1. A transmission line consisting of two concentric circular cylinders of metal with conductivity $\sigma$ and skin depth $\delta$, as shown in Figure 1, is filled with a uniform lossless dielectric with electric permittivity $\epsilon$ and magnetic permeability $\mu$. A TEM mode is propagated along this line. Denote the peak value of the azimuthal magnetic field at the surface of the inner conductor by $H_0$.

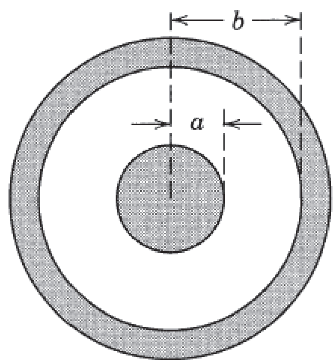


Figure 1: A transmission line consisting of two concentric circular cylinders of metal with inner radius $a$ and outer radius $b$.

(a) Compute the time-averaged power flow along the line in terms of $H_0$, $a$, $b$, $\epsilon$, and $\mu$.

Choose the $z$-axis to be the symmetry axis of the transmission line. TEM waves have following properties. The dispersion relation for TEM waves is given by eq. (8.27) of Jackson, $k = \omega\sqrt{\mu\epsilon}$. The TEM fields are transverse ($E_z = B_z = 0$) with

$$\vec{\boldsymbol{\nabla}}_\perp \times \vec{\boldsymbol{E}}_{\rm TEM} = 0\,, \qquad \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{E}}_{\rm TEM} = 0\,, \tag{1}$$

and [cf. eq.(8.28) of Jackson]:

$$\vec{\boldsymbol{B}}_{\rm TEM} = \sqrt{\mu\epsilon}\,\hat{\boldsymbol{z}} \times \vec{\boldsymbol{E}}_{\rm TEM}\,, \tag{2}$$

for waves propagating along the positive $z$ direction.

We shall employ SI units in this computation. First, we compute the time-averaged power flowing along the transmission line. This is given by

$$\int_A da\,\hat{\boldsymbol{z}} \cdot {\rm Re}\,\vec{\boldsymbol{S}}\,, \tag{3}$$

where $\vec{\boldsymbol{S}}$ is the complex Poynting vector,

$$\vec{\boldsymbol{S}} = \frac{1}{2}\vec{\boldsymbol{E}} \times \vec{\boldsymbol{H}}^*\,. \tag{4}$$

Using $\vec{\boldsymbol{B}} = \mu\vec{\boldsymbol{H}}$, we obtain

$$\vec{\boldsymbol{S}} = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\,\vec{\boldsymbol{E}}_\perp \times (\hat{\boldsymbol{z}} \times \vec{\boldsymbol{E}}_\perp) = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\,\hat{\boldsymbol{z}}|\vec{\boldsymbol{E}}_\perp|^2\,, \tag{5}$$

after expanding the triple cross-product, $\vec{\boldsymbol{E}}_\perp \times (\hat{\boldsymbol{z}} \times \vec{\boldsymbol{E}}_\perp) = \hat{\boldsymbol{z}}|\vec{\boldsymbol{E}}_\perp|^2 - \vec{\boldsymbol{E}}_\perp(\hat{\boldsymbol{z}} \cdot \vec{\boldsymbol{E}}_\perp)$, and noting that $\hat{\boldsymbol{z}} \cdot \vec{\boldsymbol{E}}_\perp = 0$.

To proceed, we evaluate $\vec{\boldsymbol{E}}_\perp$ which has the following form:

$$\vec{\boldsymbol{E}}_\perp(x,y,z,t) = \vec{\boldsymbol{E}}_\perp(x,y)e^{i(kz-\omega t)}\,,$$

where $\vec{\boldsymbol{E}}_\perp(x,y)$ satisfies eq. (1). That is $\vec{\boldsymbol{E}}_\perp(x,y)$ satisfies the equations of two-dimensional electrostatics. From $\vec{\boldsymbol{\nabla}}_\perp \times \vec{\boldsymbol{E}}_\perp = 0$, it follows that we can write

$$\vec{\boldsymbol{E}}_\perp = -\vec{\boldsymbol{\nabla}}_\perp\Phi\,.$$

Plugging this into $\vec{\boldsymbol{\nabla}}_\perp \cdot \vec{\boldsymbol{E}}_\perp = 0$ yields Laplace's equation in two-dimensions,

$$\vec{\boldsymbol{\nabla}}_\perp^2\Phi = 0\,.$$

This is easily solved in polar coordinates, where $x = \rho\cos\phi$ and $= \rho\sin\phi$. Then,

$$\vec{\boldsymbol{\nabla}}_\perp^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\,.$$

Due to the azimuthal symmetry of the problem, $\Phi$ is independent of the angle $\phi$. Thus, Laplace's equation reduces to

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) = 0\,.$$

That is,

$$\rho\frac{\partial\Phi}{\partial\rho} = C\,,$$

for some constant $C$. This equation is easily solved and we find that

$$\Phi = C\ln\rho\,,$$

up to an arbitrary additive constant (which drops out when we compute the electric field). Thus,

$$\vec{\boldsymbol{E}}_\perp = -\vec{\boldsymbol{\nabla}}_\perp\Phi = -\hat{\boldsymbol{\rho}}\,\frac{\partial\Phi}{\partial\rho} = -\frac{C}{\rho}\,\hat{\boldsymbol{\rho}}\,.$$

Eq. (2) then yields,

$$\vec{\boldsymbol{H}}_\perp = \frac{1}{\mu}\vec{\boldsymbol{B}}_\perp = \sqrt{\frac{\epsilon}{\mu}}\,\hat{\boldsymbol{z}} \times \vec{\boldsymbol{E}}_\perp = -C\,\sqrt{\frac{\epsilon}{\mu}}\,\frac{\hat{\boldsymbol{z}} \times \hat{\boldsymbol{\rho}}}{\rho} = -C\,\sqrt{\frac{\epsilon}{\mu}}\,\frac{\hat{\boldsymbol{\phi}}}{\rho}\,.$$

