Thomas Precession and the BMT equation

1. Covariant description of spin angular momentum

The theory of electromagnetic interactions is Lorentz invariant and conserves energy, momentum and angular momentum. In the relativistic formulation, we find that the momentum four-vector P^μ and the angular momentum tensor $J^{\mu\nu}$ are conserved. The angular momentum tensor is antisymmetric and thus consists of six degrees of freedom, which consist of[1]

$$J^i = \tfrac{1}{2}\epsilon^{ijk}J_{jk}\,, \qquad\qquad K^i = J^{0i}\,,$$

where $i, j, k \in \{1, 2, 3\}$. We identify the J^i as the components of the angular momentum three-vector.

The angular momentum tensor can be decomposed into a sum of two antisymmetric tensors,

$$J^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu}\,, \tag{1}$$

where

$$L^{\mu\nu} = x^\mu P^\nu - x^\nu P^\mu\,. \tag{2}$$

In particular, if we define $L^i = \frac{1}{2}\epsilon^{ijk}L_{jk}$, then it follows that $\vec{\boldsymbol{L}} = \vec{\boldsymbol{x}} \times \vec{\boldsymbol{P}}$, which we recognize as the orbital angular momentum. Note that the definition of the orbital angular momentum depends on the choice of the origin of the coordinate system, i.e. $L^{\mu\nu}$ changes under $x^\mu \to x^\mu + a^\mu$. The contribution to $J^{\mu\nu}$ that is independent of the choice of the origin of the coordinate system can be regarded as the intrinsic spin angular momentum. Thus, the spin angular momentum three-vector resides inside $S^{\mu\nu}$ in eq. (1),

$$S^i = \tfrac{1}{2}\epsilon^{ijk}S_{jk}\,.$$

It is convenient to introduce the Pauli-Lubański vector w^μ,

$$w^\mu \equiv -\tfrac{1}{2}\epsilon^{\mu\nu\rho\lambda}J_{\nu\rho}P_\lambda\,, \tag{3}$$

where $\epsilon^{0123} = 1$. Explicitly,

$$w^\mu = (\vec{\boldsymbol{J}}\cdot\vec{\boldsymbol{P}}\,;\; P^0\vec{\boldsymbol{J}} + \vec{\boldsymbol{K}} \times \vec{\boldsymbol{P}})\,. \tag{4}$$

In particular, in light of the identity $\epsilon^{\mu\nu\rho\lambda}P_\lambda P_\mu = 0$, it follows that

$$w\cdot P \equiv w_\mu P^\mu = g_{\mu\nu}w^\mu P^\mu = 0\,, \tag{5}$$

[1] In these notes, we employ the Einstein summation convention, where any twice-repeated index in a single term is implicitly summed over.

which is a Lorentz invariant result that is satisfied in all reference frames. In these notes, we shall consider only massive particles. In the rest frame of the particle, $P^0 = m$, $\vec{\boldsymbol{P}} = 0$, and $\vec{\boldsymbol{J}} = \vec{\boldsymbol{S}}$. Hence, $w^\mu = (0\,;\, m\vec{\boldsymbol{S}})$. Thus, a covariant treatment of spin angular momentum is achieved by identify the spin angular momentum three-vector by the space component of the four-vector, w^μ/m in the rest frame.

It is convenient to define the normalized spin four-vector by

$$S^\mu = \frac{w^\mu}{(-w^2)^{1/2}}\,. \tag{6}$$

Note that we can evaluate the Lorentz invariant quantity $w^2 \equiv w{\cdot}w$ in any reference frame. Indeed, in the rest frame $w^2 = -m^2\vec{\boldsymbol{S}}^2$. The spin four-vector satisfies

$$S{\cdot}P = 0\,, \qquad\qquad S{\cdot}S = -1\,, \tag{7}$$

which are Lorentz invariant equations and thus hold in any reference frame. In the rest frame of the particle,

$$S^\mu = (0\,;\,\hat{\boldsymbol{s}})\,, \tag{8}$$

where $\hat{\boldsymbol{s}}$ is the direction of the intrinsic spin angular momentum as measured in the particle rest frame.

The question that these notes address is the following. Assume that the particle is observed in the laboratory frame to be in motion along some trajectory. We shall consider arbitrary motion, which allows for the possibility that the particle has a nonzero acceleration. In this case, how does $\hat{\boldsymbol{s}}$ evolve in time as viewed from the laboratory frame?

2. Spin precession in the absence of an external torque

Consider a point particle with an intrinsic spin angular momentum. Let K be the laboratory frame and let K' be the rest frame of the particle. In the most general case, the particle in motion is accelerating. Thus, we must be careful about defining the reference frame K'. Indeed, the best we can do is to define an instantaneous rest frame for the particle at each time t (as measured in reference frame K). This rest frame will change (as viewed from the laboratory frame K) as the particle moves along its trajectory. At a given time t, the particle is moving at velocity $\vec{\boldsymbol{v}}(t) = c\vec{\boldsymbol{\beta}}(t)$ as measured by an observer in the laboratory frame.[2]

In this section, we consider a particle whose motion is governed by some external force (hence its acceleration), but where all external torques are absent. In this case, in the rest frame of the particle, $d\hat{\boldsymbol{s}}/dt' = 0$, where t' is the time as measured in K'. Our first objective is to describe the equation of motion of the spin in a Lorentz covariant fashion (which can then be applied in any reference frame). To achieve this goal, we will make use of the proper time τ.

[2]Formally, the instantaneous rest frame at time t as measured in the laboratory frame K is the reference frame obtained from K by boosting (without rotation) with boost velocity $\vec{\boldsymbol{\beta}}$. The corresponding matrix representation of this boost is given in eq. (22).

In reference frame K', we can identify $\tau = t'$, and it follows that

$$S'^{\mu} = (0\,;\,\hat{\boldsymbol{s}})\,, \qquad\qquad \frac{dS'^{\mu}}{d\tau} = \left(\frac{dS'^{0}}{d\tau}\,;\,\vec{\boldsymbol{0}}\right)\,. \tag{9}$$

