1. The energy and the linear momentum of a distribution of electromagnetic fields in vacuum is given (in SI units) by

$$U = \frac{\epsilon_0}{2}\int d^3x\,(\vec{\boldsymbol{E}}^{\,2} + c^2\vec{\boldsymbol{B}}^{\,2})\,, \tag{1}$$

$$\vec{\boldsymbol{P}} = \epsilon_0\int d^3x\,\vec{\boldsymbol{E}}\times\vec{\boldsymbol{B}}\,, \tag{2}$$

where the integration is over all space. Consider an expansion of the electric field in terms of plane waves:

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}},t) = \sum_\lambda\int\frac{d^3k}{(2\pi)^3}\left[E_0(\vec{\boldsymbol{k}},\lambda)\,\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\,e^{i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}-\omega t)} + \text{c.c.}\right]\,, \tag{3}$$

where $E_0(\vec{\boldsymbol{k}},\lambda)$ is a complex amplitude and c.c. stands for "complex conjugate" of the preceding term. The polarization vector satisfies:

$$\hat{\boldsymbol{\epsilon}}_\lambda(-\vec{\boldsymbol{k}}) = \hat{\boldsymbol{\epsilon}}^*_\lambda(\vec{\boldsymbol{k}})\,. \tag{4}$$

(a) Show that $\vec{\boldsymbol{P}}$ can be written as

$$\vec{\boldsymbol{P}} = \frac{2\epsilon_0}{c}\sum_\lambda\int\frac{d^3k}{(2\pi)^3}\,|E_0(\vec{\boldsymbol{k}},\lambda)|^2\,\hat{\boldsymbol{k}}\,. \tag{5}$$

Note that all time dependence has canceled out. Explain.

Consider the Coulomb gauge, where $\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{A}} = 0$ [cf. eq. (6.21) of Jackson]. In the absence of external sources ($\rho = \vec{\boldsymbol{J}} = 0$), we also have $\Phi = 0$ [cf. eq. (6.23) of Jackson]. Using eq. (6.9) of Jackson, the electric and magnetic fields are given by,

$$\vec{\boldsymbol{E}} = -\frac{\partial\vec{\boldsymbol{A}}}{\partial t}\,, \qquad \vec{\boldsymbol{B}} = \vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{A}}\,. \tag{6}$$

If we write

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{x}},t) = \int\frac{d^3k}{(2\pi)^3}\left[\vec{\boldsymbol{a}}(\vec{\boldsymbol{k}})\,e^{i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}-\omega t)} + \vec{\boldsymbol{a}}^*(\vec{\boldsymbol{k}})\,e^{-i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}-\omega t)}\right]\,,$$

where $\omega = kc$ and

$$\vec{\boldsymbol{a}}(\vec{\boldsymbol{k}}) = \sum_\lambda a_\lambda(\vec{\boldsymbol{k}})\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\,,$$

then the vector amplitudes in the plane wave expansion of the electric and magnetic fields, obtained from eq. (6), are given by:

$$\vec{\boldsymbol{E}}_0(\vec{\boldsymbol{k}}) = ikc\,\vec{\boldsymbol{a}}(\vec{\boldsymbol{k}})\,, \qquad \vec{\boldsymbol{B}}_0(\vec{\boldsymbol{k}}) = i\vec{\boldsymbol{k}}\times\vec{\boldsymbol{a}}(\vec{\boldsymbol{k}}) = \frac{1}{c}\,\hat{\boldsymbol{k}}\times\vec{\boldsymbol{E}}_0(\vec{\boldsymbol{k}})\,, \tag{7}$$

where

$$\vec{\boldsymbol{E}}_0(\vec{\boldsymbol{k}}) = \sum_\lambda E_0(\vec{\boldsymbol{k}},\lambda)\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\,.$$

That is,

$$\vec{\boldsymbol{B}}(\vec{\boldsymbol{x}},t) = \frac{1}{c}\sum_\lambda \int \frac{d^3k}{(2\pi)^3}\left[E_0(\vec{\boldsymbol{k}},\lambda)\,\hat{\boldsymbol{k}}\times\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\,e^{i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}-\omega t)}+\text{c.c.}\right]. \tag{8}$$

