The velocity and momentum four-vectors

The velocity four-vector of a particle is defined by:

$$u^\mu = \frac{dx^\mu}{d\tau} = (\gamma c\,;\,\gamma\vec{\boldsymbol{v}}\,)\,, \tag{1}$$

where $d\tau = \gamma^{-1}dt$ is the differential proper time (which is a scalar quantity). Note that $\gamma \equiv (1 - |\vec{\boldsymbol{v}}|^2/c^2)^{-1/2}$, where $|\vec{\boldsymbol{v}}|$ is the magnitude of the velocity vector $\vec{\boldsymbol{v}} \equiv d\vec{\boldsymbol{x}}/dt$ that appears in eq. (1). The squared magnitude of the velocity four-vector,

$$u^2 \equiv g_{\mu\nu}u^\mu u^\nu = c^2 \tag{2}$$

is a Lorentz invariant. Note that it is most easily evaluated in the rest frame of the particle where $\vec{\boldsymbol{v}} = 0$, in which case $u^\mu = c\,(1\,;\,\vec{\boldsymbol{0}})$.

Let us now consider the following question. Suppose that the velocity vector is $u^\mu = (u^0\,;\,\vec{\boldsymbol{u}})$ in an inertial frame K. That is $u^0 \equiv \gamma c$ and $\vec{\boldsymbol{u}} \equiv \gamma\vec{\boldsymbol{v}}$, where $\vec{\boldsymbol{v}}$ is the velocity as measured in K. Note that γ depends implicitly on $\vec{\boldsymbol{v}}$ and is also frame dependent. A second inertial frame K' is defined to be moving with relative velocity $\vec{\boldsymbol{w}}$ with respect to K. Note that I have chosen a different symbol for the relative velocity to avoid confusion with $\vec{\boldsymbol{v}}$ which is the velocity of the particle in the reference frame K.

We wish to relate the four vector u^μ which describes the velocity of the particle in K and the corresponding four vector u'^μ which describes the velocity of the particle in K'. This is accomplished by a Lorentz boost:

$$u'^\mu = \Lambda^\mu{}_\nu u^\nu \tag{3}$$

with the boost matrix $\Lambda^\mu{}_\nu$ given by

$$\Lambda = \begin{pmatrix} \gamma_w & -\gamma_w\,\vec{\boldsymbol{\beta}}_w \\ -\gamma_w\,\vec{\boldsymbol{\beta}}_w & \delta^{ij} + (\gamma_w - 1)\dfrac{\beta_w^i\beta_w^j}{|\vec{\boldsymbol{\beta}}_w|^2} \end{pmatrix}\,, \tag{4}$$

where $\vec{\boldsymbol{\beta}}_w \equiv \vec{\boldsymbol{w}}/c$ and $\gamma_w \equiv (1 - |\vec{\boldsymbol{\beta}}|^2)^{-1/2}$. Then, eqs. (3) and (4) imply that:

$$u'^0 = \gamma_w(u^0 - \vec{\boldsymbol{\beta}}_{\boldsymbol{w}} \cdot \vec{\boldsymbol{u}}\,)\,, \tag{5}$$

$$\vec{\boldsymbol{u}}\,' = \vec{\boldsymbol{u}} + \frac{(\gamma_w - 1)}{|\vec{\boldsymbol{\beta}}_w|^2}(\vec{\boldsymbol{\beta}}_{\boldsymbol{w}} \cdot \vec{\boldsymbol{u}})\,\vec{\boldsymbol{\beta}}_w - \gamma_w\,\vec{\boldsymbol{\beta}}_w\,u^0\,. \tag{6}$$

Dividing these two equations yields:

$$\frac{\vec{\boldsymbol{u}}'}{u'^0} = \frac{1}{u^0 - \vec{\boldsymbol{\beta}}_{\boldsymbol{w}} \cdot \vec{\boldsymbol{u}}} \left[\frac{\vec{\boldsymbol{u}}}{\gamma_w} + \frac{(\gamma_w - 1)}{\gamma_w |\vec{\boldsymbol{\beta}}_w|^2} (\vec{\boldsymbol{\beta}}_{\boldsymbol{w}} \cdot \vec{\boldsymbol{u}})\, \vec{\boldsymbol{\beta}}_w - \vec{\boldsymbol{\beta}}_w\, u^0 \right] . \tag{7}$$

Substituting $\vec{\boldsymbol{\beta}}_w \equiv \vec{\boldsymbol{w}}/c$, $u^0 = \gamma c$, $\vec{\boldsymbol{u}} = \gamma \vec{\boldsymbol{v}}$, and $\vec{\boldsymbol{u}}'/u'^0 = \vec{\boldsymbol{v}}'/c$ in eq. (7), we arrive at:

$$\vec{\boldsymbol{v}}' = \frac{1}{1 - \dfrac{\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}}}{c^2}} \left[\frac{\vec{\boldsymbol{v}}}{\gamma_w} + \frac{(\gamma_w - 1)}{|\vec{\boldsymbol{w}}|^2 \gamma_w} (\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}})\, \vec{\boldsymbol{w}} - \vec{\boldsymbol{w}} \right] . \tag{8}$$