Note that the spatial component of $dS'^{\mu}/d\tau$ is zero due to the assumption of torque free motion (since the time measured in the rest frame can be identified with the proper time). Somewhat more surprising is the fact that $dS'^{0}/d\tau \neq 0$ even though $S'^{0} = 0$. This is true because the particle is accelerating, which means that the instantaneous rest frame K' at time $\tau + d\tau$ does not coincide with K' at time τ. To demonstrate this assertion more explicitly, we make use of eq. (7) with $P^{\mu} = mu^{u}$, where u^{μ} is the velocity four-vector, to obtain

$$S\!\cdot\! u = 0\,. \tag{10}$$

Taking the derivative of this equation with respect to proper time yields

$$u\!\cdot\!\frac{dS}{d\tau} = -S\!\cdot\!\frac{du}{d\tau}\,. \tag{11}$$

The right hand side above can be evaluated by making use of eq. (28) of the class handout entitled *Examples of four-vectors*,

$$\begin{aligned}\frac{du}{d\tau} = \left(\gamma c\frac{d\gamma}{dt}\,;\,\gamma\frac{d}{dt}(\gamma\vec{\boldsymbol{v}})\right) &= \left(\frac{\gamma^4}{c}\vec{\boldsymbol{v}}\boldsymbol{\cdot}\frac{d\vec{\boldsymbol{v}}}{dt}\,;\,\gamma^2\frac{d\vec{\boldsymbol{v}}}{dt} + \frac{\gamma^4}{c^2}\left(\vec{\boldsymbol{v}}\boldsymbol{\cdot}\frac{d\vec{\boldsymbol{v}}}{dt}\right)\vec{\boldsymbol{v}}\right) \\ &= \frac{\gamma^3}{c^2}\left(\vec{\boldsymbol{v}}\boldsymbol{\cdot}\frac{d\vec{\boldsymbol{v}}}{dt}\right)u + \left(0\,;\,\gamma^2\frac{d\vec{\boldsymbol{v}}}{dt}\right)\,,\end{aligned} \tag{12}$$

where we have made use of the identity,

$$\frac{d\gamma}{dt} = \frac{d}{dt}\left(1-\beta^2\right)^{-1/2} = \gamma^3\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\frac{d\vec{\boldsymbol{\beta}}}{dt}\,. \tag{13}$$

Let us now evaluate eq. (11) in the instantaneous rest frame K', where $u'^{\mu} = (c\,;\,\vec{\boldsymbol{0}})$, $\vec{\boldsymbol{v}}\,' = 0$ and $\gamma = 1$, in which case $S'^{\mu} = (0\,;\,\hat{\boldsymbol{s}})$ and

$$\frac{du'}{d\tau} = \left(0\,;\,\frac{d\vec{\boldsymbol{v}}\,'}{d\tau}\right)\,. \tag{14}$$

Evaluating eq. (11) in the rest frame K' by making use of eqs. (9) and (14), it follows that:

$$\frac{dS'^{0}}{d\tau} = \hat{\boldsymbol{s}}\boldsymbol{\cdot}\frac{d\vec{\boldsymbol{\beta}}\,'}{d\tau}\,. \tag{15}$$

In particular, even though $\vec{\boldsymbol{\beta}}\,' = 0$ in the instantaneous rest frame, if the particle is accelerating then $d\vec{\boldsymbol{\beta}}\,'/d\tau \neq 0$ due to the fact that the instantaneous rest frame K' at time $\tau + d\tau$ does not coincide with K' at time τ, as previously noted.

Our first goal is to obtain a covariant expression for $dS^\mu/d\tau$. In light of eq. (9),

$$\frac{dS'^{\,\mu}}{d\tau} = \kappa u'^{\,\mu}\,, \tag{16}$$

where κ is a Lorentz invariant quantity and $u'^\mu = (c\,;\,\vec{\boldsymbol{0}})$. Since eq. (16) is already in a Lorentz covariant form, this equation must also be true when expressed in terms of quantities measured in the laboratory reference frame K . Thus, we can conclude that

$$\frac{dS^\mu}{d\tau} = \kappa u^\mu\,. \tag{17}$$

One can determine κ as follows. If we multiply the first equation above by u_μ, we obtain

$$u\!\cdot\!\frac{dS}{d\tau} = \kappa u\!\cdot\! u = \kappa c^2\,.$$

Using eq. (11), the above equation is equivalent to

$$S\!\cdot\!\frac{du}{d\tau} = -\kappa c^2\,.$$

Inserting this result back into eq. (17) yields[3]

$$\frac{dS^\mu}{d\tau} = -\frac{1}{c^2}\left(S\!\cdot\!\frac{du}{d\tau}\right)u^\mu\,, \tag{18}$$

which is the covariant equation for the spin four vector. It is straightforward to check that eq. (18) yields the expected results [namely, eqs. (9) and (15)] in the instantaneous rest frame K'.

It is convenient to rewrite eq. (18) in a more explicit fashion. Using $S\!\cdot\! u = 0$, it follows from eq. (12) that

$$S\!\cdot\!\frac{du}{d\tau} = -\gamma^2 c\,\vec{\boldsymbol{S}}\!\cdot\!\frac{d\vec{\boldsymbol{\beta}}}{dt}\,, \tag{19}$$

after writing $\vec{\boldsymbol{v}} = c\vec{\boldsymbol{\beta}}$. Inserting the result of eq. (19) into eq. (18),

$$\frac{dS^\mu}{d\tau} = \frac{\gamma^2}{c}\left(\vec{\boldsymbol{S}}\!\cdot\!\frac{d\vec{\boldsymbol{\beta}}}{dt}\right)u^\mu\,. \tag{20}$$

[3]Since $S\!\cdot\! u = 0$, eq. (18) can be rewritten in the following form,

$$\frac{dS^\mu}{d\tau} + \frac{1}{c^2}\left(u^\mu\frac{du^\nu}{d\tau} - u^\nu\frac{du^\mu}{d\tau}\right)S_\nu = 0\,.$$

The above equation is known as the equation for Fermi-Walker transport, and describes the trajectory of a relativistic spinning particle (e.g. a gyroscope) in the absence of external torques. It has a natural extension in the theory of general relativity.

Finally, using $u^\mu = (\gamma c\,;\,\gamma\vec{\boldsymbol{v}})$ and $d\tau = \gamma^{-1}dt$, we end up with

$$\frac{dS^0}{dt} = \gamma^2\,\vec{\boldsymbol{S}}\cdot\frac{d\vec{\boldsymbol{\beta}}}{dt}\,, \qquad\qquad \frac{d\vec{\boldsymbol{S}}}{dt} = \gamma^2\left(\vec{\boldsymbol{S}}\cdot\frac{d\vec{\boldsymbol{\beta}}}{dt}\right)\vec{\boldsymbol{\beta}}\,. \tag{21}$$

Ultimately, we would like to know how $\hat{\boldsymbol{s}}$ evolves in time as viewed from reference frame K. We can accomplish this by boosting from reference frame K back into the instantaneous rest frame of the particle (reference frame K'). Since at a given time t, the particle is moving at velocity $\vec{\boldsymbol{v}}(t) = c\vec{\boldsymbol{\beta}}(t)$ as measured by an observer in the laboratory frame, we employ the boost matrix,

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\vec{\boldsymbol{\beta}} \\ & \\ -\gamma\vec{\boldsymbol{\beta}} & \delta_{ij} + (\gamma-1)\dfrac{\beta_i\beta_j}{\beta^2} \end{pmatrix}\,. \tag{22}$$

In particular, in light of eq. (8),

$$\vec{\boldsymbol{S}}' = \hat{\boldsymbol{s}} = \vec{\boldsymbol{S}} + \frac{\gamma^2}{\gamma+1}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{S}})\vec{\boldsymbol{\beta}} - \gamma\vec{\boldsymbol{\beta}}S^0\,, \tag{23}$$

$$S'^{\,0} = 0 = \gamma(S^0 - \vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{S}})\,, \tag{24}$$

after noting the identity,

$$\frac{\gamma-1}{\beta^2} = \frac{\gamma^2}{\gamma+1}\,. \tag{25}$$

Thus,

$$S^0 = \vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{S}}\,. \tag{26}$$