The peak value of the azimuthal magnetic field at the surface of the inner conductor (at $\rho = a$) is denoted by $H_0$. That is,

$$H_0\,e^{i(kz-\omega t)} = -C\,\sqrt{\frac{\epsilon}{\mu}}\,\frac{1}{a}\,.$$

Thus, we identify

$$C = -\sqrt{\frac{\mu}{\epsilon}} H_0\, a e^{i(kz-\omega t)}\,.$$

Summarizing our results so far, we have found the following results for the transverse electric and magnetic fields,[1]

$$\vec{\boldsymbol{E}} = \sqrt{\frac{\mu}{\epsilon}} H_0 \frac{a}{\rho}\, \hat{\boldsymbol{\rho}}\, e^{i(kz-\omega t)}\,, \qquad\qquad \vec{\boldsymbol{H}} = H_0 \frac{a}{\rho}\, \hat{\boldsymbol{\phi}}\, e^{i(kz-\omega t)}\,. \tag{6}$$

Inserting these results into eq. (5) yields,

$$\vec{\boldsymbol{S}} = \frac{1}{2}\sqrt{\frac{\mu}{\epsilon}} |H_0|^2 \frac{a^2}{\rho^2}\, \hat{\boldsymbol{z}}\,.$$

Hence, eq. (3) yields,

$$P = \int_A da\, \hat{\boldsymbol{z}} \boldsymbol{\cdot} \operatorname{Re} \vec{\boldsymbol{S}} = \frac{1}{2}\sqrt{\frac{\mu}{\epsilon}} |H_0|^2 a^2 \int_0^{2\pi} d\phi \int_a^b \rho d\rho\, \frac{1}{\rho^2}\,.$$

Evaluating the integral, the end result is

$$P = \sqrt{\frac{\mu}{\epsilon}}\, \pi a^2\, |H_0|^2 \ln\left(\frac{a}{b}\right)\,. \tag{7}$$

(b) The transmitted power is attenuated along the line has the form

$$P(z) = P_0 e^{-2\gamma z}\,.$$

Determine the explicit expression for $\gamma$ in terms of the conductivity $\sigma$, the skin depth $\delta$, and the parameters $a$, $b$, $\epsilon$, $\mu$.

We shall make use of eq. (8.12) of Jackson,

$$\frac{dP_{\rm loss}}{da} = \frac{\mu_c \omega \delta}{4} |\vec{\boldsymbol{H}}_{\parallel}|^2\,,$$

where $\vec{\boldsymbol{H}}_{\parallel}$ is the magnetic field parallel to the conductor at its surface, $\delta$ is the skin depth, which is given in SI units by eq. (8.8) of Jackson,

$$\delta = \sqrt{\frac{2}{\mu_c \omega \sigma}}\,, \tag{8}$$

and $\mu_c$ is the magnetic permeability of the conductor. The attenuation constant is given by eq. (8.57) of Jackson,

$$\gamma = -\frac{1}{2P}\frac{dP}{dz} = \frac{1}{2P}\frac{dP_{\rm loss}}{dz}\,, \tag{9}$$

[1] Since $E_z = H_z = 0$, there is no need to retain the $\perp$ subscript on the electric and magnetic fields.

where the $P$ that appears in the denominator is the time-averaged power flow along the transmission line in the infinite conductivity limit. The power loss per distance traveled along the transmission line is

$$\frac{dP_{\rm loss}}{dz} = \oint_C \frac{dP_{\rm loss}}{da}\, d\ell\,,$$

integrated around the cross-sectional loop obtained by cutting the surface of the conductor with a plane at fixed $z$. For the cylindrical symmetry of this problem, $d\ell = \rho d\phi$, and we must integrate around the cross-sectional loop at $\rho = a$ and $\rho = b$. The rest is plug-in. Using the explicit form for the magnetic field given in eq. (6),

$$\frac{dP_{\rm loss}}{dz} = \frac{\mu_c\omega\delta}{4}|H_0|^2 a^2 \left\{ \int_0^{2\pi} \frac{\rho\, d\rho}{\rho^2}\bigg|_{\rho=a} + \int_0^{2\pi} \frac{\rho\, d\rho}{\rho^2}\bigg|_{\rho=b} \right\} = \frac{\mu_c\omega\delta}{4}|H_0|^2\, 2\pi a^2 \left(\frac{1}{a}+\frac{1}{b}\right). \quad (10)$$

We can now compute $\gamma$ given by eq. (9). Plugging in $P$ given by eq. (7) obtained in part (a), the end result is

$$\gamma = \frac{\mu_c\omega\delta}{4}\sqrt{\frac{\epsilon}{\mu}}\,\frac{\left(\dfrac{1}{a}+\dfrac{1}{b}\right)}{\ln\left(\dfrac{b}{a}\right)} = \frac{1}{2\sigma\delta}\sqrt{\frac{\epsilon}{\mu}}\,\frac{\left(\dfrac{1}{a}+\dfrac{1}{b}\right)}{\ln\left(\dfrac{b}{a}\right)}\,,$$

after using eq. (8) to eliminate $\mu_c$ from the above expression.

<u>*REMARK:*</u> This problem was taken from parts (a) and (b) of Jackson, problem 8.2. For completeness, parts (c) and (d) of Jackson, problem 8.2 and their solutions are given in Appendix A.

2. A classical point magnetic dipole moment $\vec{\boldsymbol{\mu}}$ at rest has a vector potential (in gaussian units), which is given by

$$\vec{\boldsymbol{A}} = \frac{\vec{\boldsymbol{\mu}}\times\vec{\boldsymbol{r}}}{r^3}$$

and no scalar potential ($\Phi = 0$). Suppose that the point magnetic dipole moment $\vec{\boldsymbol{\mu}}$ now moves with velocity $\vec{\boldsymbol{v}}$.