Inserting eqs. (3) and (8) into eq. (2) [taking care to employ different dummy variables in the sums and integrals], and expanding out the resulting expression,

$$\begin{aligned}\vec{\boldsymbol{P}} = \frac{\epsilon_0}{(2\pi)^6\,c}\sum_\lambda\sum_{\lambda'}\int d^3k\,d^3k'\,d^3x\Big\{& E_0(\vec{\boldsymbol{k}},\lambda)E_0(\vec{\boldsymbol{k}}',\lambda')\,\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\times[\hat{\boldsymbol{k}}'\times\hat{\boldsymbol{\epsilon}}_{\lambda'}(\vec{\boldsymbol{k}}')]e^{i(\vec{\boldsymbol{k}}+\vec{\boldsymbol{k}}')\cdot\vec{\boldsymbol{x}}}\,e^{-i(\omega+\omega')t}\\ &+E_0^*(\vec{\boldsymbol{k}},\lambda)E_0^*(\vec{\boldsymbol{k}}',\lambda')\,\hat{\boldsymbol{\epsilon}}^*_\lambda(\vec{\boldsymbol{k}})\times[\hat{\boldsymbol{k}}'\times\hat{\boldsymbol{\epsilon}}^*_{\lambda'}(\vec{\boldsymbol{k}}')]e^{-i(\vec{\boldsymbol{k}}+\vec{\boldsymbol{k}}')\cdot\vec{\boldsymbol{x}}}\,e^{i(\omega+\omega')t}\\ &+E_0(\vec{\boldsymbol{k}},\lambda)E_0^*(\vec{\boldsymbol{k}}',\lambda')\,\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\times[\hat{\boldsymbol{k}}'\times\hat{\boldsymbol{\epsilon}}^*_{\lambda'}(\vec{\boldsymbol{k}}')]e^{-i(\vec{\boldsymbol{k}}-\vec{\boldsymbol{k}}')\cdot\vec{\boldsymbol{x}}}\,e^{-i(\omega-\omega')t}\\ &+E_0^*(\vec{\boldsymbol{k}},\lambda)E_0(\vec{\boldsymbol{k}}',\lambda')\,\hat{\boldsymbol{\epsilon}}^*_\lambda(\vec{\boldsymbol{k}})\times[\hat{\boldsymbol{k}}'\times\hat{\boldsymbol{\epsilon}}_{\lambda'}(\vec{\boldsymbol{k}}')]e^{-i(\vec{\boldsymbol{k}}-\vec{\boldsymbol{k}}')\cdot\vec{\boldsymbol{x}}}\,e^{i(\omega-\omega')t}\Big\}\,,\end{aligned} \tag{9}$$

where $\omega\equiv kc$ and $\omega'\equiv k'c$. In our notation, $k\equiv|\vec{\boldsymbol{k}}|$ and $k'\equiv|\vec{\boldsymbol{k}}'|$.

We may now perform the integral over $\vec{\boldsymbol{x}}$, using

$$\frac{1}{(2\pi)^3}\int d^3x\,e^{i(\vec{\boldsymbol{k}}\pm\vec{\boldsymbol{k}}')\cdot\vec{\boldsymbol{x}}} = \delta^3(\vec{\boldsymbol{k}}\pm\vec{\boldsymbol{k}}')\,, \tag{10}$$

and then use the delta function to facilitate the integration over $\vec{\boldsymbol{k}}'$. Then eq. (9) reduces to