Inserting eq. (26) back into eq. (23) yields (after some algebraic simplification),

$$\hat{\boldsymbol{s}} = \vec{\boldsymbol{S}} - \frac{\gamma}{\gamma+1}\vec{\boldsymbol{\beta}}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{S}})\,. \tag{27}$$

We are now ready to compute $d\hat{\boldsymbol{s}}/dt$. Taking the time derivative of eq. (27),

$$\frac{d\hat{\boldsymbol{s}}}{dt} = \frac{d\vec{\boldsymbol{S}}}{dt} - \vec{\boldsymbol{\beta}}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{S}})\frac{d}{dt}\left(\frac{\gamma}{\gamma+1}\right) - \frac{\gamma}{\gamma+1}\left\{\vec{\boldsymbol{\beta}}\left(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\frac{d\vec{\boldsymbol{S}}}{dt} + \vec{\boldsymbol{S}}\boldsymbol{\cdot}\frac{d\vec{\boldsymbol{\beta}}}{dt}\right) + (\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{S}})\frac{d\vec{\boldsymbol{\beta}}}{dt}\right\}\,. \tag{28}$$

We can simplify the above expression by making use of eq. (21) for $d\vec{\boldsymbol{S}}/dt$. In particular, note that

$$\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\frac{d\vec{\boldsymbol{S}}}{dt} = \gamma^2\beta^2\left(\vec{\boldsymbol{S}}\boldsymbol{\cdot}\frac{d\vec{\boldsymbol{\beta}}}{dt}\right)\,. \tag{29}$$

Combining the results of eqs. (21) and (29), it then follows that

$$\frac{d\vec{\boldsymbol{S}}}{dt} - \frac{\gamma}{\gamma+1}\vec{\boldsymbol{\beta}}\left(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\frac{d\vec{\boldsymbol{S}}}{dt}\right) = \gamma\vec{\boldsymbol{\beta}}\left(\vec{\boldsymbol{S}}\boldsymbol{\cdot}\frac{d\vec{\boldsymbol{\beta}}}{dt}\right)\,, \tag{30}$$

after making use of eq. (25). Using eq. (30), we can simplify eq. (28) to obtain

$$\frac{d\hat{\boldsymbol{s}}}{dt} = \frac{\gamma^2}{\gamma+1}\vec{\boldsymbol{\beta}}\left(\vec{\boldsymbol{S}}\cdot\frac{d\vec{\boldsymbol{\beta}}}{dt}\right) - \vec{\boldsymbol{\beta}}(\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{S}})\frac{d}{dt}\left(\frac{\gamma}{\gamma+1}\right) - \frac{\gamma}{\gamma+1}(\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{S}})\frac{d\vec{\boldsymbol{\beta}}}{dt}\,. \tag{31}$$

Our strategy now is to rewrite $\vec{\boldsymbol{S}}\cdot d\vec{\boldsymbol{\beta}}/dt$ and $\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{S}}$ in terms of $\hat{\boldsymbol{s}}$. To accomplish this, we first take the dot product of eq. (27) with $\vec{\boldsymbol{\beta}}$ and make use of eq. (25) to obtain

$$\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{S}} = \gamma\vec{\boldsymbol{\beta}}\cdot\hat{\boldsymbol{s}}\,. \tag{32}$$

Inserting this result back into eqs. (26) and (27) yields

$$S^0 = \gamma\vec{\boldsymbol{\beta}}\cdot\hat{\boldsymbol{s}}\,, \qquad\qquad \vec{\boldsymbol{S}} = \hat{\boldsymbol{s}} + \frac{\gamma^2}{\gamma+1}\vec{\boldsymbol{\beta}}(\vec{\boldsymbol{\beta}}\cdot\hat{\boldsymbol{s}})\,. \tag{33}$$

Hence,

$$\vec{\boldsymbol{S}}\cdot\frac{d\vec{\boldsymbol{\beta}}}{dt} = \hat{\boldsymbol{s}}\cdot\frac{d\vec{\boldsymbol{\beta}}}{dt} + \frac{\gamma^2}{\gamma+1}(\vec{\boldsymbol{\beta}}\cdot\hat{\boldsymbol{s}})\vec{\boldsymbol{\beta}}\cdot\frac{d\vec{\boldsymbol{\beta}}}{dt}\,. \tag{34}$$

Plugging eqs. (32) and (34) back into eq. (31) yields

$$\frac{d\hat{\boldsymbol{s}}}{dt} = \frac{\gamma^2}{\gamma+1}\left[\hat{\boldsymbol{s}}\cdot\frac{d\vec{\boldsymbol{\beta}}}{dt} + \frac{\gamma^2}{\gamma+1}(\vec{\boldsymbol{\beta}}\cdot\hat{\boldsymbol{s}})\vec{\boldsymbol{\beta}}\cdot\frac{d\vec{\boldsymbol{\beta}}}{dt}\right]\vec{\boldsymbol{\beta}} - \frac{\gamma^2}{\gamma+1}(\vec{\boldsymbol{\beta}}\cdot\hat{\boldsymbol{s}})\frac{d\vec{\boldsymbol{\beta}}}{dt} - \gamma\vec{\boldsymbol{\beta}}(\vec{\boldsymbol{\beta}}\cdot\hat{\boldsymbol{s}})\frac{d}{dt}\left(\frac{\gamma}{\gamma+1}\right)\,. \tag{35}$$

The rest is algebra. In particular, note that in light of eq. (13),

$$\frac{d}{dt}\left(\frac{\gamma}{\gamma+1}\right) = \frac{1}{(\gamma+1)^2}\,\frac{d\gamma}{dt} = \frac{\gamma^3}{(\gamma+1)^2}\,\vec{\boldsymbol{\beta}}\cdot\frac{d\vec{\boldsymbol{\beta}}}{dt}\,. \tag{36}$$

Plugging this back into eq. (35), we see that two of the terms exactly cancel and we are left with

$$\frac{d\hat{\boldsymbol{s}}}{dt} = \frac{\gamma^2}{\gamma+1}\left[\left(\hat{\boldsymbol{s}}\cdot\frac{d\vec{\boldsymbol{\beta}}}{dt}\right)\vec{\boldsymbol{\beta}} - (\vec{\boldsymbol{\beta}}\cdot\hat{\boldsymbol{s}})\frac{d\vec{\boldsymbol{\beta}}}{dt}\right]\,. \tag{37}$$

We recognize the right hand side above as a triple cross product,

$$\frac{d\hat{\boldsymbol{s}}}{dt} = \frac{\gamma^2}{\gamma+1}\,\hat{\boldsymbol{s}}\times\left(\vec{\boldsymbol{\beta}}\times\frac{d\vec{\boldsymbol{\beta}}}{dt}\right)\,, \tag{38}$$

which is the desired result.

