Physics 171 MIDTERM SOLUTIONS Fall 2001

1. Explicitly, we have

$$F^{i0} = -F^{0i} = E^i$$
$$F^{ij} = -\epsilon^{ijk}B^k$$

(a) Thus,

$$F^{\mu\nu}F_{\mu\nu} = F^{0i}F_{0i} + F^{i0}F_{i0} + F^{ij}F_{ij}$$
$$= -F^{0i}F^{0i} - F^{i0}F^{i0} + F^{ij}F^{ij}$$

since $F_{\mu\nu} = \eta_{\mu\alpha}\eta_{\nu\beta}F^{\alpha\beta}$, with $\eta_{\mu\alpha} = \mathrm{diag}(1,-1,-1,-1)$

Hence,

$$F^{\mu\nu}F_{\mu\nu} = -2E^iE^i + (\epsilon^{ijk}B^k)(\epsilon^{ijl}B^l)$$
$$= -2E^iE^i + 2\delta^{kl}B^kB^l$$
$$= 2(\vec{B}^2 - \vec{E}^2)$$

where I have used $\epsilon^{ijk}\epsilon^{ijl} = 2\delta^{kl}$.

(b) The second invariant is
$$F^{\mu\nu}\tilde{F}_{\mu\nu}$$
where $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$

On problem set 1, problem 4, I showed that $\tilde{F}^{\mu\nu}$ is obtained from $F^{\mu\nu}$ by making the replacements $\vec{E} \to \vec{B}$ and $\vec{B} \to -\vec{E}$. That is,

$$\tilde{F}^{i0} = -\tilde{F}^{0i} = B^i$$
$$\tilde{F}^{ij} = \epsilon^{ijk}E^k$$

Following the steps of part (a),

$$F^{\mu\nu}\tilde{F}_{\mu\nu} = F^{0i}\tilde{F}_{0i} + F^{i0}\tilde{F}_{i0} + F^{ij}\tilde{F}_{ij}$$
$$= -F^{0i}\tilde{F}^{0i} - F^{i0}\tilde{F}^{i0} + F^{ij}\tilde{F}^{ij}$$
$$= -2E^iB^i - (\epsilon^{ijk}E^k)(\epsilon^{ijl}B^l) = -4\vec{E}\cdot\vec{B}$$

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2. Given $V_\mu = g_{\mu\nu}V^\nu$

$$V_{\mu;\alpha} = (g_{\mu\nu}V^\nu)_{;\alpha} = g_{\mu\nu;\alpha}V^\nu + g_{\mu\nu}V^\nu{}_{;\alpha}$$
$$= g_{\mu\nu}V^\nu{}_{;\alpha} \qquad \text{since } g_{\mu\nu;\alpha} = 0$$
$$= g_{\mu\nu}\left[\frac{\partial V^\nu}{\partial x^\alpha} + \Gamma^\nu_{\rho\alpha}V^\rho\right]$$
$$= g_{\mu\nu}\frac{\partial V^\nu}{\partial x^\alpha} + g_{\mu\nu}\Gamma^\nu_{\rho\alpha}V^\rho$$

At this point, use

$$\frac{\partial}{\partial x^\alpha}g_{\mu\rho} = \Gamma^\nu_{\rho\alpha}g_{\mu\nu} + \Gamma^\nu_{\mu\alpha}g_{\rho\nu}$$

a result obtained in class (which is equivalent to the condition $g_{\mu\rho;\alpha} = 0$). Thus, insert

$$\Gamma^\nu_{\rho\alpha}g_{\mu\nu} = \frac{\partial}{\partial x^\alpha}g_{\mu\rho} - \Gamma^\nu_{\mu\alpha}g_{\rho\nu}$$

above to obtain:

$$V_{\mu;\alpha} = g_{\mu\nu}\frac{\partial V^\nu}{\partial x^\alpha} + V^\rho\frac{\partial}{\partial x^\alpha}g_{\mu\rho} - \Gamma^\nu_{\mu\alpha}g_{\rho\nu}V^\rho$$

In the second term on the right hand side, replace the dummy index ρ by ν. In the third term on the right hand side, use $g_{\rho\nu}V^\rho = V_\nu$.