$$\begin{aligned}\vec{\boldsymbol{P}} = \frac{\epsilon_0}{c}\sum_\lambda\sum_{\lambda'}\int\frac{d^3k}{(2\pi)^3}\Big\{&-E_0(\vec{\boldsymbol{k}},\lambda)E_0(-\vec{\boldsymbol{k}},\lambda')\,\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\times[\hat{\boldsymbol{k}}\times\hat{\boldsymbol{\epsilon}}^*_{\lambda'}(\vec{\boldsymbol{k}})]e^{-2i\omega t}\\ &-E_0^*(\vec{\boldsymbol{k}},\lambda)E_0^*(-\vec{\boldsymbol{k}},\lambda')\,\hat{\boldsymbol{\epsilon}}^*_\lambda(\vec{\boldsymbol{k}})\times[\hat{\boldsymbol{k}}\times\hat{\boldsymbol{\epsilon}}_{\lambda'}(\vec{\boldsymbol{k}})]e^{2i\omega t}\\ &+E_0(\vec{\boldsymbol{k}},\lambda)E_0^*(\vec{\boldsymbol{k}},\lambda')\,\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\times[\hat{\boldsymbol{k}}\times\hat{\boldsymbol{\epsilon}}^*_{\lambda'}(\vec{\boldsymbol{k}})]\\ &+E_0^*(\vec{\boldsymbol{k}},\lambda)E_0(\vec{\boldsymbol{k}},\lambda')\,\hat{\boldsymbol{\epsilon}}^*_\lambda(\vec{\boldsymbol{k}})\times[\hat{\boldsymbol{k}}\times\hat{\boldsymbol{\epsilon}}_{\lambda'}(\vec{\boldsymbol{k}})]\Big\}\,,\end{aligned} \tag{11}$$

where we have used eq. (4) to write:[1]

$$\hat{\boldsymbol{\epsilon}}_{\lambda'}(\vec{\boldsymbol{k}}')\delta^3(\vec{\boldsymbol{k}}+\vec{\boldsymbol{k}}') = \hat{\boldsymbol{\epsilon}}_{\lambda'}(-\vec{\boldsymbol{k}})\delta^3(\vec{\boldsymbol{k}}+\vec{\boldsymbol{k}}') = \hat{\boldsymbol{\epsilon}}^*_{\lambda'}(\vec{\boldsymbol{k}})\delta^3(\vec{\boldsymbol{k}}+\vec{\boldsymbol{k}}')\,. \tag{12}$$

We can now make use of the vector identity,

$$\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\times[\hat{\boldsymbol{k}}\times\hat{\boldsymbol{\epsilon}}^*_{\lambda'}(\vec{\boldsymbol{k}})] = \hat{\boldsymbol{k}}[\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\cdot\hat{\boldsymbol{\epsilon}}^*_{\lambda'}(\vec{\boldsymbol{k}})] - \hat{\boldsymbol{\epsilon}}^*_{\lambda'}(\vec{\boldsymbol{k}})[\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})] = \hat{\boldsymbol{k}}\,\delta_{\lambda\lambda'}\,, \tag{13}$$

[1] Recall that for any well-behaved function $f(\vec{\boldsymbol{k}},\vec{\boldsymbol{k}}')$ we have $f(\vec{\boldsymbol{k}},\vec{\boldsymbol{k}}')\delta^3(\vec{\boldsymbol{k}}\pm\vec{\boldsymbol{k}}') = f(\vec{\boldsymbol{k}},\pm\vec{\boldsymbol{k}})\delta^3(\vec{\boldsymbol{k}}\pm\vec{\boldsymbol{k}}')$, due to the presence of the delta function. For example, $\omega'\delta^3(\vec{\boldsymbol{k}}\pm\vec{\boldsymbol{k}}') = k'c\,\delta^3(\vec{\boldsymbol{k}}\pm\vec{\boldsymbol{k}}') = kc\,\delta^3(\vec{\boldsymbol{k}}\pm\vec{\boldsymbol{k}}') = \omega\delta^3(\vec{\boldsymbol{k}}\pm\vec{\boldsymbol{k}}')$, since $|\pm\vec{\boldsymbol{k}}| = k$.

while employing the properties of the polarization vector,

$$\hat{\boldsymbol{k}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}}) = 0\,, \qquad \text{and} \qquad \hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}^*_{\lambda'}(\vec{\boldsymbol{k}}) = \delta_{\lambda\lambda'}\,. \tag{14}$$