Note that eq. (37) has the form of a spin precession equation, which is traditionally rewritten in the form,

$$\frac{d\hat{\boldsymbol{s}}}{dt} = \vec{\boldsymbol{\omega}}_T\times\hat{\boldsymbol{s}}\,,$$

where

$$\vec{\boldsymbol{\omega}}_T = \frac{\gamma^2}{\gamma+1}\left(\frac{d\vec{\boldsymbol{\beta}}}{dt}\times\vec{\boldsymbol{\beta}}\right) = \frac{\gamma^2}{\gamma+1}\,\frac{\vec{\boldsymbol{a}}\times\vec{\boldsymbol{v}}}{c^2}\,,$$

where $\vec{\boldsymbol{a}} = d\vec{\boldsymbol{v}}/dt$. This is the phenomenon of Thomas precession, and ω_T is the Thomas precession (angular) frequency. Note that $d\hat{\boldsymbol{s}}/dt$ is expressed in terms of $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{v}}$. The latter two quantities are the acceleration and the velocity of the particle as measured in the laboratory frame K. That is, $\hat{\boldsymbol{s}}$ which is perceived to be time-independent in the instantaneous (comoving) rest frame is seen to precess when viewed from the laboratory frame K.

3. Comments on the analysis of Section 2

In Section 2, we focused on $\hat{\boldsymbol{s}}$, which is the direction of the spin three-vector in the instantaneous rest frame. However, we could have performed the same analysis with the Pauli-Lubański vector w^μ. In particular, as shown in Section 1, $w^\mu = (0\,;\, m\vec{\boldsymbol{S}}')$ in the rest frame (where rest frame quantities are denoted with primes). In the absence of an external torque, $d\vec{\boldsymbol{S}}'/dt = 0$ in which case we can write

$$\frac{dw'^{\mu}}{d\tau} = \left(\frac{dw'^{0}}{d\tau}\,;\,\vec{\boldsymbol{0}}\right) ,$$

following the same analysis used to obtain eq. (9). Thus,

$$\frac{dw^\mu}{d\tau} = -\frac{1}{c^2}\left(w\cdot\frac{du}{d\tau}\right)u^\mu\,, \tag{39}$$

in the same way that eq. (18) was derived. Using $P^\mu = mu^\mu$, it follows from eq. (5) that $w\cdot u = 0$. Hence, eq. (39) yields

$$\frac{d}{d\tau}w^2 = 2w\cdot\frac{dw^\mu}{d\tau} = 0\,.$$

That is, w^2 (which is a Lorentz invariant quantity) is a constant in time. Using the form of w^μ in the instantaneous rest frame, it follows that $|\vec{\boldsymbol{S}}'|^2$ is a constant in time. Hence, it is sufficient to focus on the time dependence of $\hat{\boldsymbol{s}}$, as we did in Section 2.

To get better sense of the meaning of $d\vec{\boldsymbol{\beta}}'/dt$ in eq. (15), let us compute $dS'^{0}/d\tau$ by evaluating the four-vectors of eq. (20) in the instantaneous rest frame K'. It then immediately follows that

$$\frac{dS'^{0}}{d\tau} = \gamma^2\vec{\boldsymbol{S}}\cdot\frac{d\vec{\boldsymbol{\beta}}}{dt}\,.$$

Using eq. (34), it follows that

$$\frac{dS'^{0}}{d\tau} = \gamma^2\hat{\boldsymbol{s}}\boldsymbol{\cdot}\left[\frac{d\vec{\boldsymbol{\beta}}}{dt} + \frac{\gamma^2}{\gamma+1}\left(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\frac{d\vec{\boldsymbol{\beta}}}{dt}\right)\vec{\boldsymbol{\beta}}\right]\,.$$

Comparing this result with that of eq. (15), we conclude that

$$\frac{d\vec{\boldsymbol{\beta}}'}{dt} = \gamma^2\left[\frac{d\vec{\boldsymbol{\beta}}}{dt} + \frac{\gamma^2}{\gamma+1}\left(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\frac{d\vec{\boldsymbol{\beta}}}{dt}\right)\vec{\boldsymbol{\beta}}\right]\,. \tag{40}$$

The meaning of eq. (40) is as follows. In reference frame K, perform a boost (without rotation) such that the particle is at rest in the instantaneous reference frame K'. In particular, this means that if the particle velocity four-vector is $u^\mu = (\gamma c\,;\,\gamma\vec{\boldsymbol{v}})$ in reference frame K, then

$$u'^\mu = \Lambda^\mu{}_\nu u^\nu = (c\,;\,\vec{\boldsymbol{0}})\,,$$

as expected for the rest frame, where Λ is given by eq. (22). Using the same boost matrix, we boost $du/d\tau$ given by eq. (12) to the instantaneous rest frame K'. The result of this calculation is

$$\left.\frac{du}{d\tau}\right|_{K'} = \Lambda^\mu{}_\nu\frac{du^\nu}{d\tau} = \left(0\,;\,\gamma^2 c\left(\frac{d\vec{\boldsymbol{\beta}}}{dt} + \left(\frac{\gamma-1}{\beta^2}\right)\left(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\frac{d\vec{\boldsymbol{\beta}}}{dt}\right)\vec{\boldsymbol{\beta}}\right)\right).$$

In light of eq. (25),

$$\left.\frac{du}{d\tau}\right|_{K'} = \left(0\,;\,c\frac{d\vec{\boldsymbol{\beta}}'}{dt}\right).$$

That is, if the acceleration of the particle is observed in the laboratory frame, then boosting the acceleration four-vector back to the instantaneous rest frame will lead to a non-zero result, precisely because the instantaneous rest frame is not a constant in time.

4. The Bargmann-Michel-Telegdi (BMT) equation

A particle with charge e and intrinsic spin also has an intrinsic magnetic moment. In the instantaneous rest frame of the particle K', the magnetic moment is given by

$$\vec{\boldsymbol{m}}' = \frac{ge}{2mc}\vec{\boldsymbol{S}}'\,, \tag{41}$$

where $\vec{\boldsymbol{S}}'$ is the spin vector of the particle[4] and g is the so-called g factor of the particle (which can be experimentally measured). Using Newton's laws of motion, the torque $\vec{\boldsymbol{N}}'$ in reference frame K' is given by

$$\vec{\boldsymbol{N}}' = \frac{d\vec{\boldsymbol{S}}'}{dt'}\,. \tag{42}$$

In the presence of an external electromagnetic field, Jackson eq. (5.71) yields,

$$\vec{\boldsymbol{N}}' = \vec{\boldsymbol{m}}' \times \vec{\boldsymbol{B}}'\,. \tag{43}$$

Since the spin is an intrinsic property of the particle (see footnote 4 below), we are free to normalize the spin vector as we did in eqs. (6)–(8). Consequently, in the presence of an external magnetic field, the time dependence of $\hat{\boldsymbol{s}}$ is determined by eqs. (41)–(43),

$$\frac{d\hat{\boldsymbol{s}}}{d\tau} = \frac{ge}{2mc}\hat{\boldsymbol{s}} \times \vec{\boldsymbol{B}}'\,,$$