(a) Compute and compare the interaction energy between the moving magnetic dipole and static external fields $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ and the interaction energy computed in the rest frame of the magnetic moment. Explain why the two results agree (or disagree). Assume that $v \ll c$ and keep only terms up to $\mathcal{O}(v/c)$.

Following part (b) of problem 11.27 of Jackson, we define $K'$ to be the rest frame of the magnetic dipole with dipole moment $\vec{\boldsymbol{\mu}}'$. In frame $K$, the magnetic dipole (or equivalently frame $K'$) is moving at velocity $\vec{\boldsymbol{v}} = \vec{\boldsymbol{\beta}}c$. In the solutions to problem 11.27, we showed that to order $\mathcal{O}(v/c)$, the magnetic dipole moment in frame $K$ is $\vec{\boldsymbol{\mu}} = \vec{\boldsymbol{\mu}}'$. In addition, there is also an electric dipole moment in frame $K$ given by $\vec{\boldsymbol{p}} = \vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{\mu}}$.

To compute the interaction energies, we need expressions for the electric and magnetic fields in both reference frames. Using eq. (11.149) of Jackson,

$$\vec{\boldsymbol{E}}' = \gamma(\vec{\boldsymbol{E}} + \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{B}}) - \frac{\gamma^2}{\gamma+1}\,\vec{\boldsymbol{\beta}}(\vec{\boldsymbol{\beta}} \boldsymbol{\cdot} \vec{\boldsymbol{E}})\,,$$

$$\vec{\boldsymbol{B}}' = \gamma(\vec{\boldsymbol{B}} - \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{E}}) - \frac{\gamma^2}{\gamma+1}\,\vec{\boldsymbol{\beta}}(\vec{\boldsymbol{\beta}} \boldsymbol{\cdot} \vec{\boldsymbol{B}})\,.$$

Keeping only terms up to $\mathcal{O}(v/c)$, we have

$$\vec{\boldsymbol{E}}' = \vec{\boldsymbol{E}} + \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{B}} + \mathcal{O}\left(\frac{v^2}{c^2}\right)\,, \qquad\qquad \vec{\boldsymbol{B}}' = \vec{\boldsymbol{B}} - \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{E}} + \mathcal{O}\left(\frac{v^2}{c^2}\right)\,.$$

In frame $K$, the interaction energy is [cf. eqs.(4.24) and (5.72) of Jackson]:

$$U = -\vec{\boldsymbol{p}} \boldsymbol{\cdot} \vec{\boldsymbol{E}} - \vec{\boldsymbol{\mu}} \boldsymbol{\cdot} \vec{\boldsymbol{B}} = -(\vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{\mu}}) \boldsymbol{\cdot} \vec{\boldsymbol{E}} - \vec{\boldsymbol{\mu}} \boldsymbol{\cdot} \vec{\boldsymbol{B}} + \mathcal{O}\left(\frac{v^2}{c^2}\right)\,.$$

The first term on the right hand side above can be rewritten with the help of vector identities,

$$-(\vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{\mu}}) \boldsymbol{\cdot} \vec{\boldsymbol{E}} = (\vec{\boldsymbol{\mu}} \times \vec{\boldsymbol{\beta}}) \boldsymbol{\cdot} \vec{\boldsymbol{E}} = \vec{\boldsymbol{\mu}} \boldsymbol{\cdot} (\vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{E}})\,.$$

Hence, the interaction energy in frame $K$ is given by

$$U = -\vec{\boldsymbol{\mu}} \boldsymbol{\cdot} (\vec{\boldsymbol{B}} - \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{E}}) + \mathcal{O}\left(\frac{v^2}{c^2}\right)\,. \tag{11}$$

In frame $K'$, the interaction energy is given by

$$U' = -\vec{\boldsymbol{\mu}}' \boldsymbol{\cdot} \vec{\boldsymbol{B}}' = -\vec{\boldsymbol{\mu}} \boldsymbol{\cdot} (\vec{\boldsymbol{B}} - \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{E}}) + \mathcal{O}\left(\frac{v^2}{c^2}\right)\,,$$

where we have used $\vec{\boldsymbol{\mu}} = \vec{\boldsymbol{\mu}}'$ and $\vec{\boldsymbol{B}}' = \vec{\boldsymbol{B}} - \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{E}}$, where terms of order $\mathcal{O}(v^2/c^2)$ have been neglected. The two computations agree to first order in $v/c$. The interaction energy is not a Lorentz invariant. Indeed, we expect that $U' = \gamma U$. But if we drop terms of $\mathcal{O}(v^2/c^2)$, then $\gamma \simeq 1$ in which case $U' \simeq U$ as obtained above.

(b) Compute the *exact* expression for the scalar potential generated by a point magnetic dipole $\vec{\boldsymbol{\mu}}$ moving with velocity $\vec{\boldsymbol{v}}$. (Do *not* assume that $v \ll c$.) Express your answer in terms of the angle between $\hat{\boldsymbol{n}}$ and $\vec{\boldsymbol{v}}$, where $\hat{\boldsymbol{n}}$ is a unit vector pointing from the magnetic dipole to the point at which the scalar potential is measured.

The four-vector potential is $A^\mu = (\Phi\,;\,\vec{\boldsymbol{A}})$ transforms just like any four-vector [cf. eq. (11.19) of Jackson],

$$\Phi' = \gamma(\Phi - \vec{\boldsymbol{\beta}} \boldsymbol{\cdot} \vec{\boldsymbol{A}})\,, \qquad\qquad \vec{\boldsymbol{A}}' = \vec{\boldsymbol{A}} + \frac{(\gamma-1)}{\beta^2}\,(\vec{\boldsymbol{\beta}} \boldsymbol{\cdot} \vec{\boldsymbol{A}})\vec{\boldsymbol{\beta}} - \gamma\beta\Phi\,.$$