This result can be rewritten as:

$$\boxed{\vec{\boldsymbol{v}}' = \frac{1}{1 - \dfrac{\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}}}{c^2}} \left[\frac{1}{\gamma_w} \left(\vec{\boldsymbol{v}} - \frac{\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}}}{|\vec{\boldsymbol{w}}|^2}\, \vec{\boldsymbol{w}} \right) - \left(1 - \frac{\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}}}{|\vec{\boldsymbol{w}}|^2} \right) \vec{\boldsymbol{w}} \right] .} \tag{9}$$

This is the law of addition of velocities. As a simple check, consider the vase of $\vec{\boldsymbol{w}} = \vec{\boldsymbol{v}}$. In this case, the frame K' is moving at velocity $\vec{\boldsymbol{v}}$ with respect to frame K. Since the particle is moving at velocity $\vec{\boldsymbol{v}}$ in frame K, this means that the particle is at rest in frame K'. Thus, we conclude that $\vec{\boldsymbol{v}}' = 0$, as confirmed by eq. (9).

The law of addition of velocity simplifies in the simple case where $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{w}}$ are parallel. In this case, $\vec{\boldsymbol{v}} \propto \vec{\boldsymbol{w}}$, and the constant of proportionality is easily determined,[1]

$$\vec{\boldsymbol{v}} = \left(\frac{\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}}}{|\vec{\boldsymbol{w}}|^2} \right) \vec{\boldsymbol{w}} . \tag{10}$$

Hence, eq. (9) simplifies immediately to:

$$\vec{\boldsymbol{v}}' = \frac{\vec{\boldsymbol{v}} - \vec{\boldsymbol{w}}}{1 - \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}}/c^2} . \tag{11}$$

In the non-relativistic limit, $\gamma_w \simeq 1$ and eqs. (9) and (11) both reduce to the expected form: $\vec{\boldsymbol{v}}' = \vec{\boldsymbol{v}} - \vec{\boldsymbol{w}}$.

[1] One can check the correctness of eq. (10) by taking the dot product of both sides of the equation with $\vec{\boldsymbol{w}}$.

The momentum four-vector (also called the four-momentum) is related in a simple way to the velocity four-vector:

$$p^\mu = mu^\mu = (E/c\,;\,\vec{\boldsymbol{p}})\,, \tag{12}$$

where [using eq. (1)]

$$\vec{\boldsymbol{p}} = \gamma m\vec{\boldsymbol{v}}\,, \tag{13}$$

$$E = \gamma mc^2\,. \tag{14}$$

Note that by dividing these two equations, one deduces an expression for the particle velocity:

$$\vec{\boldsymbol{v}} = \frac{\vec{\boldsymbol{p}}\,c^2}{E}\,. \tag{15}$$

Again, it must be emphasized that $\vec{\boldsymbol{v}}$, which appears both explicitly and implicitly in $\gamma \equiv (1 - |\vec{\boldsymbol{v}}|^2/c^2)^{-1/2}$ in eqs. (13)—(15) corresponds to the velocity of the particle. Thus, in the rest frame of the particle, $\vec{\boldsymbol{v}} = 0$ and $\gamma = 1$, which implies that the rest frame four-momentum is given by $p^\mu = mc(1\,;\,\vec{\mathbf{0}}\,)$.

Furthermore, the mass m is a scalar quantity (which is Lorentz invariant); it corresponds to the rest energy of the particle divided by c^2. This also follows from the observation[2] that the Lorentz invariant scalar $p_\mu p^\mu = m^2c^2$. Finally, by noting that

$$g_{\mu\nu}p^\mu p^\nu = (p^0)^2 - |\vec{\boldsymbol{p}}|^2 = m^2c^2\,, \tag{16}$$

and inserting $p^0 = E/c$, one obtains an expression for the relativistic energy:

$$E^2 = c^2|\vec{\boldsymbol{p}}|^2 + m^2c^4\,. \tag{17}$$

Taking the square root, and expanding out resulting expression in the limit of $|\vec{\boldsymbol{v}}| \ll c$ yields:

$$E \simeq mc^2 + \frac{|\vec{\boldsymbol{p}}|^2}{2m}\,, \tag{18}$$

which we recognize as the sum of the rest energy and the non-relativistic kinetic energy. More generally, the relativistic energy can be written as $E = mc^2 + T$, which defines the relativistic kinetic energy as:

$$T = \sqrt{c^2|\vec{\boldsymbol{p}}|^2 + m^2c^4} - mc^2\,. \tag{19}$$

[2]Since Lorentz scalars do not depend on the reference frame, I may evaluate it in any frame. By choosing the rest frame of the particle, the computation is trivial.