# Evaluation of some integrals over solid angles—Part 2

In computing the total power liberated by an accelerating charge moving with velocity $\vec{\boldsymbol{\beta}} \equiv \vec{\boldsymbol{v}}/c$ and acceleration $\vec{\boldsymbol{a}} \equiv d\vec{\boldsymbol{v}}/dt$, we need to compute three integrals over the solid angle $\Omega$,

$$I_1 = \int \frac{(\hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{a}})^2}{(1 - \vec{\boldsymbol{\beta}} \cdot \hat{\boldsymbol{n}})^5}\, d\Omega\,, \tag{1}$$

$$I_2 = \int \frac{d\Omega}{(1 - \vec{\boldsymbol{\beta}} \cdot \hat{\boldsymbol{n}})^3}\,, \tag{2}$$

$$I_3 = \int \frac{\hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{a}}}{(1 - \vec{\boldsymbol{\beta}} \cdot \hat{\boldsymbol{n}})^4}\, d\Omega\,. \tag{3}$$

Let's begin with $I_2$. Choose the $z$ axis to lie along the direction of $\vec{\boldsymbol{\beta}}$. Then, it follow that $\vec{\boldsymbol{\beta}} \cdot \hat{\boldsymbol{n}} = 1 - \beta\cos\theta$, where $\beta \equiv |\vec{\boldsymbol{\beta}}|$. Writing $d\Omega = d\cos\theta\, d\phi$ and introducing $w \equiv \cos\theta$, it follows that

$$I_2 = 2\pi \int_{-1}^{1} \frac{dw}{(1 - \beta w)^3} = \frac{2\pi}{\beta} \int_{1-\beta}^{1+\beta} \frac{dy}{y^3} = -\frac{\pi}{\beta}\left[\frac{1}{(1+\beta)^2} - \frac{1}{(1-\beta)^2}\right]\,, \tag{4}$$

after changing the integration variable to $y = 1 - \beta w$. Hence,

$$\boxed{I_2 = \frac{4\pi}{(1-\beta^2)^2}} \tag{5}$$

Next, we can take the derivative of $I_2$ with respect to $\vec{\boldsymbol{\beta}}$ by making use of eq. (2),

$$\frac{\partial I_2}{\partial \vec{\boldsymbol{\beta}}} = 3 \int \frac{\hat{\boldsymbol{n}}\, d\Omega}{(1 - \vec{\boldsymbol{\beta}} \cdot \hat{\boldsymbol{n}})^4}\,. \tag{6}$$

Thus, we can identify

$$I_3 = \tfrac{1}{3}\vec{\boldsymbol{a}} \cdot \frac{\partial I_2}{\partial \vec{\boldsymbol{\beta}}}\,. \tag{7}$$

We can evaluate the right hand side of eq. (7) by using the result obtained in eq. (4). Since eq. (4) is a function of $\beta = |\vec{\boldsymbol{\beta}}|$, we can use the chain rule to write

$$\frac{\partial I_2}{\partial \vec{\boldsymbol{\beta}}} = \frac{\partial \beta}{\partial \vec{\boldsymbol{\beta}}} \frac{\partial I_2}{\partial \beta} = \frac{\vec{\boldsymbol{\beta}}}{\beta} \frac{\partial I_2}{\partial \beta}\,. \tag{8}$$

To obtain the last step above, we noted that $\beta = (\vec{\boldsymbol{\beta}} \cdot \vec{\boldsymbol{\beta}})^{1/2}$. Hence, it follows that

$$\frac{\partial\beta}{\partial\vec{\boldsymbol{\beta}}} = \frac{\partial}{\partial\vec{\boldsymbol{\beta}}}(\vec{\boldsymbol{\beta}} \cdot \vec{\boldsymbol{\beta}})^{1/2} = \tfrac{1}{2}(\vec{\boldsymbol{\beta}} \cdot \vec{\boldsymbol{\beta}})^{-1/2}\frac{\partial}{\partial\vec{\boldsymbol{\beta}}}(\vec{\boldsymbol{\beta}} \cdot \vec{\boldsymbol{\beta}}) = (\vec{\boldsymbol{\beta}} \cdot \vec{\boldsymbol{\beta}})^{-1/2}\vec{\boldsymbol{\beta}} = \frac{\vec{\boldsymbol{\beta}}}{\beta}\,. \tag{9}$$

Finally, we can use eq. (4) to evaluate $\partial I_2/\partial\beta$ ,

$$\frac{\partial I_2}{\partial\beta} = \frac{16\pi\beta}{(1-\beta^2)^3}\,. \tag{10}$$

Hence, we end up with

$$\boxed{I_3 = \frac{16\pi}{3}\,\frac{\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{\beta}}}{(1-\beta^2)^3}\,.} \tag{11}$$