[4] In quantum mechanics, the eigenvalue of $\vec{\boldsymbol{S}}'^2$ when acting on a state vector of the particle is $\hbar^2 s(s+1)$, where s is the intrinsic spin of the particle, which is either an integer or half integer.

in the instantaneous rest frame K' where $\vec{\boldsymbol{B}}\,'$ is the magnetic field measured in frame K' and the proper time τ corresponds to the time t' measured in the instantaneous rest frame. Eq. (9) is now replaced by

$$S'^{\,\mu} = (0\,;\,\hat{\boldsymbol{s}})\,, \qquad\qquad \frac{dS'^{\,\mu}}{d\tau} = \left(\frac{dS'^{\,0}}{d\tau}\,;\,\frac{ge}{2mc}\hat{\boldsymbol{s}}\times\vec{\boldsymbol{B}}\,'\right)\,.$$

The calculation of $dS'^{\,0}/d\tau$ which resulted in eq. (15) is still valid since it only relies on the relation $S\!\cdot\!u = 0$. Hence, it follows that

$$\frac{dS'^{\,\mu}}{d\tau} = \left(\hat{\boldsymbol{s}}\boldsymbol{\cdot}\frac{d\vec{\boldsymbol{\beta}}\,'}{d\tau}\,;\,\frac{ge}{2mc}\hat{\boldsymbol{s}}\times\vec{\boldsymbol{B}}\,'\right)\,. \tag{44}$$

We can write the above equation in a Lorentz covariant form as follows. Recall that the electric and magnetic fields reside in the electromagnetic field strength tensor $F^{\mu\nu}$. In particular, $F^{j0} = -F^{0j} = E^j$ and $F^{ij} = -\epsilon^{ijk}B^k$. Hence, in the instantaneous rest frame K',

$$F'^{\,\mu\nu}S'_\nu = (-F'^{\,0j}\hat{\boldsymbol{s}}^j\,;\,-F'^{\,ij}\hat{\boldsymbol{s}}^j) = (\vec{\boldsymbol{E}}\,'\boldsymbol{\cdot}\hat{\boldsymbol{s}}\,;\,\hat{\boldsymbol{s}}\times\vec{\boldsymbol{B}}\,')\,,$$

after noting that $S'^{\,0} = 0$ and $S'_j = -S'^{\,j} = -\hat{\boldsymbol{s}}^j$. It follows that we can rewrite eq. (44) as

$$\frac{dS'^{\,\mu}}{d\tau} = \frac{ge}{2mc}F'^{\,\mu\nu}S'_\nu + \left(\hat{\boldsymbol{s}}\boldsymbol{\cdot}\frac{d\vec{\boldsymbol{\beta}}\,'}{d\tau} - \frac{ge}{2mc}\hat{\boldsymbol{s}}\boldsymbol{\cdot}\vec{\boldsymbol{E}}\,'\,;\,\vec{\boldsymbol{0}}\right)\,. \tag{45}$$

Once again we note that $u'^{\mu} = (c\,;\,\vec{\boldsymbol{0}})$ in the instantaneous rest frame K', which allows us to write

$$\frac{dS'^{\,\mu}}{d\tau} = \frac{ge}{2mc}F'^{\,\mu\nu}S'_\nu + \kappa u'^{\,\mu}\,, \tag{46}$$

where κ is a Lorentz invariant constant [which will differ from the corresponding constant that appears in eq. (16)]. Since eq. (46) is already in a Lorentz covariant form, this equation must also be true when expressed in terms of quantities measured in the laboratory reference frame K . Thus, we can conclude that

$$\frac{dS^\mu}{d\tau} = \frac{ge}{2mc}F^{\mu\nu}S_\nu + \kappa u^\mu\,. \tag{47}$$

Multiplying both sides of the equation by u_μ and using $u_\mu u^\mu = c^2$, we can solve for κ,

$$\kappa = \frac{1}{c^2}u\!\cdot\!\frac{dS}{d\tau} - \frac{ge}{2mc^3}F^{\mu\nu}u_\mu S_\nu\,.$$

Employing eq. (11), and using the antisymmetry of $F^{\mu\nu}$,

$$\kappa = -\frac{1}{c^2}S\!\cdot\!\frac{du}{d\tau} + \frac{ge}{2mc^3}S_\alpha F^{\alpha\beta}u_\beta\,.$$

Finally, inserting this result into eq. (47) yields

$$\frac{dS^\mu}{d\tau} = \frac{ge}{2mc}\left[F^{\mu\nu}S_\nu + \frac{1}{c^2}u^\mu S_\alpha F^{\alpha\beta}u_\beta\right] - \frac{1}{c^2}\left(S\!\cdot\!\frac{du}{d\tau}\right)u^\mu\,. \tag{48}$$

which is the generalization of eq. (18) in the presence of an electromagnetic field. This reproduces eq. (11.162) of Jackson.

Finally, if the acceleration of the charged particle with spin is due entirely to the dynamics of the electromagnetic fields and if gradient forces can be neglected, then we may employ the equations of motion[5]

$$\frac{du^\alpha}{d\tau} = \frac{e}{mc} F^{\alpha\beta} u_\beta \,. \tag{49}$$

Inserting this result into eq. (48), we end up with

$$\frac{dS^\mu}{d\tau} = \frac{e}{mc}\left[\frac{g}{2} F^{\mu\nu} S_\nu + \frac{1}{c^2}\left(\frac{g}{2} - 1\right) u^\mu S_\alpha F^{\alpha\beta} u_\beta\right] , \tag{50}$$

which is the celebrated BMT equation given in eq. (11.164) of Jackson. Note that in the derivation presented here, we did not need to invoke a variety of conditions imposed by Jackson such as linearity and the absence of higher time derivatives.

As in Section 2, it is useful to use eq. (48) to derive an equation for $d\hat{\boldsymbol{s}}/dt$, which was previously given in eq. (28) and is repeated here for the convenience of the reader,

$$\frac{d\hat{\boldsymbol{s}}}{dt} = \frac{d\vec{\boldsymbol{S}}}{dt} - \vec{\boldsymbol{\beta}}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{S}})\frac{d}{dt}\left(\frac{\gamma}{\gamma+1}\right) - \frac{\gamma}{\gamma+1}\left\{\vec{\boldsymbol{\beta}}\left(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\frac{d\vec{\boldsymbol{S}}}{dt} + \vec{\boldsymbol{S}}\boldsymbol{\cdot}\frac{d\vec{\boldsymbol{\beta}}}{dt}\right) + (\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{S}})\frac{d\vec{\boldsymbol{\beta}}}{dt}\right\} . \tag{51}$$