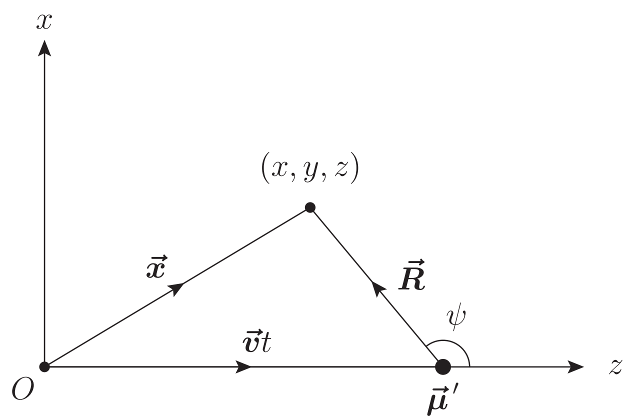


Figure 2: A magnetic dipole is moving at constant velocity $\vec{\boldsymbol{v}} = \vec{\boldsymbol{\beta}}c$ in the $z$-direction as seen from reference frame $K$. The origin of the laboratory frame $K$ is denoted by $O$. The angle $\psi$ is defined so that $\hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{R}} = \cos\psi$. By convention, we take $0 \leq \psi \leq \pi$. Note that the specific choice of $x$, $y$ and $z$ axes is arbitrary and not used in the solution to this problem.

The inverse transformation is obtained simply by changing the sign of $\vec{\boldsymbol{\beta}}$,

$$\Phi = \gamma(\Phi' + \vec{\boldsymbol{\beta}} \cdot \vec{\boldsymbol{A}}\,')\,, \qquad\qquad \vec{\boldsymbol{A}} = \vec{\boldsymbol{A}}\,' + \frac{(\gamma-1)}{\beta^2}\,(\vec{\boldsymbol{\beta}} \cdot \vec{\boldsymbol{A}}\,')\vec{\boldsymbol{\beta}} + \gamma\beta\Phi'\,. \tag{12}$$

In the rest frame, we have a magnetic dipole moment but no electric dipole moment, which corresponds to

$$\vec{\boldsymbol{A}}\,' = \frac{\vec{\boldsymbol{\mu}}\,' \times \vec{\boldsymbol{r}}\,'}{r'^3}\,, \qquad\qquad \Phi' = 0\,, \tag{13}$$

where $r' \equiv |\vec{\boldsymbol{r}}\,'|$. The coordinate $\vec{\boldsymbol{r}}\,'$ in frame $K$ is related to the coordinate $\vec{\boldsymbol{r}}$ in frame $K$ by eq. (11.19) of Jackson,

$$\vec{\boldsymbol{r}}\,' = \vec{\boldsymbol{r}} + \frac{(\gamma-1)}{\beta^2}\,(\vec{\boldsymbol{\beta}} \cdot \vec{\boldsymbol{r}})\vec{\boldsymbol{\beta}} - \gamma\vec{\boldsymbol{\beta}}ct\,, \tag{14}$$

where $t$ is the time measured in frame $K$.

The moving magnetic dipole as seen from the laboratory frame $K$ is shown in Figure 1. It is convenient to define $\vec{\boldsymbol{R}}$ to be the vector in frame $K$ that points from the location of the magnetic dipole at time $t$ to the location of the observer. It follows that

$$\vec{\boldsymbol{R}} = \vec{\boldsymbol{r}} - \vec{\boldsymbol{v}}t\,, \tag{15}$$

where $\vec{\boldsymbol{R}} \equiv R\hat{\boldsymbol{n}}$ and $\hat{\boldsymbol{n}}$ is the unit vector that points in the direction of $\vec{\boldsymbol{R}}$. Using eq. (14), it follows that

$$\vec{\boldsymbol{r}}\,' = \vec{\boldsymbol{R}} + \frac{(\gamma-1)}{\beta^2}\,(\vec{\boldsymbol{\beta}} \cdot \vec{\boldsymbol{R}})\vec{\boldsymbol{\beta}}\,.$$

Note that $\vec{\boldsymbol{\beta}} \cdot (\vec{\boldsymbol{\mu}}\,' \times \vec{\boldsymbol{r}}\,') = \vec{\boldsymbol{\beta}} \cdot (\vec{\boldsymbol{\mu}}\,' \times \vec{\boldsymbol{R}})$, since $\vec{\boldsymbol{\beta}} \cdot (\vec{\boldsymbol{\mu}}\,' \times \vec{\boldsymbol{\beta}}) = 0$. Hence, eqs. (12) and (13) yield

$$\Phi = \gamma\vec{\boldsymbol{\beta}} \cdot \vec{\boldsymbol{A}}\,' = \frac{\gamma\vec{\boldsymbol{\beta}} \cdot (\vec{\boldsymbol{\mu}}\,' \times \vec{\boldsymbol{r}}\,')}{r'^3} = \frac{\gamma\vec{\boldsymbol{\beta}} \cdot (\vec{\boldsymbol{\mu}}\,' \times \vec{\boldsymbol{R}})}{r'^3}\,. \tag{16}$$

The last step involves relating $r'$ to $R \equiv |\vec{\boldsymbol{R}}|$. In eq. (13) of the class handout entitled, *The electromagnetic fields of a uniformly moving charge*, I showed that

$$r'^3 = \gamma^3 R^3 (1 - \beta^2 \sin^2 \psi)^{3/2}\,, \tag{17}$$

where $\psi$ is the angle between $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{R}}$ as seen in frame $K$ (cf. Figure 2). Inserting eq. (17) into the denominator of eq. (16) and using $\vec{\boldsymbol{v}} = c\vec{\boldsymbol{\beta}}$ and $\vec{\boldsymbol{R}} = R\hat{\boldsymbol{n}}$, we end up with

$$\Phi = \frac{\vec{\boldsymbol{v}} \cdot (\vec{\boldsymbol{\mu}}' \times \hat{\boldsymbol{n}})}{c\gamma^2 R^2 (1 - \beta^2 \sin^2 \psi)^{3/2}}\,.$$

Note that we have expressed our answer in terms of the magnetic dipole moment as measured in the rest frame $K'$. As discussed in the solution to Jackson problem 11.27 (in Solution Set 2), the exact relationship between $\vec{\boldsymbol{\mu}}$ and $\vec{\boldsymbol{\mu}}'$ is somewhat complicated, although as noted in part (a) above, the two dipole moments coincide if terms of $\mathcal{O}(v^2/c^2)$ are dropped.