Using eq. (13) allows us to perform the sum over λ' in eq. (11), which yields

$$\vec{\boldsymbol{P}} = \frac{\epsilon_0}{c}\sum_\lambda \int \frac{d^3k}{(2\pi)^3}\,\hat{\boldsymbol{k}}\Big\{-E_0(\vec{\boldsymbol{k}},\lambda)E_0(-\vec{\boldsymbol{k}},\lambda)e^{-2i\omega t} - E_0^*(\vec{\boldsymbol{k}},\lambda)E_0^*(-\vec{\boldsymbol{k}},\lambda)e^{2i\omega t} + 2|E_0(\vec{\boldsymbol{k}},\lambda)|^2\Big\}\,. \tag{15}$$

Noting that $\omega = kc$ where $k \equiv |\vec{\boldsymbol{k}}|$ and $\hat{\boldsymbol{k}} \equiv \vec{\boldsymbol{k}}/k$, it follows that

$$\int d^3k\,\hat{\boldsymbol{k}}\,E_0(\vec{\boldsymbol{k}},\lambda)E_0(-\vec{\boldsymbol{k}},\lambda)e^{-2ikct} = 0\,,$$

since the integrand is an odd function under $\vec{\boldsymbol{k}} \to -\vec{\boldsymbol{k}}$. That is, if we denote the integrand by $f(\vec{\boldsymbol{k}}) \equiv \hat{\boldsymbol{k}}\,E_0(\vec{\boldsymbol{k}},\lambda)E_0(-\vec{\boldsymbol{k}},\lambda)e^{-2ikct}$, then $f(\vec{\boldsymbol{k}}) = -f(-\vec{\boldsymbol{k}})$. It follows that

$$\int f(\vec{\boldsymbol{k}})\,d^3k = -\int f(-\vec{\boldsymbol{k}})\,d^3k = -\int f(\vec{\boldsymbol{k}})\,d^3k = 0\,, \tag{16}$$

after making a change of integration variables $\vec{\boldsymbol{k}} \to -\vec{\boldsymbol{k}}$ and noting that the absolute value of the determinant of the corresponding Jacobian matrix is one. In the final step above, we used the fact that a quantity that is equal to its negative must be zero. Hence, eq. (15) yields

$$\vec{\boldsymbol{P}} = \frac{2\epsilon_0}{c}\sum_\lambda \int \frac{d^3k}{(2\pi)^3}\,\hat{\boldsymbol{k}}\,|E_0(\vec{\boldsymbol{k}},\lambda)|^2\,, \tag{17}$$

which confirms the result of eq. (5).

Note that $\vec{\boldsymbol{P}}$ given in eq. (17) is explicitly time-independent. This is simply an expression of the conservation of momentum, $d\vec{\boldsymbol{P}}/dt = 0$. This is a consequence of eq. (6.122) of Jackson. Since $\rho = \vec{\boldsymbol{J}} = 0$ for a free electromagnetic field, we have $\vec{\boldsymbol{P}}_{\rm mech} = 0$, in which case

$$\frac{d\vec{\boldsymbol{P}}}{dt} = \frac{\vec{\boldsymbol{P}}_{\rm field}}{dt} = \oint_S da\,\hat{\boldsymbol{n}}\boldsymbol{\cdot}\overset{\leftrightarrow}{\boldsymbol{T}} = 0\,,$$

where $\overset{\leftrightarrow}{\boldsymbol{T}}$ is the Maxwell stress tensor. The unit vector $\hat{\boldsymbol{n}}$ is the outward normal to the surface S, where S is the surface of infinity. For any finite energy field configuration, the stress tensor vanishes at the surface of infinity and we recover $d\vec{\boldsymbol{P}}/dt = 0$ as expected.

(b) Obtain the corresponding expression for the total energy U. Employing the photon interpretation for each mode $(\vec{\boldsymbol{k}},\lambda)$ of the electromagnetic field, justify the statement that photons are massless.