We follow the same strategy employed in Section 2. First, we use eqs. (20) and (48) and $d\tau = \gamma^{-1} dt$ to write,

$$\frac{dS^0}{dt} = \frac{ge}{2\gamma mc}\left[\vec{\boldsymbol{S}}\boldsymbol{\cdot}\vec{\boldsymbol{E}} + \frac{\gamma}{c} S_\alpha F^{\alpha\beta} u_\beta\right] + \gamma^2\, \vec{\boldsymbol{S}}\boldsymbol{\cdot}\frac{d\vec{\boldsymbol{\beta}}}{dt} ,$$

$$\frac{d\vec{\boldsymbol{S}}}{dt} = \frac{ge}{2\gamma mc}\left[S_0 \vec{\boldsymbol{E}} + \vec{\boldsymbol{S}} \times \vec{\boldsymbol{B}} + \frac{\gamma}{c}\vec{\boldsymbol{\beta}}\, S_\alpha F^{\alpha\beta} u_\beta\right] + \gamma^2\, \vec{\boldsymbol{S}}\boldsymbol{\cdot}\frac{d\vec{\boldsymbol{\beta}}}{dt}\vec{\boldsymbol{\beta}} ,$$

where

$$S_\alpha F^{\alpha\beta} u_\beta = \gamma c\big[S_0 \vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{E}} - \vec{\boldsymbol{S}}\boldsymbol{\cdot}(\vec{\boldsymbol{E}} + \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{B}})\big] .$$

It follows that

$$\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\frac{d\vec{\boldsymbol{S}}}{dt} = \frac{\gamma ge}{2mc}\big[S_0 \vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{E}} - \vec{\boldsymbol{S}}\boldsymbol{\cdot}(\vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{B}}) - \beta^2 \vec{\boldsymbol{S}}\boldsymbol{\cdot}\vec{\boldsymbol{E}}\big] + \gamma^2\beta^2 \vec{\boldsymbol{S}}\boldsymbol{\cdot}\frac{d\vec{\boldsymbol{\beta}}}{dt} .$$

It is convenient to evaluate the following combination of terms (as we did in Section 2),

$$\frac{d\vec{\boldsymbol{S}}}{dt} - \frac{\gamma}{\gamma+1}\vec{\boldsymbol{\beta}}\left(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\frac{d\vec{\boldsymbol{S}}}{dt}\right) = \frac{ge}{2\gamma mc}\left\{S_0 \vec{\boldsymbol{E}} + \vec{\boldsymbol{S}} \times \vec{\boldsymbol{B}} + \frac{\gamma^2}{\gamma+1}\vec{\boldsymbol{\beta}}\big[S_0 \vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{E}} - \vec{\boldsymbol{S}}\boldsymbol{\cdot}(\vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{B}})\,\big] - \gamma\vec{\boldsymbol{\beta}}(\vec{\boldsymbol{S}}\boldsymbol{\cdot}\vec{\boldsymbol{E}})\right\}$$

$$+\gamma\vec{\boldsymbol{\beta}}\left(\vec{\boldsymbol{S}}\boldsymbol{\cdot}\frac{d\vec{\boldsymbol{\beta}}}{dt}\right) .$$

[5] Eq. (49) neglects the fact that a charged particle with spin also possesses an intrinsic magnetic moment. In this case, there will be an extra contribution to the three-vector force given by eq. (5.69) of Jackson, $\vec{\boldsymbol{F}} = \vec{\boldsymbol{\nabla}}(\vec{\boldsymbol{m}}\boldsymbol{\cdot}\vec{\boldsymbol{B}})$. This so-called gradient force can be shown to be subdominant as compared to the Lorentz force, $\vec{\boldsymbol{F}} = q\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}/c$. For further details, see Ref. 5.

With the help of the above result and eqs. (32), (34) and (36), eq. (51) yields,

$$\frac{d\hat{\boldsymbol{s}}}{dt} = \frac{ge}{2\gamma mc}\Big\{S_0\vec{\boldsymbol{E}} + \vec{\boldsymbol{S}}\times\vec{\boldsymbol{B}} + \frac{\gamma^2}{\gamma+1}\vec{\boldsymbol{\beta}}\big[S_0\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{E}} - \vec{\boldsymbol{S}}\boldsymbol{\cdot}(\vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{B}})\big] - \gamma\vec{\boldsymbol{\beta}}(\vec{\boldsymbol{S}}\boldsymbol{\cdot}\vec{\boldsymbol{E}})\Big\}$$
$$+\frac{\gamma^2}{\gamma+1}\left[\left(\hat{\boldsymbol{s}}\boldsymbol{\cdot}\frac{d\vec{\boldsymbol{\beta}}}{dt}\right)\vec{\boldsymbol{\beta}} - (\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\hat{\boldsymbol{s}})\frac{d\vec{\boldsymbol{\beta}}}{dt}\right].$$

Finally, using eq. (33) to write $\vec{\boldsymbol{S}}$ in terms of $\hat{\boldsymbol{s}}$,

$$\frac{d\hat{\boldsymbol{s}}}{dt} = \frac{ge}{2mc}\Big\{(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\hat{\boldsymbol{s}})\vec{\boldsymbol{E}} - (\hat{\boldsymbol{s}}\boldsymbol{\cdot}\vec{\boldsymbol{E}})\vec{\boldsymbol{\beta}} + \frac{1}{\gamma}\left[\hat{\boldsymbol{s}}\times\vec{\boldsymbol{B}} + \frac{\gamma^2}{\gamma+1}\big[(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\hat{\boldsymbol{s}})\vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{B}} - \hat{\boldsymbol{s}}\boldsymbol{\cdot}(\vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{B}})\vec{\boldsymbol{\beta}}\big]\right]\Big\}$$
$$+\frac{\gamma^2}{\gamma+1}\left[\left(\hat{\boldsymbol{s}}\boldsymbol{\cdot}\frac{d\vec{\boldsymbol{\beta}}}{dt}\right)\vec{\boldsymbol{\beta}} - (\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\hat{\boldsymbol{s}})\frac{d\vec{\boldsymbol{\beta}}}{dt}\right].$$

We recognize,

$$\hat{\boldsymbol{s}}\times(\vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{E}}) = (\hat{\boldsymbol{s}}\boldsymbol{\cdot}\vec{\boldsymbol{E}})\vec{\boldsymbol{\beta}} - (\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\hat{\boldsymbol{s}})\vec{\boldsymbol{E}}\,,$$
$$\hat{\boldsymbol{s}}\times\big[\vec{\boldsymbol{\beta}}\times(\vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{B}})\big] = \hat{\boldsymbol{s}}\boldsymbol{\cdot}(\vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{B}})\vec{\boldsymbol{\beta}} - (\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\hat{\boldsymbol{s}})\vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{B}}\,.$$

Hence, we end up with

$$\frac{d\hat{\boldsymbol{s}}}{dt} = \hat{\boldsymbol{s}}\times\left\{\frac{ge}{2mc}\left[\frac{1}{\gamma}\vec{\boldsymbol{B}} - \frac{\gamma}{\gamma+1}\vec{\boldsymbol{\beta}}\times(\vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{B}}) - \vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{E}}\right] + \frac{\gamma^2}{\gamma+1}\vec{\boldsymbol{\beta}}\times\frac{d\vec{\boldsymbol{\beta}}}{dt}\right\}.$$