3. A magnetic dipole $\vec{\boldsymbol{m}}$ undergoes precessional motion with angular frequency $\omega$ and angle $\vartheta_0$ with respect to the $z$-axis as shown in Fig. 3. That is, the time-dependence of the azimuthal angle is $\varphi_0(t) = \varphi_0 - \omega t$. Electromagnetic radiation is emitted by the precessing dipole.

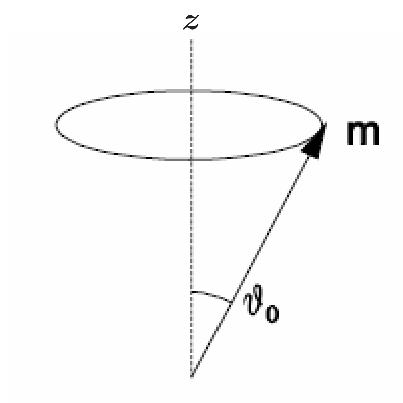


Figure 3: A magnetic dipole $\vec{\boldsymbol{m}}$ undergoes precessional motion with angular frequency $\omega$ and angle $\vartheta_0$ with respect to the $z$-axis.

(a) Write out an explicit expression for the time-dependent magnetic dipole vector $\vec{\boldsymbol{m}}$ in terms of its magnitude $m_0$, the angles $\vartheta_0$ and $\varphi_0$ and the time $t$. Show that $\vec{\boldsymbol{m}}$ consists of the sum of a time-dependent term and a time-independent term. Verify that the time-dependent term can be written as $\mathrm{Re}(\vec{\boldsymbol{\mu}}\, e^{-i\omega t})$, for some suitably chosen complex vector $\vec{\boldsymbol{\mu}}$.

In light of Fig. 3, the magnetic dipole moment vector is given by:

$$\begin{aligned}\vec{\boldsymbol{m}} &= m_0\Big[\hat{\boldsymbol{x}} \sin\vartheta_0 \cos(\varphi_0 - \omega t) + \hat{\boldsymbol{y}} \sin\vartheta_0 \sin(\varphi_0 - \omega t) + \hat{\boldsymbol{z}} \cos\vartheta_0\Big] \qquad (18)\\ &= \mathrm{Re}\Big[m_0 \sin\vartheta_0\, e^{i\varphi_0} (\hat{\boldsymbol{x}} - i\hat{\boldsymbol{y}}) e^{-i\omega t}\Big] + m_0 \cos\vartheta_0\, \hat{\boldsymbol{z}}\,.\end{aligned}$$

Thus, we can write the time-dependent term of $\vec{\boldsymbol{m}}$ as $\mathrm{Re}(\vec{\boldsymbol{\mu}}\, e^{-i\omega t})$, where

$$\vec{\boldsymbol{\mu}} = m_0 \sin\varphi_0\, e^{i\varphi_0}(\hat{\boldsymbol{x}} - i\hat{\boldsymbol{y}})\,. \tag{19}$$

(b) Compute the angular distribution of the time-averaged radiated power, with respect to the $z$-axis defined in the above figure.

The angular distribution of the time-averaged power is given by eq. (9.21) of Jackson in SI units,

$$\frac{dP}{d\Omega} = \frac{1}{2}\,\mathrm{Re}\big[r^2\,\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{E}}\times\vec{\boldsymbol{H}}^{\,*}\big]\,.$$

The magnetic and electric fields of the magnetic dipole are given by eqs. (9.35) and (9.36) of Jackson. Keeping only the leading terms of $\mathcal{O}(1/r)$, we see that

$$\vec{\boldsymbol{H}} = -\frac{1}{Z_0}\vec{\boldsymbol{E}}\times\hat{\boldsymbol{n}}\,,$$

where $Z_0 = \sqrt{\mu_0/\epsilon_0}$ is the impedance of free space. It follows that

$$\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{E}}\times\vec{\boldsymbol{H}}^{\,*} = -\frac{1}{Z_0}\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{E}}\times(\vec{\boldsymbol{E}}^{\,*}\times\hat{\boldsymbol{n}}) = \frac{1}{Z_0}\Big[|\vec{\boldsymbol{E}}|^2 - |\vec{\boldsymbol{E}}\cdot\hat{\boldsymbol{n}}|^2\Big] = \frac{1}{Z_0}|\vec{\boldsymbol{E}}|^2\,,$$

since $\vec{\boldsymbol{E}}\cdot\hat{\boldsymbol{n}} = 0$ (due to the transverse nature of electromagnetic radiation). Hence,

$$\frac{dP}{d\Omega} = \frac{r^2}{2Z_0}|\vec{\boldsymbol{E}}|^2\,, \tag{20}$$

where the leading $\mathcal{O}(1/r)$ term of eq. (9.36) of Jackson, applied to the complex magnetic moment vector $\vec{\boldsymbol{\mu}}$, yields

$$\vec{\boldsymbol{E}} = -\frac{Z_0}{4\pi}k^2(\vec{\boldsymbol{n}}\times\vec{\boldsymbol{\mu}})\frac{e^{ikr}}{r}\,. \tag{21}$$

Inserting this result into eq. (20), we end up with

$$\frac{dP}{d\Omega} = \frac{Z_0}{32\pi^2}\,k^4|\hat{\boldsymbol{n}}\times\vec{\boldsymbol{\mu}}|^2\,. \tag{22}$$