The total energy is given (in SI units) by

$$U = \frac{\epsilon_0}{2}\int d^3x\,(\vec{\boldsymbol{E}}^{\,2} + c^2\vec{\boldsymbol{B}}^{\,2})\,. \tag{18}$$

We first compute

$$\int \vec{\boldsymbol{E}}^{\,2}\, d^3x = \frac{1}{(2\pi)^6}\sum_\lambda\sum_{\lambda'}\int d^3k d^3k' d^3x \biggl\{\Bigl[E_0(\vec{\boldsymbol{k}},\lambda)E_0(\vec{\boldsymbol{k}}',\lambda')\,\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_{\lambda'}(\vec{\boldsymbol{k}}')\,e^{i(\vec{\boldsymbol{k}}+\vec{\boldsymbol{k}}')\boldsymbol{\cdot}\vec{\boldsymbol{x}}}\,e^{-i(\omega+\omega')t}+\text{c.c.}\Bigr]$$
$$+\Bigl[E_0(\vec{\boldsymbol{k}},\lambda)E_0^*(\vec{\boldsymbol{k}}',\lambda')\,\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}^*_{\lambda'}(\vec{\boldsymbol{k}}')\,e^{i(\vec{\boldsymbol{k}}-\vec{\boldsymbol{k}}')\boldsymbol{\cdot}\vec{\boldsymbol{x}}}\,e^{-i(\omega-\omega')t}+\text{c.c.}\Bigr]\biggr\},$$

where the computation is similar to that of part (a). Integrating over $\vec{\boldsymbol{x}}$ and using eq. (12) as we did in part (a), it follows that

$$\int \vec{\boldsymbol{E}}^{\,2}\, d^3x = \sum_\lambda\sum_{\lambda'}\int\frac{d^3k}{(2\pi)^3}\biggl\{E_0(\vec{\boldsymbol{k}},\lambda)E_0(\vec{\boldsymbol{k}},\lambda')\,\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_{\lambda'}(\vec{\boldsymbol{k}})\,e^{-2i\omega t}+E_0(\vec{\boldsymbol{k}},\lambda)E_0^*(\vec{\boldsymbol{k}},\lambda')\,\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}^*_{\lambda'}(\vec{\boldsymbol{k}})+\text{c.c.}\biggr\}.$$

Summing over λ' using eq. (14), we obtain

$$\int \vec{\boldsymbol{E}}^{\,2}\, d^3x = \sum_\lambda\int\frac{d^3k}{(2\pi)^3}\Bigl[E_0(\vec{\boldsymbol{k}},\lambda)E_0(\vec{\boldsymbol{k}},\lambda)\,e^{-2i\omega t}+\text{c.c.}\Bigr]+2\sum_\lambda\int\frac{d^3k}{(2\pi)^3}|E_0(\vec{\boldsymbol{k}},\lambda)|^2\,. \tag{19}$$

Next, we compute $\int c^2\vec{\boldsymbol{B}}^{\,2}d^3x$. The only difference in the computation compared to the one above is that $\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})$ is replaced by $\hat{\boldsymbol{k}}\times\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})$ and $\hat{\boldsymbol{\epsilon}}_{\lambda'}(\vec{\boldsymbol{k}}')$ is replaced by $\hat{\boldsymbol{k}}'\times\hat{\boldsymbol{\epsilon}}_{\lambda'}(\vec{\boldsymbol{k}}')$. Thus, instead of obtaining the factor $\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_{\lambda'}(\vec{\boldsymbol{k}}')\,\delta^3(\vec{\boldsymbol{k}}+\vec{\boldsymbol{k}}')$ after the integration over $\vec{\boldsymbol{x}}$, we now have [cf. footnote 1]:

$$\begin{aligned}[\hat{\boldsymbol{k}}\times\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})]\boldsymbol{\cdot}[\hat{\boldsymbol{k}}'\times\hat{\boldsymbol{\epsilon}}_{\lambda'}(\vec{\boldsymbol{k}}')]\,\delta^3(\vec{\boldsymbol{k}}+\vec{\boldsymbol{k}}') &= [\hat{\boldsymbol{k}}\boldsymbol{\cdot}\hat{\boldsymbol{k}}'][\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_{\lambda'}(\vec{\boldsymbol{k}}')]-[\hat{\boldsymbol{k}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_{\lambda'}(\vec{\boldsymbol{k}}')][\hat{\boldsymbol{k}}'\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})]\,\delta^3(\vec{\boldsymbol{k}}+\vec{\boldsymbol{k}}')\\ &= \bigl\{-\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_{\lambda'}(-\vec{\boldsymbol{k}})+[\hat{\boldsymbol{k}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_{\lambda'}(-\vec{\boldsymbol{k}})][\hat{\boldsymbol{k}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})]\bigr\}\delta^3(\vec{\boldsymbol{k}}+\vec{\boldsymbol{k}}')\\ &= -\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}^*_{\lambda'}(\vec{\boldsymbol{k}})\,\delta^3(\vec{\boldsymbol{k}}+\vec{\boldsymbol{k}}')\,,\end{aligned}$$

after using eqs. (12) and (14). Similarly, instead of obtaining the factor $\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_{\lambda'}(\vec{\boldsymbol{k}}')\,\delta^3(\vec{\boldsymbol{k}}-\vec{\boldsymbol{k}}')$ after the integration over $\vec{\boldsymbol{x}}$, we now have:

$$[\hat{\boldsymbol{k}}\times\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})]\boldsymbol{\cdot}[\hat{\boldsymbol{k}}'\times\hat{\boldsymbol{\epsilon}}_{\lambda'}(\vec{\boldsymbol{k}}')]\,\delta^3(\vec{\boldsymbol{k}}-\vec{\boldsymbol{k}}') = \hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}^*_{\lambda'}(\vec{\boldsymbol{k}})\,\delta^3(\vec{\boldsymbol{k}}-\vec{\boldsymbol{k}}')\,.$$

Hence, it follows that:

$$\int c^2\vec{\boldsymbol{B}}^{\,2}\, d^3x = -\sum_\lambda\int\frac{d^3k}{(2\pi)^3}\Bigl[E_0(\vec{\boldsymbol{k}},\lambda)E_0(\vec{\boldsymbol{k}},\lambda)\,e^{-2i\omega t}+\text{c.c.}\Bigr]+2\sum_\lambda\int\frac{d^3k}{(2\pi)^3}|E_0(\vec{\boldsymbol{k}},\lambda)|^2\,. \tag{20}$$

Adding eqs. (19) and (20) yields

$$U = 2\epsilon_0\sum_\lambda\int\frac{d^3k}{(2\pi)^3}|E_0(\vec{\boldsymbol{k}},\lambda)|^2\,. \tag{21}$$

Note that U given in eq. (21) is explicitly time-independent. This is simply an expression of the conservation of momentum, $dU/dt=0$. This is a consequence of eq. (6.111) of Jackson. Since $\rho=\vec{\boldsymbol{J}}=0$ for a free electromagnetic field, we have $\vec{\boldsymbol{P}}_{\rm mech}=0$, in which case

$$\frac{dU}{dt}=\frac{U_{\rm field}}{dt}=-\oint_S da\,\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{S}}=0\,,$$

where $\vec{\boldsymbol{S}}$ is the Poynting vector. For any finite energy field configuration, the Poynting vector vanishes at the surface of infinity and we recover $dU/dt = 0$ as expected.

Finally, consider a fixed wave number vector $\vec{\boldsymbol{k}}_0$, for which $E_0(\vec{\boldsymbol{k}}, \lambda) \equiv E_0(\lambda)\, \delta^3(\vec{\boldsymbol{k}} - \vec{\boldsymbol{k}}_0)$. Then, eqs. (17) and (21) yield

$$U = 2\epsilon_0 \sum_\lambda |E_0(\lambda)|^2 \,, \qquad\qquad \vec{\boldsymbol{P}} = \hat{\boldsymbol{k}}_0 \frac{2\epsilon_0}{c} \sum_\lambda |E_0(\lambda)|^2 = \frac{\hat{\boldsymbol{k}}_0}{c}\, U \,.$$

That is, $U = Pc$. Comparing this result to the relativistic relation between the energy and momentum of a particle, $E = \sqrt{p^2c^2 + m^2c^4}$, we conclude that photons are massless.