Finally, using $\vec{\boldsymbol{\beta}}\times(\vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{B}}) = (\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{B}})\vec{\boldsymbol{\beta}} - \beta^2\vec{\boldsymbol{B}}$, and making use of eq. (25), we obtain

$$\frac{d\hat{\boldsymbol{s}}}{dt} = \hat{\boldsymbol{s}}\times\left\{\frac{ge}{2mc}\left[\vec{\boldsymbol{B}} - \frac{\gamma}{\gamma+1}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{B}})\vec{\boldsymbol{\beta}} - \vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{E}}\right] + \frac{\gamma^2}{\gamma+1}\vec{\boldsymbol{\beta}}\times\frac{d\vec{\boldsymbol{\beta}}}{dt}\right\}, \qquad (52)$$

which is the generalization of eq. (38) in the presence of an electromagnetic field.[6]

If the acceleration of the charged particle with spin is due entirely to the dynamics of the electromagnetic fields and if gradient forces can be neglected, then we may employ eq. (49). This yields eq. (11.168) of Jackson,

$$\frac{d\vec{\boldsymbol{\beta}}}{dt} = \frac{e}{\gamma mc}\left[\vec{\boldsymbol{E}} + \vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{B}} - \vec{\boldsymbol{\beta}}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{E}})\right].$$

Inserting this into eq. (52) yields,

$$\frac{d\hat{\boldsymbol{s}}}{dt} = \frac{e}{mc}\,\hat{\boldsymbol{s}}\times\left\{\left(\frac{g}{2} - 1 + \frac{1}{\gamma}\right)\vec{\boldsymbol{B}} - \left(\frac{g}{2} - 1\right)\frac{\gamma}{\gamma+1}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{B}})\vec{\boldsymbol{\beta}} - \left(\frac{g}{2} - \frac{\gamma}{\gamma+1}\right)\vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{E}}\right\}, \qquad (53)$$

which is the Thomas equation. This result is also obtained in eq. (11.170) of Jackson.

[6]Note that by setting $g = 0$ we recover the Thomas precession obtained in Section 2 [cf. eq. (38)].

For completeness, we note that the Thomas equation can also be written in the following alternative form,

$$\frac{d\hat{\boldsymbol{s}}}{dt} = \frac{ge}{2mc}\,\hat{\boldsymbol{s}} \times \left(\frac{1}{\gamma}\vec{\boldsymbol{B}} - \frac{1}{\gamma+1}\vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{E}}\right) - (g-2)\left(\frac{e}{2mc}\right)\frac{\gamma}{1+\gamma}\,\hat{\boldsymbol{s}} \times \left[\vec{\boldsymbol{\beta}} \times (\vec{\boldsymbol{E}} + \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{B}})\right]. \quad (54)$$

As an example, consider the special case of the motion of a particle with charge e with velocity $\vec{\boldsymbol{v}}$ in a uniform magnetic field $\vec{\boldsymbol{B}}$ such that $\vec{\boldsymbol{B}}\cdot\vec{\boldsymbol{v}} = 0$. In this case, the particle moves in a circle that lies in the plane perpendicular to $\vec{\boldsymbol{B}}$. The velocity $\vec{\boldsymbol{v}}$ precesses about $\vec{\boldsymbol{B}}$ such that [cf. eqs. (12.38) and (12.39) of Jackson]:

$$\frac{d\vec{\boldsymbol{v}}}{dt} = \vec{\boldsymbol{v}} \times \vec{\boldsymbol{\omega}}_{\boldsymbol{B}}\,, \quad \text{with } \vec{\boldsymbol{\omega}}_B = \frac{e\vec{\boldsymbol{B}}}{\gamma mc}. \quad (55)$$

Assuming that there is no electric field ($\vec{\boldsymbol{E}} = 0$), then eq. (54) yields,

$$\frac{d\hat{\boldsymbol{s}}}{dt} = \frac{e}{\gamma mc}\left[1 + \tfrac{1}{2}(g-2)\gamma\right]\hat{\boldsymbol{s}} \times \vec{\boldsymbol{B}}\,, \quad (56)$$

after employing $\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{B}} = 0$ and $\beta^2 = (\gamma^2 - 1)/\gamma^2$. That is,

$$\frac{d\hat{\boldsymbol{s}}}{dt} = \hat{\boldsymbol{s}} \times \vec{\boldsymbol{\omega}}\,, \qquad \text{with } \vec{\boldsymbol{\omega}} = \left[1 + \tfrac{1}{2}(g-2)\gamma\right]\vec{\boldsymbol{\omega}}_B. \quad (57)$$

Thus, if $g = 2$ then the spin and velocity vectors would precess at precisely the same rate.

The g factor of the electron and the muon are measured experimentally and are found to be very close to $g = 2$, but there is a small correction. Indeed, one of the famous early predictions of quantum electrodynamics was that

$$\frac{g-2}{2} \simeq \frac{\alpha}{2\pi} \simeq 0.0011614\,, \quad (58)$$

where $\alpha \equiv e^2/(4\pi\hbar c) \simeq 1/137$. Thus, $\hat{\boldsymbol{s}}$ and $\vec{\boldsymbol{v}}$ precess at slightly different angular frequencies and eventually become out of phase during the motion, which provides an experimental measure of $g-2$. The quantum field theory predictions for $g-2$ of the electron and muon are now known to far greater accuracy than the simple formula given in eq. (58). The most precise measurements of $g-2$ of the muon at Fermilab has revealed a very slight discrepancy with the theoretical prediction. Whether this is a statistical fluke or evidence for new physics phenomena beyond the Standard Model of particle physics remains to be seen.

5. Spin-Orbit interaction and the hydrogen atom

We can now apply the results of the previous section to derive the spin-orbit interaction of the hydrogen atom Hamiltonian. The electron is attracted to the nucleus of charge Ze via the central Coulomb potential $V(r)$. In particular,[7]

$$e\vec{\boldsymbol{E}} = -\vec{\boldsymbol{\nabla}}V(r) = -\hat{\boldsymbol{n}}\frac{dV(r)}{dr}\,, \qquad \vec{\boldsymbol{B}} = 0\,, \quad (59)$$

[7]Note that since $\vec{\boldsymbol{B}} = 0$, the gradient forces associated with the magnetic dipole moment are absent.

where $\vec{\boldsymbol{x}} = r\hat{\boldsymbol{n}}$ and $V(r) = -Ze^2/r$. Plugging these results into the Thomas equation given by eq. (53),

$$\frac{d\hat{\boldsymbol{s}}}{dt} = -\frac{e}{mc}\left(\frac{g}{2} - \frac{\gamma}{\gamma+1}\right)\hat{\boldsymbol{s}} \times (\vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{E}})\,. \tag{60}$$