The squared magnitude of the cross product above is easily computed,

$$|\hat{\boldsymbol{n}}\times\vec{\boldsymbol{\mu}}|^2 = (\hat{\boldsymbol{n}}\times\vec{\boldsymbol{\mu}})\cdot(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{\mu}}^{\,*}) = |\vec{\boldsymbol{\mu}}|^2 - |\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{\mu}}|^2\,,$$

since $\hat{\boldsymbol{n}}$ is a unit vector. Explicitly, $\vec{\boldsymbol{\mu}}$ is given by eq. (19) and

$$\hat{\boldsymbol{n}} = \sin\theta\cos\phi\,\hat{\boldsymbol{x}} + \sin\theta\sin\phi\,\hat{\boldsymbol{y}} + \cos\theta\,\hat{\boldsymbol{z}}\,.$$

Hence, it follows that

$$|\vec{\boldsymbol{\mu}}|^2 = 2m_0^2\sin^2\vartheta_0\,, \qquad |\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{\mu}}| = m_0\sin\vartheta_0\sin\theta\,,$$

and

$$|\hat{\boldsymbol{n}} \times \vec{\boldsymbol{\mu}}|^2 = m_0^2 \sin^2\vartheta_0 (2 - \sin^2\theta) = m_0^2 \sin^2\vartheta_0 (1 + \cos^2\theta)\,.$$

Thus, the angular distribution of the time-averaged radiated power is given by[2]

$$\frac{dP}{d\Omega} = \frac{Z_0 m_0^2 \sin^2\vartheta_0}{32\pi^2}\, k^4 (1 + \cos^2\theta)\,. \tag{23}$$

An alternative technique for computing the angular distribution of the time-averaged radiated power which makes use of the real physical magnetic dipole moment is given in Appendix B.

(c) Compute the total power radiated.

Integrating eq. (23) over solid angles,

$$\int d\Omega (1 + \cos^2\theta) = 2\pi \int_{-1}^{1} (1 + \cos^2\theta)\, d\cos\theta = \frac{16\pi}{3}\,. \tag{24}$$

Hence,

$$P = \frac{Z_0 m_0^2 k^4 \sin^2\vartheta_0^2}{6\pi}\,. \tag{25}$$

(d) What is the polarization of the radiation measured by an observer located along the positive $z$-axis far from the precessing dipole? How would your answer change if the observer were located in the $x$–$y$ plane?

The polarization is determined from the electric field given in eq. (21). Thus, we must evaluate $\hat{\boldsymbol{n}} \times \vec{\boldsymbol{\mu}}$,

$$\begin{aligned}\hat{\boldsymbol{n}} \times \vec{\boldsymbol{\mu}} &= \det \begin{pmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ m_0 \sin\vartheta_0 e^{i\varphi_0} & -i m_0 \sin\vartheta_0 e^{i\varphi_0} & 0 \end{pmatrix} \\ &= i m_0 e^{i\varphi_0} \sin\vartheta_0 \cos\theta (\hat{\boldsymbol{x}} - i\hat{\boldsymbol{y}}) - i m_0 \sin\vartheta_0 \sin\theta\, e^{i(\varphi_0 - \phi)}\, \hat{\boldsymbol{z}}\,. \end{aligned} \tag{26}$$

The polarization depends on the location of the observer. If the observer is located on the positive $z$-axis then $\theta = 0$. In this case, $\hat{\boldsymbol{n}} = \hat{\boldsymbol{z}}$ and $\vec{\boldsymbol{E}} \propto \hat{\boldsymbol{x}} - i\hat{\boldsymbol{y}}$, which corresponds to right-circularly polarized light [cf. p. 300 of Jackson]. If the observer is located in the $x$–$y$ plane, the $\theta = \frac{1}{2}\pi$. In this case, $\hat{\boldsymbol{n}} = \hat{\boldsymbol{x}}\cos\phi + \hat{\boldsymbol{y}}\sin\phi$ and $\vec{\boldsymbol{E}} \propto \hat{\boldsymbol{z}}$, which corresponds to linearly polarized light in the $z$-direction.

<u>*REMARK*</u>: If $\theta = \pi$, then $\hat{\boldsymbol{n}} = -\hat{\boldsymbol{z}}$ and $\vec{\boldsymbol{E}} \propto \hat{\boldsymbol{x}} - i\hat{\boldsymbol{y}}$, which corresponds to left-circularly polarized light. For any other value of $\theta \neq 0, \frac{1}{2}\pi$ or $\pi$, the radiation is elliptically polarized.

[2] To obtain the angular distribution of the time-averaged radiated power in gaussian units, one must replace $Z_0 \to 4\pi/c$ and $m_0 \to m_0 c$ in eq. (23).

## **APPENDIX A**: Solution to parts (c) and (d) of Jackson, problem 8.2

Refer to problem 1 of this exam. The solutions to parts (a) and (b) have been given above.

(c) The characteristic impedance $Z_0$ of the transmission line is defined as the ratio of the voltage between the cylinders to the axial current flowing in one of them at any position $z$. Show that the impedance for this problem is

$$Z_0 = \frac{1}{2\pi}\sqrt{\frac{\mu}{\epsilon}}\,\ln\left(\frac{b}{a}\right).$$

The voltage difference between the conducting cylinders is

$$V = \int_a^b \vec{\boldsymbol{E}}\cdot d\vec{\boldsymbol{\ell}} = \int_a^b \vec{\boldsymbol{E}}\cdot\hat{\boldsymbol{\rho}}\,d\rho\,,$$

where $\vec{\boldsymbol{E}}$ is given by eq. (6). Thus,

$$V = \sqrt{\frac{\mu}{\epsilon}} H_0\, a\, e^{i(kz-\omega t)} \int_a^b \frac{d\rho}{\rho} = \sqrt{\frac{\mu}{\epsilon}} H_0\, a \ln\left(\frac{b}{a}\right)\, e^{i(kz-\omega t)}\,. \tag{27}$$

