in the equation for H]. Thus,

$$\rho_r(t) R^4(t) = \rho_{c0} R^4(t_0)$$

or

$$\frac{\rho_r}{\rho_{c0}} = \frac{R^4(t_0)}{R^4(t)} = \frac{1}{a^4(t)}$$

Thus,

$$H^2 = \frac{H_0^2}{a^4(t)}$$

and so

$$d_H = \frac{c}{H_0} = ct_H$$

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③ In class, we showed that

$$1 + z = \frac{R(t_0)}{R(t)}$$

For an Einstein-de Sitter (matter dominated) universe,

$$R(t) \propto t^{2/3}$$

Thus,

$$1 + z = \left(\frac{t_0}{t}\right)^{2/3}$$

If $z = 1$, then $\left(\frac{t_0}{t}\right)^{2/3} = 2$. Thus,

$$\frac{t}{t_0} = 2^{-3/2} = 0.354$$

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(4) In class, we showed that a light signal emitted from $\sigma = \sigma_H$ (where $\sigma$ is the dimensionless radial coordinate of the Robertson-Walker metric) at $t=0$ and travels on a radial trajectory to our location, $\sigma=0$ at $t=t_0$ satisfies:

$$\int_0^{t_0} \frac{c\,dt}{R(t)} = \int_0^{\sigma_H} \frac{d\sigma}{\sqrt{1-k\sigma^2}}$$

This defines the particle horizon, since any light signal emitted at $t=0$ from a value of $\sigma > \sigma_H$ has not yet had a chance to reach $\sigma=0$ at $t=t_0$

Assume that $\Omega_T \le 1$. If $\Omega_T < 1$ then $k=-1$, while if $\Omega_T = 1$ then $k=0$. We can solve for $\sigma_H$.

$$\int_0^{t_0} \frac{c\,dt}{R(t)} = \begin{cases} \sinh^{-1}\sigma_H\,, & k=-1 \\ \sigma_H\,, & k=0 \end{cases}$$

so that

$$\sigma_H = \begin{cases} \sinh\left(\int_0^{t_0} \frac{c\,dt}{R(t)}\right), & k=-1 \\ \int_0^{t_0} \frac{c\,dt}{R(t)}\,, & k=0 \end{cases}$$

It will be convenient to follow the analysis of problem 2 by writing

$$\int_0^{t_0} \frac{c\,dt}{R(t)} = c\int_0^{R(t_0)} \frac{dR}{R^2 H}$$

Using the expression for $H$ derived in class:

$$H = H_0\left[\frac{\Omega_m}{a^3} + \frac{\Omega_r}{a^4} + \Omega_\Lambda + \frac{(1-\Omega_T)}{a^2}\right]^{1/2}, \qquad a(t) \equiv \frac{R(t)}{R(t_0)}$$

we obtain

$$\int_0^{t_0} \frac{c\,dt}{R(t)} = c\int_0^{R(t_0)} \frac{dR}{R^2 H_0}\left[\frac{\Omega_m}{a^3} + \frac{\Omega_r}{a^4} + \Omega_\Lambda + \frac{(1-\Omega_T)}{a^2}\right]^{-1/2}$$



In this problem, we are asked whether the entire universe will eventually come within our horizon. Thus, we take $t_0 \to \infty$ and examine the behavior of $\sigma_H$.

Note that for an open universe, $R(t_0) \to \infty$ as $t_0 \to \infty$. Thus,

$$\int_0^{\infty} \frac{c\,dt}{R(t)} = \frac{c}{H_0}\int_0^{\infty} \frac{dR}{R^2}\left[\frac{\Omega_m}{a^3} + \frac{\Omega_r}{a^4} + \Omega_\Lambda + \frac{(1-\Omega_T)}{a^2}\right]^{-1/2}$$

Changing variables to $a(t) = \frac{R(t)}{R_0(t)}$, we note that $\frac{dR}{R^2} = \frac{da}{R(t_0)\,a^2}$. Thus,

$$\boxed{\int_0^{\infty} \frac{c\,dt}{R(t)} = \frac{c}{R(t_0)H_0}\int_0^{\infty} da\left[\Omega_r + \Omega_m a + \Omega_\Lambda a^4 + (1-\Omega_T)a^2\right]^{-1/2}}$$

(a) If $\Omega_\Lambda = 0$, then

$$\int_0^{\infty} \frac{c\,dt}{R(t)} = \frac{c}{R(t_0)H_0}\int_0^{\infty} \frac{da}{\sqrt{\Omega_r + \Omega_m a + (1-\Omega_T)a^2}} = \infty$$

The integral diverges since at large $a$, it behaves as $\int^{\infty} \frac{da}{a} = \ln a\Big|^{\infty} = \infty$

Thus, $\sigma_H = \infty$. That is, all of the universe will eventually come within our horizon.

(b) If $\Omega_\Lambda > 0$, then

$$\int_0^{\infty} \frac{c\,dt}{R(t)} = \frac{c}{R(t_0)H_0}\int_0^{\infty} \frac{da}{\sqrt{\Omega_r + \Omega_m a + (1-\Omega_T)a^2 + \Omega_\Lambda a^4}} < \infty$$

The integral converges since at large $a$, it behaves as $\int^{\infty} \frac{da}{a^2}$. Note the importance of $\Omega_\Lambda > 0$; otherwise there would exist a singularity due to the vanishing of the denominator of the integrand for some finite positive value of $a$.

Thus,

$$\sigma_H = \begin{cases} \sinh\left(\int_0^\infty \frac{c\,dt}{R(t)}\right), & k=-1 \\ \int_0^\infty \frac{c\,dt}{R(t)}, & k=0 \end{cases}$$

is finite. Thus, the universe possesses an event horizon beyond which we will never see (namely, all values of $\sigma > \sigma_H$).