2. [Jackson, problem 7.27] The angular momentum of a distribution of electromagnetic fields in vacuum (in SI units) is given by

$$\vec{\boldsymbol{L}} = \frac{1}{\mu_0 c^2} \int d^3x\, \vec{\boldsymbol{x}} \times (\vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}}) \,, \tag{22}$$

where the integration is over all space.

(a) For fields produced a finite time in the past (and so localized to a finite region of space) show that, provided the magnetic field is eliminated in favor of the vector potential $\vec{\boldsymbol{A}}$, the angular momentum can be written in the form

$$\vec{\boldsymbol{L}} = \frac{1}{\mu_0 c^2} \int d^3x \left[\vec{\boldsymbol{E}} \times \vec{\boldsymbol{A}} + \sum_{\ell=1}^{3} E_\ell (\vec{\boldsymbol{x}} \times \vec{\boldsymbol{\nabla}}) A_\ell \right] . \tag{23}$$

The first term above is sometimes identified with the "spin" of the photon and the second with the "orbital" angular momentum because of the presence of the angular momentum operator $\vec{\boldsymbol{L}}_{\rm op} = -i(\vec{\boldsymbol{x}} \times \vec{\boldsymbol{\nabla}})$.

The magnetic field can be written in terms of the vector potential, $\vec{\boldsymbol{B}} = \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}$. Hence, we need to evaluate $\vec{\boldsymbol{x}} \times [\vec{\boldsymbol{E}} \times (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}})]$. Using the Einstein summation convention, where there is an implicit summation over a pair of identical indices, we can write $(\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}})_i = \epsilon_{ijk} a_j b_k$, where the indices take on the values $i, j, k = 1, 2, 3$ and there is an implicit sum over j and k. The Levi-Civita tensor is defined as

$$\epsilon_{ijk} = \begin{cases} +1 \,, & \text{if } (i,j,k) \text{ is an even permutation of } (1,2,3), \\ -1 \,, & \text{if } (i,j,k) \text{ is an odd permutation of } (1,2,3), \\ \ \ 0 \,, & \text{otherwise.} \end{cases}$$

Thus, it follows that

$$\big\{\vec{\boldsymbol{x}} \times [\vec{\boldsymbol{E}} \times (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}})]\big\}_i = \epsilon_{ijk} x_j [\vec{\boldsymbol{E}} \times (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}})]_k = \epsilon_{ijk} x_j \epsilon_{k\ell m} E_\ell (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}})_m = \epsilon_{ijk} x_j \epsilon_{k\ell m} E_\ell \epsilon_{mpq} \nabla_p A_q \,,$$

where $\vec{\boldsymbol{x}} \equiv (x_1, x_2, x_3)$ and $\nabla_p \equiv \partial/\partial x_p$. We now employ the following ϵ-identity,

$$\epsilon_{k\ell m}\epsilon_{mpq} = \delta_{kp}\delta_{\ell q} - \delta_{kq}\delta_{\ell p}\,.$$

Hence, it follows that

$$\big\{\vec{\boldsymbol{x}}\times[\vec{\boldsymbol{E}}\times(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{A}})]\big\}_i = \epsilon_{ijk}x_j E_\ell(\delta_{kp}\delta_{\ell q}-\delta_{kq}\delta_{\ell p})\nabla_p A_q = \epsilon_{ijk}x_j E_\ell\nabla_k A_\ell - \epsilon_{ijk}x_j E_\ell \nabla_\ell A_k\,. \quad (24)$$