Thus,

$$V_{\mu;\alpha} = g_{\mu\nu}\frac{\partial V^\nu}{\partial x^\alpha} + V^\nu\frac{\partial}{\partial x^\alpha}g_{\mu\nu} - \Gamma^\nu_{\mu\alpha}V_\nu$$
$$= \frac{\partial}{\partial x^\alpha}(g_{\mu\nu}V^\nu) - \Gamma^\nu_{\mu\alpha}V_\nu$$

or

$$\boxed{V_{\mu;\alpha} = \frac{\partial V_\mu}{\partial x^\alpha} - \Gamma^\nu_{\mu\alpha}V_\nu}$$

③ Given

$$t = t'$$
$$x = x'\cos\omega t' - y'\sin\omega t'$$
$$y = x'\sin\omega t' + y'\cos\omega t'$$
$$z = z'$$

(a) We compute:

$$dt = dt'$$
$$dx = dx'\cos\omega t' - dy'\sin\omega t' - \omega x'\sin\omega t'\,dt' - \omega y'\cos\omega t'\,dt'$$
$$dy = dx'\sin\omega t' + dy'\cos\omega t' + \omega x'\cos\omega t'\,dt' - \omega y'\sin\omega t'\,dt'$$
$$dz = dz'$$

Hence,

$$\begin{aligned} dx^2+dy^2 &= \left[(dx'-\omega y'dt')^2 + (dy'+\omega x'dt')^2\right]\cos^2\omega t' \\ &\quad + \left[(-dy'-\omega x'dt')^2 + (dx'-\omega y'dt')^2\right]\sin^2\omega t' \\ &\quad + 2\left[(dx'-\omega y'dt')(dy'+\omega x'dt') + (-dy'-\omega x'dt')(dx'-\omega y'dt')\right]\sin\omega t'\cos\omega t' \\ &= dx'^2+dy'^2+\omega^2(x'^2+y'^2)dt'^2 - 2\omega y'dx'dt' + 2\omega x'dy'dt' \end{aligned}$$

Thus,

$$\begin{aligned} ds^2 &= c^2dt^2 - dx^2 - dy^2 - dz^2 \\ &= [c^2-\omega^2(x'^2+y'^2)]dt'^2 + 2\omega y'dx'dt' - 2\omega x'dy'dt' - dx'^2 - dy'^2 - dz'^2. \end{aligned}$$

(b) For simplicity in notation, we henceforth omit the primes on the coordinates, and write:

$$ds^2 = [c^2-\omega^2(x^2+y^2)]dt^2 + 2\omega y\,dx\,dt - 2\omega x\,dy\,dt - dx^2 - dy^2 - dz^2.$$

To write down the geodesic equations, we therefore consider the Lagrangian

$$L = [c^2-\omega^2(x^2+y^2)]\dot t^2 + 2\omega y\dot x\dot t - 2\omega x\dot y\dot t - \dot x^2 - \dot y^2 - \dot z^2$$

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot t}\right) - \frac{\partial L}{\partial t} = 0 \quad\Longrightarrow\quad \frac{d}{ds}\Big[[c^2-\omega^2(x^2+y^2)]\dot t + \omega y\dot x - \omega x\dot y\Big] = 0$$

That is, $$\boxed{[c^2-\omega^2(x^2+y^2)]\ddot t - 2\omega^2\dot t(x\dot x + y\dot y) + \omega y\ddot x - \omega x\ddot y = 0}$$

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot x}\right) - \frac{\partial L}{\partial x} = 0 \quad\Longrightarrow\quad \frac{d}{ds}(\omega y\dot t - \dot x) + \omega^2 x\dot t^2 - \omega\dot y\dot t = 0$$

That is, $$\boxed{\ddot x - \omega^2 x\dot t^2 - 2\omega\dot y\dot t - \omega y\ddot t = 0}$$

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot y}\right) - \frac{\partial L}{\partial y} = 0 \quad\Longrightarrow\quad \frac{d}{ds}(-\omega x\dot t - \dot y) + \omega^2 y\dot t^2 - \omega\dot x\dot t = 0$$

That is, $$\boxed{\ddot y - \omega^2 y\dot t^2 + 2\omega\dot x\dot t + \omega x\ddot t = 0}$$

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot z}\right) - \frac{\partial L}{\partial z} = 0 \quad\Longrightarrow\quad \boxed{\ddot z = 0}$$

The top three boxed equations are coupled. To simplify, multiply the second equation by ωy and the third equation by $-\omega x$. Add the two results to obtain:

$$\omega y\ddot x - \omega x\ddot y - 2\omega^2\dot t(x\dot x + y\dot y) - \omega^2(x^2+y^2)\ddot t = 0$$

Inserting this result into the first equation immediately yields

$$\boxed{\ddot t = 0}$$

We can then insert this result back into the second and third equation to obtain

$$\ddot x - \omega^2 x\dot t^2 - 2\omega\dot y\dot t = 0$$
$$\ddot y - \omega^2 y\dot t^2 + 2\omega\dot x\dot t = 0$$

Remark: It is easy to see using the original coordinates that $\ddot t = 0$. Since $t' = t$ and for $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$ we see that the geodesic equations in these coordinates is trivial, namely $\ddot x^\mu = 0$. Thus $\ddot t = 0$ and $t' = t$ yields $\ddot t' = 0$.