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5 (a) Let us return to the expression for $H$ used in problem 4:

$$H = H_0\left[\frac{\Omega_m}{a^3} + \frac{\Omega_r}{a^4} + \Omega_\Lambda + \frac{(1-\Omega_T)}{a^2}\right]^{1/2}$$

We take $\Omega_T = \Omega_m + \Omega_\Lambda = 1$ and $\Omega_r = 0$ (since the universe is matter dominated over nearly its entire evolution since the big bang). Hence,

$$H = H_0\left[\frac{\Omega_m}{a^3} + \Omega_\Lambda\right]^{1/2}$$

or since $\Omega_\Lambda = 1-\Omega_m$,

$$H = H_0\left[1-\Omega_m + \frac{\Omega_m}{a^3}\right]^{1/2}$$

Since $H = \frac{\dot{R}}{R} = \frac{\dot{a}}{a}$ [since $a(t) = R(t)/R(t_0)$],

$$\frac{da}{dt} = H_0\, a\left[1-\Omega_m + \frac{\Omega_m}{a^3}\right]^{1/2}$$

Integrate from $t=0$ to $t=t_0$ (the present time). Noting that $a(t_0)=1$, we obtain

$$H_0 t_0 = \int_0^1 \frac{da}{a\left[1-\Omega_m + \frac{\Omega_m}{a^3}\right]^{1/2}}$$

Let $x = \frac{1}{a^3}$. Then $\frac{da}{a} = -\frac{1}{3}\frac{dx}{x}$ and

$$H_0 t_0 = \frac{1}{3}\int_1^\infty \frac{dx}{x}\,\frac{1}{\sqrt{1-\Omega_m + \Omega_m x}}$$

We consult the integral tables. There are two cases:

Case 1: $\Omega_m < 1$

$$H_0 t_0 = \frac{1}{3}\frac{1}{\sqrt{1-\Omega_m}} \ln\left(\frac{\sqrt{1-\Omega_m+\Omega_m x} - \sqrt{1-\Omega_m}}{\sqrt{1-\Omega_m+\Omega_m x} + \sqrt{1-\Omega_m}}\right)\Bigg|_1^\infty$$

$$= -\frac{1}{3}\frac{1}{\sqrt{1-\Omega_m}} \ln\left(\frac{1-\sqrt{1-\Omega_m}}{1+\sqrt{1-\Omega_m}}\right)$$

If I multiply the numerator and the denominator of the argument of the logarithm by $1+\sqrt{1-\Omega_m}$ and note that

$$(1-\sqrt{1-\Omega_m})(1+\sqrt{1-\Omega_m}) = 1-(1-\Omega_m) = \Omega_m$$

then,

$$H_0 t_0 = -\frac{1}{3}\frac{1}{\sqrt{1-\Omega_m}} \ln\left[\left(\frac{\sqrt{\Omega_m}}{1+\sqrt{1-\Omega_m}}\right)^2\right]$$

which can be written as:

$$\boxed{t_0 = \frac{2}{3}H_0^{-1}\frac{1}{\sqrt{1-\Omega_m}} \ln\left(\frac{1+\sqrt{1-\Omega_m}}{\sqrt{\Omega_m}}\right), \quad \Omega_m < 1}$$

Case 2: $\Omega_m > 1$

$$H_0 t_0 = \frac{2}{3}\frac{1}{\sqrt{\Omega_m-1}} \tan^{-1}\left(\frac{\sqrt{1-\Omega_m+\Omega_m x}}{\sqrt{\Omega_m-1}}\right)\Bigg|_1^\infty$$

$$= \frac{2}{3}\frac{1}{\sqrt{\Omega_m-1}}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{1}{\sqrt{\Omega_m-1}}\right)\right]$$

after using $\tan^{-1}\infty = \frac{\pi}{2}$. If we note the identity

$$\tan^{-1} z = \frac{\pi}{2} - \tan^{-1}\left(\frac{1}{z}\right) \quad \text{for } z>0$$

then we can write

$$\boxed{t_0 = \frac{2}{3}H_0^{-1}\frac{1}{\sqrt{\Omega_m-1}}\tan^{-1}\left(\sqrt{\Omega_m-1}\right), \quad \Omega_m > 1}$$



Note that in the limit of $\Omega_m = 1$, both formulae reduce to the well known result:

$$\boxed{t_0 = \frac{2}{3}H_0^{-1}, \quad \Omega_m = 1}$$

All the results quoted above are true only when $\Omega_T = \Omega_m + \Omega_\Lambda = 1$.

(b) We now apply the result of part (a) for the case $\Omega_m = 0.3$ and $\Omega_\Lambda = 0.7$. Then, the $\Omega_m < 1$ formula applies. We take $h = 0.7$ so that

$$H_0 = (100 \text{ km s}^{-1}\text{ Mpc}^{-1})\, h$$
$$= 70 \text{ km s}^{-1}\text{ Mpc}^{-1}$$

and

$$H_0^{-1} = (9.77813 \text{ Gyr})\, h^{-1}$$
$$= 13.97 \text{ Gyr}$$

using the astrophysical constants table from the particle data group handout.

Thus,

$$t_0 = \frac{2}{3}(13.97\text{ Gyr})\frac{1}{\sqrt{0.7}}\ln\left(\frac{1+\sqrt{0.7}}{\sqrt{0.3}}\right)$$

$$= \frac{2}{3}(13.97\text{ Gyr})(1.446)$$

$$\boxed{t_0 = 13.47 \text{ Gyr}}$$

That is, the age of the universe is roughly 13.5 billion years.