The axial current $I$ flowing at any position $z$ is determined by Ampère's law,

$$I = \oint_C \vec{\boldsymbol{H}}\cdot d\vec{\boldsymbol{\ell}}.$$

Here, $d\vec{\boldsymbol{\ell}} = \rho d\phi\,\vec{\boldsymbol{\phi}}$ and $\vec{\boldsymbol{H}} = H_0\, a\, e^{i(kz-\omega t)}\,\hat{\boldsymbol{\phi}}/\rho$. Hence, it follows that

$$I = 2\pi a H_0\, e^{i(kz-\omega t)}\,. \tag{28}$$

Using the results of eqs. (27) and (28), the characteristic impedance is therefore given by:

$$Z_0 = \frac{V}{I} = \frac{1}{2\pi}\sqrt{\frac{\mu}{\epsilon}}\,\ln\left(\frac{b}{a}\right).$$

(d) Show that the series resistance and inductance per unit length of the transmission line are

$$R = \frac{1}{2\pi\sigma\delta}\left(\frac{1}{a}+\frac{1}{b}\right), \qquad L = \frac{\mu}{2\pi}\ln\left(\frac{b}{a}\right) + \frac{\mu_c\delta}{4\pi}\left(\frac{1}{a}+\frac{1}{b}\right),$$

where $\mu_c$ is the permeability of the conductor. The correction to the inductance comes from the penetration of the flux into the conductors by a distance of order $\delta$.

We first treat the case of infinite conductivity. Then, there is no resistance. The inductance is easily computed from first principles. Consider a closed loop $C$ with induced emf,

$$\mathcal{E} = \oint_C \vec{\boldsymbol{E}}\cdot d\vec{\boldsymbol{\ell}} = -\frac{d\Phi}{dt}\,,$$

where

$$\Phi \equiv \int_A \vec{\boldsymbol{B}} \boldsymbol{\cdot} \hat{\boldsymbol{n}}\, da$$

is the magnetic flux through the area $A$ (with outward normal $\hat{\boldsymbol{n}}$) capping the closed circuit $C$. The inductance $\mathscr{L}$ is defined by

$$\mathcal{E} = -\mathscr{L}\, \frac{dI}{dt}\, .$$

It follows that

$$\frac{d\Phi}{dt} = \mathscr{L}\, \frac{dI}{dt}\, ,$$

from which we obtain $\Phi = \mathscr{L} I$.

We will make use of the results of eqs. (6) and (28),

$$\vec{\boldsymbol{B}} = \mu \vec{\boldsymbol{H}} = \mu H_0\, \frac{a}{\rho}\, \hat{\boldsymbol{\phi}}\, e^{i(kz-\omega t)}\, , \qquad\qquad I = 2\pi a H_0\, e^{i(kz-\omega t)}\, ,$$

and take the corresponding real parts to obtain the physical field and current. We choose an area $A$ which slices through the transmission line in the radial direction (as measured from the axis of the cylinders) with an infinitesimal width $dz$ along the $z$-direction. The normal to this area is $\hat{\boldsymbol{\phi}}$ and the curve $C$ caps the infinitesimal area. The magnetic flux per unit length is

$$\frac{d\Phi}{dz} = \int_a^b \vec{\boldsymbol{B}} \boldsymbol{\cdot} \hat{\boldsymbol{\phi}}\, d\rho = \mu H_0\, a\, e^{i(kz-\omega t)} \int_a^b \frac{d\rho}{\rho} = \mu H_0\, a\, \ln\left(\frac{b}{a}\right)\, e^{i(kz-\omega t)}\, .$$

The inductance per unit length $L$ is then defined via $d\Phi/dz = LI$. Hence,

$$L = \frac{\mu}{2\pi} \ln\left(\frac{b}{a}\right)\, . \tag{29}$$

Now, we must take into account the fact that the conductivity $\sigma$ is finite. In this case, there is Joule heating into the conducting surfaces, which results in a non-zero resistance and adds a corresponding contribution to the inductance. To perform the necessary computations, we will make use of the complex impedance. Using eq. (6.135) of Jackson,

$$\tfrac{1}{2} I^* V = -\oint_S \vec{\boldsymbol{S}} \boldsymbol{\cdot} \hat{\boldsymbol{n}}\, da\, , \tag{30}$$

where the right-hand side above is related to the power loss $P_{\rm loss}$ computed in part (b),

$$P_{\rm loss} = -\oint_S (\mathrm{Re}\ \vec{\boldsymbol{S}}) \boldsymbol{\cdot} \hat{\boldsymbol{n}}\, da\, , \tag{31}$$

where the complex Poynting vector is given by eq. (4). The complex impedance is given by

$$Z = \frac{V}{I} = \mathscr{R} - i\omega \mathscr{L}\, , \tag{32}$$

where $\mathscr{R}$ is the resistance and $\mathscr{L}$ is the inductance.

Using eqs. (8.9) and (8.10) of Jackson, the electric and magnetic field just inside the conducting surface is given by

$$\vec{\boldsymbol{H}} = \vec{\boldsymbol{H}}_\parallel \, e^{-\xi/\delta} \, e^{i\xi/\delta} \,, \qquad\qquad \vec{\boldsymbol{E}} \simeq \sqrt{\frac{\mu_c \omega}{2\sigma}} \, (1-i)(\hat{\boldsymbol{n}} \times \vec{\boldsymbol{H}}_\parallel) \, e^{-\xi/\delta} \, e^{i\xi/\delta} \,,$$

for waves propagating a distance $\xi$ into the conductor, where $\vec{\boldsymbol{H}}_\parallel$ is the tangential magnetic field (parallel to the conducting surface) that exists just outside the conducting surface. Thus, the complex Poynting vector is

$$\vec{\boldsymbol{S}} = (1-i)\mathrm{Re}\,\vec{\boldsymbol{S}} \,.$$

Inserting this result in eq. (30) and making use of eq. (31), it follows that

$$\tfrac{1}{2} I^* V = P_{\rm loss}(1-i) \,.$$

In light of eq. (32), we can write $V = ZI$ above and solve for the complex impedance $Z$,