(5)

Summarizing our results, the geodesic equations are

$$\ddot{t} = 0$$
$$\ddot{x} - \omega^2 x \dot{t}^2 - 2\omega \dot{y}\dot{t} = 0$$
$$\ddot{y} - \omega^2 y \dot{t}^2 + 2\omega \dot{x}\dot{t} = 0$$
$$\ddot{z} = 0$$

(c) Comparing the above equations with

$$\ddot{x}^\mu + \Gamma^\mu_{\alpha\beta}\, \dot{x}^\alpha \dot{x}^\beta = 0$$

we immediately read off the Christoffel symbols:

$$\Gamma^1_{00} = -\omega^2 x$$
$$\Gamma^1_{02} = \Gamma^1_{20} = -\omega$$
$$\Gamma^2_{00} = -\omega^2 y$$
$$\Gamma^2_{01} = \Gamma^2_{10} = \omega$$

and all other Christoffel symbols vanish.

(d) In the non-relativistic limit, $t \simeq \tau$. Thus we may rewrite the geodesic equations as

$$\frac{d^2x}{dt^2} = \omega^2 x + 2\omega \frac{dy}{dt}$$
$$\frac{d^2y}{dt^2} = \omega^2 y - 2\omega \frac{dx}{dt}$$
$$\frac{d^2z}{dt^2} = 0$$

Note that $\ddot{t} = 0$ is trivially solved by $dt = \gamma\, d\tau$. In the non-relativistic limit, $\gamma \simeq 1$.

(6)

The last three equations have the form:

$$\boxed{\frac{d^2\vec{r}}{dt^2} = -\vec{\omega}\times(\vec{\omega}\times\vec{r}) - 2\vec{\omega}\times\frac{d\vec{r}}{dt}}$$

where

$$\vec{\omega} = (0, 0, \omega)$$
$$\vec{r} = (x, y, z)$$

For example,

$$\vec{\omega}\times\vec{r} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \omega \\ x & y & z \end{vmatrix} = -\hat{x}(y\omega) + \hat{y}(x\omega)$$

$$-\vec{\omega}\times(\vec{\omega}\times\vec{r}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \omega \\ y\omega & -x\omega & 0 \end{vmatrix} = \omega^2(x\hat{x} + y\hat{y})$$

and

$$-\vec{\omega}\times\frac{d\vec{r}}{dt} = \omega\left[\hat{x}\frac{dy}{dt} - \hat{y}\frac{dx}{dt}\right]$$

Thus,

$$\frac{d^2x}{dt^2} = \omega^2 x + 2\omega\frac{dy}{dt}$$
$$\frac{d^2y}{dt^2} = \omega^2 y - 2\omega\frac{dx}{dt}$$
$$\frac{d^2z}{dt^2} = 0$$

as claimed.

<u>Interpretation</u>:

$-\vec{\omega}\times(\vec{\omega}\times\vec{r})$	centifugal acceleration
$-2\vec{\omega}\times\frac{d\vec{r}}{dt}$	Coriolos acceleration

④ (a) For fixed r, θ and ϕ, we have

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2$$

Thus, we identify $ds^2 = c^2 dT^2$. That is,

$$\boxed{dT = \left(1 - \frac{2GM}{c^2 r}\right)^{1/2} dt}$$

(b) For fixed t, θ, ϕ, we have

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2$$

Thus, we identify $ds^2 = -dR^2$. That is,

$$\boxed{dR = \left(1 - \frac{2GM}{c^2 r}\right)^{-1/2} dr}$$

(c) The geodesic equations can be obtained from

$$L = \left(1 - \frac{2GM}{c^2 r}\right) c^2 \dot{t}^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} \dot{r}^2 - r^2 \dot{\theta}^2 - r^2 \sin^2\theta\, \dot{\phi}^2$$

Then,

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{t}}\right) - \frac{\partial L}{\partial t} = 0 \quad \Rightarrow \quad \frac{d}{ds}\left[\left(1 - \frac{2GM}{c^2 r}\right) \dot{t}\right] = 0$$

Since $s = c\tau$, we can write this equation as

$$\frac{d}{d\tau}\left[\left(1 - \frac{2GM}{c^2 r}\right) \frac{dt}{d\tau}\right] = 0$$

This means that

$$\left(1 - \frac{2GM}{c^2 r}\right) \frac{dt}{d\tau} = \text{constant}$$

We identify

$$E = mc^2 \left(1 - \frac{2GM}{c^2 r}\right) \frac{dt}{d\tau}$$

Since in the limit of $G = 0$ (no gravitational field), we recover the special relativity result

$$E = \gamma mc^2$$

since $dt/d\tau = \gamma$ in special relativity.