Finally, we can use eq. (2) to obtain

$$\frac{\partial I_2}{\partial\beta_i} = 3\int\frac{\hat{n}_i\,d\Omega}{(1-\vec{\boldsymbol{\beta}} \cdot \hat{\boldsymbol{n}})^4}\,, \tag{12}$$

$$\frac{\partial^2 I_2}{\partial\beta_i\partial\beta_j} = 12\int\frac{\hat{n}_i\hat{n}_j\,d\Omega}{(1-\vec{\boldsymbol{\beta}} \cdot \hat{\boldsymbol{n}})^5}\,, \tag{13}$$

after two successive differentiations. Hence, we can identify,

$$I_1 = \frac{1}{12}\sum_{i,j}a_i a_j\,\frac{\partial^2 I_2}{\partial\beta_i\partial\beta_j}\,. \tag{14}$$

Note that eqs. (9) and (10) are equivalent to

$$\frac{dI_2}{d\beta_i} = \frac{16\pi\beta_i}{(1=\beta^2)^3}\,. \tag{15}$$

The second derivative can now be easily evaluated with the help of eq. (9),

$$\begin{aligned}\frac{\partial^2 I_2}{\partial\beta_i\partial\beta_j} &= \frac{16\pi\delta_{ij}}{(1-\beta^2)^3} + 16\pi\beta_i\frac{\beta_j}{\beta}\frac{\partial}{\partial\beta}\left(\frac{1}{(1-\beta^2)^3}\right) \\ &= \frac{16\pi\delta_{ij}}{(1-\beta^2)^3} + \frac{96\pi\beta_i\beta_j}{(1-\beta^2)^4}\,.\end{aligned} \tag{16}$$

Hence, eq. (14) yields,

$$\boxed{I_1 = \frac{4\pi}{3}\,\frac{|\vec{\boldsymbol{a}}|^2}{(1-\beta^2)^3} + \frac{8\pi(\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{\beta}})^2}{(1-\beta^2)^4}\,.} \tag{17}$$

There is an alternative technique for evaluating $I_1$ and $I_3$. We can define the following two integrals,

$$J_{ij} = \int \frac{\hat{n}_i \hat{n}_j \, d\Omega}{(1 - \vec{\boldsymbol{\beta}} \cdot \hat{\boldsymbol{n}})^5}\,, \tag{18}$$

$$K_i = \int \frac{\hat{n}_i \, d\Omega}{(1 - \vec{\boldsymbol{\beta}} \cdot \hat{\boldsymbol{n}})^4}\,. \tag{19}$$

By the covariance properties of Euclidean tensors, it follows that

$$J_{ij} = c_1 \delta_{ij} + c_2 \beta_i \beta_j\,, \tag{20}$$

$$K_i = \kappa \beta_i\,. \tag{21}$$

Consider first the evaluation of $K_i$. Multiplying by $\beta_i$ and summing over $i$ yields

$$\kappa \beta^2 = \int \frac{\vec{\boldsymbol{\beta}} \cdot \hat{\boldsymbol{n}} \, d\Omega}{(1 - \vec{\boldsymbol{\beta}} \cdot \hat{\boldsymbol{n}})^4}\,. \tag{22}$$

The integral above is now easily evaluated by employing the same method used to obtain eq. (4). Thus, we can obtain an explicit expression for $\kappa$. I will leave it as an exercise for the reader to show that

$$\kappa = \frac{16\pi}{3} \frac{1}{(1 - \beta^2)^3}\,. \tag{23}$$

Likewise, to evaluate $J_{ij}$, we first multiply by $\delta_{ij}$ and sum over $i$ and $j$ to get one equation. A second equation is obtained by multiplying by $\beta_i \beta_j$ and summing over $i$ and $j$, Thus, we get two equations for the two unknowns $c_1$ and $c_2$,

$$3c_1 + c_2 \beta^2 = \int \frac{d\Omega}{(1 - \vec{\boldsymbol{\beta}} \cdot \hat{\boldsymbol{n}})^5}\,, \tag{24}$$

$$c_1 \beta^2 + c_2 \beta^4 = \int \frac{(\vec{\boldsymbol{\beta}} \cdot \hat{\boldsymbol{n}})^2 \, d\Omega}{(1 - \vec{\boldsymbol{\beta}} \cdot \hat{\boldsymbol{n}})^5}\,. \tag{25}$$

Again, the two integrals above are easily evaluated by employing the same method used to obtain eq. (4). One can now solve for $c_1$ and $c_2$. I will leave it as an exercise for the reader to carry out the remaining computations to obtain,

$$c_1 = \frac{4\pi}{3} \frac{1}{(1 - \beta^2)^3}\,, \qquad c_2 = \frac{8\pi}{(1 - \beta^2)^4}\,. \tag{26}$$

Finally, we obtain

$$I_1 = \sum_{i,j} a_i a_j J_{ij}\,, \qquad I_3 = \sum_i a_i K_i\,. \tag{27}$$

Using eqs. (20), (21), (23) and (26), we recover the results obtained in eqs. (17) and (11), respectively.