$$Z = \frac{2P_{\rm loss}}{|I|^2} \, (1-i) \,.$$

Taking a derivative with respect to $z$,

$$\frac{dZ}{dz} = \frac{2(1-i)}{|I|^2} \, \frac{dP_{\rm loss}}{dz} \,.$$

Using eqs. (10) and (28), we have $|I| = 2\pi a H_0$ and

$$\frac{dP_{\rm loss}}{dz} = \frac{\mu_c \omega \delta}{4} |H_0|^2 \, 2\pi a^2 \left(\frac{1}{a} + \frac{1}{b}\right) \,.$$

we end up with

$$\frac{dZ}{dz} = R - i\omega L = \left(\frac{1-i}{4\pi}\right) \mu_c \omega \delta \left(\frac{1}{a} + \frac{1}{b}\right) \,, \qquad (33)$$

where $R$ is the resistance per unit length and $L$ is the inductance per unit length [cf. eq. (32)] due to the penetration of the wave into the conductors. Identifying the corresponding real and imaginary parts, it follows that

$$R = \frac{\mu_c \omega \delta}{4\pi} \left(\frac{1}{a} + \frac{1}{b}\right) = \frac{1}{2\pi\sigma\delta} \left(\frac{1}{a} + \frac{1}{b}\right) \,,$$

after using eq. (8) to eliminate $\mu_c$. The contribution to the inductance obtained from the imaginary part of eq. (33), $\frac{\mu_c \delta}{4\pi}\left(\frac{1}{a} + \frac{1}{b}\right)$, must be added to the infinite conductivity result given in eq. (29), which yields the series inductance per unit length[3]

$$L = \frac{\mu}{2\pi} \ln\left(\frac{b}{a}\right) + \frac{\mu_c \delta}{4\pi} \left(\frac{1}{a} + \frac{1}{b}\right) \,.$$

[3]In circuits, given two inductors in series, the series inductance is equal to the sum of the individual inductances.

## **APPENDIX B**: An alternative method for solving problem 3(b)

Instead of evaluating eq. (22), which requires the complex magnetic moment $\vec{\boldsymbol{\mu}}$ given in eq. (19), one can instead employ the result of problem 9.7(a) of Jackson,

$$\frac{dP(t)}{d\Omega} = \frac{Z_0}{16\pi^2 c^4} |\ddot{\vec{\boldsymbol{m}}} \times \hat{\boldsymbol{n}}|^2 \,, \tag{34}$$

where $\ddot{\vec{\boldsymbol{m}}} \equiv d^2\vec{\boldsymbol{m}}/dt^2$, and $\vec{\boldsymbol{m}}$ is the time-dependent magnetic dipole moment given in eq. (18). Note that eq. (34) yields the time dependent power distribution, so to recover the results obtained in problem 1(b), we must time-average over one cycle.

For convenience, we rewrite eq. (18) here:

$$\vec{\boldsymbol{m}} = m_0 \Big[ \hat{\boldsymbol{x}} \sin\vartheta_0 \cos(\varphi_0 - \omega t) + \hat{\boldsymbol{y}} \sin\vartheta_0 \sin(\varphi_0 - \omega t) + \hat{\boldsymbol{z}} \cos\vartheta_0 \Big]$$

Taking two time derivatives, we obtain:

$$\ddot{\vec{\boldsymbol{m}}} = -m_0\omega^2 \Big[ \hat{\boldsymbol{x}} \sin\vartheta_0 \cos(\varphi_0 - \omega t) + \hat{\boldsymbol{y}} \sin\vartheta_0 \sin(\varphi_0 - \omega t) \Big] \,. \tag{35}$$

Next, we compute the square of the cross product,

$$|\ddot{\vec{\boldsymbol{m}}} \times \hat{\boldsymbol{n}}|^2 = \ddot{\vec{\boldsymbol{m}}} \cdot \ddot{\vec{\boldsymbol{m}}} - (\hat{\boldsymbol{n}} \cdot \ddot{\vec{\boldsymbol{m}}})^2 \,,$$

after using the fact that $\hat{\boldsymbol{n}}$ is a unit vector,

$$\hat{\boldsymbol{n}} = \hat{\boldsymbol{x}} \sin\theta \cos\phi + \hat{\boldsymbol{y}} \sin\theta \sin\phi + \hat{\boldsymbol{z}} \cos\theta \,. \tag{36}$$

Using eqs. (35) and (36), it follows that

$$\ddot{\vec{\boldsymbol{m}}} \cdot \ddot{\vec{\boldsymbol{m}}} = m_0^2 \omega^4 \sin^2\vartheta_0 \,,$$

and

$$\begin{aligned} \hat{\boldsymbol{n}} \cdot \ddot{\vec{\boldsymbol{m}}} &= -m_0\omega^2 \sin\vartheta_0 \sin\theta \Big[ \cos\phi \cos(\varphi - \omega t) + \sin\phi \sin(\varphi_0 - \omega t) \Big] \\ &= -m_0\omega^2 \sin\vartheta_0 \sin\theta \cos(\omega t - \varphi_0 + \phi) \,. \end{aligned}$$

Hence,

$$|\ddot{\vec{\boldsymbol{m}}} \times \hat{\boldsymbol{n}}|^2 = m_0^2 \omega^4 \sin^2\vartheta \Big[ 1 - \sin^2\theta \cos^2(\omega t - \varphi_0 + \phi) \Big] \,.$$

Inserting the above result into eq. (34) and using $\omega = kc$, we end up with

$$\frac{dP(t)}{d\Omega} = \frac{Z_0 m_0^2 \sin^2\vartheta}{16\pi^2} k^4 \Big[ 1 - \sin^2\theta \cos^2(\omega t - \varphi_0 + \phi) \Big] \,. \tag{37}$$

Time-averaging over one cycle, $\langle \cos^2(\omega t - \varphi_0 + \phi) \rangle = \frac{1}{2}$. Since $1 - \frac{1}{2}\sin^2\theta = \frac{1}{2}(1 + \cos^2\theta)$, we recover eq. (23). One can also check that the total power obtained by integrating eq. (37) over solid angles is time-independent and coincides with eq. (25).