DUE: TUESDAY, JANUARY 24, 2023

1. The energy and the linear momentum of a distribution of electromagnetic fields in vacuum is given (in SI units) by

$$U = \frac{\epsilon_0}{2}\int d^3x\,(\vec{\boldsymbol{E}}^{\,2} + c^2\vec{\boldsymbol{B}}^{\,2})\,, \qquad \vec{\boldsymbol{P}} = \epsilon_0\int d^3x\,\vec{\boldsymbol{E}}\times\vec{\boldsymbol{B}}\,,$$

where the integration is over all space. Consider an expansion of the electric field in terms of plane waves:

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}},t) = \sum_\lambda \int \frac{d^3k}{(2\pi)^3}\left[E_0(\vec{\boldsymbol{k}},\lambda)\,\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\,e^{i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}-\omega t)} + \text{c.c.}\right]\,,$$

where $E_0(\vec{\boldsymbol{k}},\lambda)$ is a complex amplitude and c.c. stands for "complex conjugate" of the preceding term. The polarization vector satisfies: $\hat{\boldsymbol{\epsilon}}_\lambda(-\vec{\boldsymbol{k}}) = \hat{\boldsymbol{\epsilon}}^*_\lambda(\vec{\boldsymbol{k}})$.

(a) Show that $\vec{\boldsymbol{P}}$ can be written as

$$\vec{\boldsymbol{P}} = \frac{2\epsilon_0}{c}\sum_\lambda\int\frac{d^3k}{(2\pi)^3}\,|E_0(\vec{\boldsymbol{k}},\lambda)|^2\,\hat{\boldsymbol{k}}\,.$$

Note that all time dependence has canceled out. Explain.

(b) Obtain the corresponding expression for the total energy U. Employing the photon interpretation for each mode $(\vec{\boldsymbol{k}},\lambda)$ of the electromagnetic field, justify the statement that photons are massless.

2. Jackson, problem 7.27

3. Jackson, problem 8.4

HINT: You will need the few zeros of the Bessel function and its derivative. See e.g., *Handbook of Mathematical Functions*, by Milton Abramowitz and Irene A. Stegun or Wolfram MathWorld (`http://mathworld.wolfram.com/BesselFunctionZeros.html`).

4. Jackson, problem 8.5

5. Jackson, problem 8.6

HINT: To complete part (b), you will need to know the conductivity σ of copper. For copper at room temperature, Jackson provides the value $\sigma^{-1} = 1.68\times 10^{-8}\ \Omega\cdot\text{m}$ on p. 220 below eq. (5.165), which implies that $\sigma = 5.96\times 10^7$ siemens/m, where the SI unit *siemens* is equivalent to an inverse-ohm [cf. Table 4 on p. 783 of Jackson].