Using $\vec{\boldsymbol{p}} = \gamma mc\vec{\boldsymbol{\beta}}$,

$$e\vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{E}} = \frac{\vec{\boldsymbol{x}} \times \vec{\boldsymbol{p}}}{\gamma mcr}\frac{dV}{dr} = \frac{\vec{\boldsymbol{L}}}{\gamma mcr}\frac{dV}{dr}\,, \tag{61}$$

where $\vec{\boldsymbol{L}} = \vec{\boldsymbol{x}} \times \vec{\boldsymbol{p}}$ is the orbital angular momentum vector. Hence,

$$\frac{d\hat{\boldsymbol{s}}}{dt} = -\frac{1}{\gamma m^2c^2}\left(\frac{g}{2} - \frac{\gamma}{\gamma+1}\right)\hat{\boldsymbol{s}} \times \vec{\boldsymbol{L}}\,\frac{1}{r}\frac{dV}{dr}\,. \tag{62}$$

In the nonrelativistic limit, $\gamma \simeq 1$, and we arrive at

$$\frac{d\hat{\boldsymbol{s}}}{dt} \simeq -\frac{g-1}{2m^2c^2}\hat{\boldsymbol{s}} \times \vec{\boldsymbol{L}}\,\frac{1}{r}\frac{dV}{dr}\,. \tag{63}$$

Let us promote eq. (63) to an equation for the quantum spin operator $\vec{\boldsymbol{S}}$, which satisfies the commutation relations,[8]

$$\left[S_i\,,\,S_j\right] = i\hbar\epsilon_{ijk}S_k\,, \tag{64}$$

where there is an implicit sum over the repeated index k. Likewise, $\vec{\boldsymbol{L}}$ is promoted to the quantum orbital angular momentum operator, which commutes with $\vec{\boldsymbol{S}}$. Then, in the Heisenberg representation,

$$\frac{d\vec{\boldsymbol{S}}}{dt} = -\frac{g-1}{2m^2c^2}\vec{\boldsymbol{S}} \times \vec{\boldsymbol{L}}\,\frac{1}{r}\frac{dV}{dr}\,. \tag{65}$$

We now employ the Heisenberg equations of motion,

$$i\hbar\frac{d\vec{\boldsymbol{S}}}{dt} = \left[\vec{\boldsymbol{S}}, H_{\rm LS}\right], \tag{66}$$

where $H_{\rm LS}$ is the spin-orbit Hamiltonian that we seek. Note that $-i\hbar\vec{\boldsymbol{S}} \times \vec{\boldsymbol{L}} = \left[\vec{\boldsymbol{S}}\,,\,\vec{\boldsymbol{S}}\boldsymbol{\cdot}\vec{\boldsymbol{L}}\right]$. One can verify this quantum operator identity by using eq. (64). Explicitly,

$$-i\hbar(\vec{\boldsymbol{S}} \times \vec{\boldsymbol{L}})_k = -i\hbar\epsilon_{ijk}S_iL_j = i\hbar_{kji}S_iL_j = \left[S_k\,,\,S_j\right]L_j = \left[S_k\,,\,\vec{\boldsymbol{S}}\boldsymbol{\cdot}\vec{\boldsymbol{L}}\right] = \left[\vec{\boldsymbol{S}}\,,\,\vec{\boldsymbol{S}}\boldsymbol{\cdot}\vec{\boldsymbol{L}}\right]_k. \tag{67}$$

Hence, we can identify:

$$\boxed{H_{\rm LS} = \frac{g-1}{2m^2c^2}\vec{\boldsymbol{S}}\boldsymbol{\cdot}\vec{\boldsymbol{L}}\,\frac{1}{r}\frac{dV}{dr}\,.} \tag{68}$$

which is the well-known spin-orbit coupling Hamiltonian that governs the hydrogen atom.

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[8]Henceforth, I will suppress the prime superscript on $\vec{\boldsymbol{S}}$, which indicates that the spin operator measures the spin in the rest frame of the electron.

In quantum mechanics textbooks, the spin-orbit interaction term is usually derived by asserting that in the rest frame of the electron, the Coulomb $\vec{\boldsymbol{E}}$-field in the laboratory frame appears as a combination of electric and magnetic fields, where

$$\vec{\boldsymbol{B}}' \simeq -\vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{E}} + \mathcal{O}(\beta^2)\,. \tag{69}$$

Hence, the electron, which is a point magnetic dipole with a dipole moment given by eq. (41), interacts with the magnetic field $\vec{\boldsymbol{B}}'$ with an interaction energy,

$$U = -\vec{\boldsymbol{m}}'\boldsymbol{\cdot}\vec{\boldsymbol{B}}' = \frac{ge}{2mc}\vec{\boldsymbol{S}}\boldsymbol{\cdot}(\vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{E}}) = \frac{g}{2m^2c^2}\vec{\boldsymbol{S}}\boldsymbol{\cdot}\vec{\boldsymbol{L}}\,\frac{1}{r}\,\frac{dV}{dr}\,, \tag{70}$$

after using eq. (59). Comparing eqs. (68) and (70), we see that the correct numerator factor is $g-1$ and not g. The origin of the -1 in eq. (68) is the Thomas precession (identified by setting $g=0$, as noted in footnote 6), which yields a correction to the kinetic energy of the electron. Since the g-factor of the electron is observed to be $g \simeq 2$, we see that by neglecting the Thomas precession in eq. (70), we have obtained a result that is twice as large as the correct result exhibited in eq. (68). It is sometimes said that it remarkable that neglecting a relativistic effect (i.e., the Thomas precession) results in a factor of two error. However, the existence of a spin orbit coupling is of $\mathcal{O}(v/c)$, so that this factor of two discrepancy refers to the relative weights of two relativistic effects, one of which has not been properly accounted for in obtaining eq. (70).

References

1. An illuminating treatment of angular momentum in the theory of relativity (including intrinsic spin) can be found in Chapter 15 of Andrew M. Steane, *Relativity Made Relatively Easy* (Oxford University Press, Oxford, UK, 2012) pp. 341–354.

2. A very nice presentation of the Thomas precession similar to the one given in these notes can be found in Anupam Garg, *Classical Electromagnetism in a Nutshell* (Princeton University Press, Princeton, NJ, 2012) pp. 576–580.

3. The role of the observer is frequently obscured, either by writing equations in a coordinate system implicitly pertaining to some specific observer or by entangling the invariance and the observer dependence of physical quantities. For example, this confusion often arises in the treatment of Thomas precession. These issues are clarified by a number of examples in relativistic kinematics and classical electrodynamics in a paper by Bruno Klajn and Ivica Smolić, *Subtleties of Invariance, Covariance and Observer Independence*, European Journal of Physics **34** (2013) 887–899.

4. The derivation of the BMT equation presented in Section 4 of these notes is inspired by Krzysztof Rębilas, American Journal of Physics **79** (2011) 1064–1067.

5. The treatment of the dynamics of charged particles with spin that takes into account the gradient forces (due to the intrinsic magnetic moment) can be found in A.O. Barut, *Electrodynamics and Classical Theory of Fields and Particles* (Dover Publications, Inc., New York, NY, 1980). In particular, see pp. 73–80.