By assumption, the initial velocity of the particle is zero at $r \to \infty$. In this limit, $\gamma = 1$ and we see that

$$E = mc^2 \qquad (r \to \infty).$$

Since E is a constant at all points along the trajectory, we conclude that $E = mc^2$ for any value of r. This implies that

$$\left(1 - \frac{2GM}{c^2 r}\right) \frac{dt}{d\tau} = 1$$

(d) Recall that for any curve $x^\mu = x^\mu(s)$ parameterized by its arclength, we have

$$g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 1 \qquad \dot{x}^\mu = \frac{dx^\mu}{ds}.$$

This simply follows from $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$. Since $L = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$ we conclude that $L = 1$. For purely radial motion, $d\theta = d\phi = 0$, and so

$$\left(1 - \frac{2GM}{c^2 r}\right) c^2 \left(\frac{dt}{ds}\right)^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} \left(\frac{dr}{ds}\right)^2 = 1$$

Putting $s = c\tau$ and solving for $\frac{dr}{dt}$,

$$\left(\frac{dr}{dt}\right)^2 = c^2 \left[1 - \frac{2GM}{c^2 r}\right]\left[1 - \frac{2GM}{c^2 r} - \left(\frac{d\tau}{dt}\right)^2\right]$$

To obtain this result, we multiplied the previous equation by $\left(\frac{ds}{dt}\right)^2 = c\left(\frac{d\tau}{dt}\right)^2$ and rearranged the resulting equation.

At this point, it is convenient to make use of the result quoted at the end of part (c).

$$\frac{d\tau}{dt} = 1 - \frac{2GM}{c^2 r}$$

Thus,

$$\left(\frac{dr}{dt}\right)^2 = c^2\left(1 - \frac{2GM}{c^2 r}\right)^2\left[1 - \left(1 - \frac{2GM}{c^2 r}\right)\right]$$

$$\left(\frac{dr}{dt}\right)^2 = \frac{2GM}{r}\left(1 - \frac{2GM}{c^2 r}\right)^2$$

Hence,

$$\boxed{\frac{dr}{dt} = -\left(1 - \frac{2GM}{c^2 r}\right)\left(\frac{2GM}{r}\right)^{1/2}}$$

Note that we have taken the negative square root since the motion is <u>towards</u> $r=0$. Inverting this equation, we have

$$\frac{dt}{dr} = -\left(\frac{r}{2GM}\right)^{1/2}\left(1 - \frac{2GM}{c^2 r}\right)^{-1}$$

$$t = \frac{-1}{(2GM)^{1/2}}\int_{r_0}^{r_s} \frac{\sqrt{r}\,dr}{1 - \frac{2GM}{c^2 r}}$$

where t is the elapsed coordinate time it takes the particle to move from r_0 to r_s. If $r_s = \frac{2GM}{c^2}$ is the Schwarzschild radius, we can write:

$$\boxed{t = \frac{-1}{(2GM)^{1/2}}\int_{r_0}^{r_s} \frac{r^{3/2}\,dr}{r - r_s}}$$

Note that for infalling trajectories, $r_0 > r_s$ and so t is positive.

But, due to the singularity in the integrand at $r = r_s$, the integral diverges logarithmically. That is,

$$t = \infty$$

meaning it takes an infinite amount of coordinate time for the particle to reach the Schwarzschild radius.

<u>Remark</u>: The coordinate time is the time measured by an observer far away from the origin ($r=0$). Thus, to this observer, the particle never reaches the Schwarzschild radius.

<u>A curious fact</u>: Note that $\frac{dr}{dt} = 0$ at $r = r_s$. This may seem quite peculiar to you. But recall from problem 5 of problem set 2, we saw that the speed of light as measured with respect to coordinate time was

$$\left(\frac{dr}{dt}\right)_{\text{light}} = c\left(1 - \frac{2GM}{c^2 r}\right)$$

for a radially directed beam. Thus, at $r = r_s = \frac{2GM}{c^2}$, $\left(\frac{dr}{dt}\right)_{\text{light}} = 0$.

Nothing travels faster than the speed of light in a vacuum, so perhaps it is not so surprising that the radial speed of the particle must vanish at $r = r_s$.

The Schwarzschild radius is the radius at which the escape velocity and the speed of light coincide. Thus, for $r < r_s$, light cannot escape (this is a black hole), and at $r = r_s$ we are at the border between possible escape and impossible escape.

(e) Using the results of parts (a) and (b),

$$\frac{dR}{dT} = \left(1 - \frac{2GM}{c^2 r}\right)^{-1}\frac{dr}{dt}$$

Plugging in the result for dr/dt obtained in part (d),

$$\boxed{\frac{dR}{dT} = -\left(\frac{2GM}{r}\right)^{1/2} = -c\left(\frac{r_s}{r}\right)^{1/2}}$$

Indeed, $|dR/dT| \longrightarrow c$ as $r \longrightarrow r_s$.