We recognize $\epsilon_{ijk}x_j E_\ell\nabla_k A_\ell = E_\ell(\vec{\boldsymbol{x}}\times\vec{\boldsymbol{\nabla}})_i A_\ell$ which corresponds to the second term in eq. (23). To obtain the first term in eq. (23) will require an integration by parts. That is, we first write:

$$\epsilon_{ijk}x_j E_\ell\nabla_\ell A_k = \epsilon_{ijk}\left[\nabla_\ell(x_j E_\ell A_k) - A_k\nabla_\ell(x_j E_k)\right]\,,$$

which is an identity that follows from the rule for differentiating products. Next, we note that

$$\epsilon_{ijk}A_k\nabla_\ell(x_j E_\ell) = \epsilon_{ijk}A_k\left[x_j(\nabla_\ell E_\ell) + E_\ell(\nabla_\ell x_j)\right] = \epsilon_{ijk}A_k E_\ell\delta_{\ell j} = \epsilon_{ijk}A_k E_j = (\vec{\boldsymbol{E}}\times\vec{\boldsymbol{A}})_i\,,$$

where we used $\nabla_\ell x_j \equiv \partial x_j/\partial x_\ell = \delta_{\ell j}$ and $\nabla_\ell E_\ell = \vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{E}} = 0$ (in vacuum). Thus, eq. (24) yields the vector identity,

$$\big\{\vec{\boldsymbol{x}}\times[\vec{\boldsymbol{E}}\times(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{A}})]\big\}_i = E_\ell(\vec{\boldsymbol{x}}\times\vec{\boldsymbol{\nabla}})_i A_\ell + (\vec{\boldsymbol{E}}\times\vec{\boldsymbol{A}})_i - \epsilon_{ijk}\nabla_\ell(x_j E_\ell A_k)\,, \quad (25)$$

where there is an implicit sum over the repeated index ℓ. An alternative proof of eq. (25) is given at the end of the solution to part (a) of this problem [see eqs. (28)–(31)].

When we integrate over all of space, we can use the divergence theorem [given in the inside cover of Jackson's textbook]:

$$\int_V d^3x\,\epsilon_{ijk}\nabla_\ell(x_j E_\ell A_k) = \oint_S da\,\epsilon_{ijk}n_\ell x_j E_\ell A_k = \oint_S da\,\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{E}}\,(\vec{\boldsymbol{x}}\times\vec{\boldsymbol{A}})_i = 0\,, \quad (26)$$

where n_ℓ is the outward normal at the surface of infinity S. Since the fields are assumed to be localized to a finite region of space, the integral above vanishes. Hence, inserting the results of eqs. (25) and (26) into eq. (22) [after putting $\vec{\boldsymbol{B}} = \vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{A}}$] immediately yields

$$\int d^3x\,\vec{\boldsymbol{x}}\times(\vec{\boldsymbol{E}}\times\vec{\boldsymbol{B}}) = \int d^3x\left[\vec{\boldsymbol{E}}\times\vec{\boldsymbol{A}} + \sum_{\ell=1}^{3}E_\ell(\vec{\boldsymbol{x}}\times\vec{\boldsymbol{\nabla}})A_\ell\right]\,.$$

Therefore, eq. (23) is proven.

<u>*REMARK:*</u> The identification of

$$\vec{\boldsymbol{L}}_{\rm spin} = \frac{1}{\mu_0 c^2}\int d^3x\,\vec{\boldsymbol{E}}\times\vec{\boldsymbol{A}}\,, \quad (27)$$

as the spin angular momentum is problematical, as eq. (27) is not invariant under gauge transformations. In fact, a gauge-invariant expression for the spin angular momentum can be constructed that reduces to eq. (27) in the radiation (Coulomb) gauge.[2]

[2] See e.g., Iwo Bialynicki-Birula and Zofia Bialynicki-Birula, Journal of Optics **13**, 064014 (2011) and references therein.

Vector identities revisited

Using the well-known vector identity, $\vec{\boldsymbol{A}}\times(\vec{\boldsymbol{B}}\times\vec{\boldsymbol{C}})=\vec{\boldsymbol{B}}(\vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{C}})-\vec{\boldsymbol{C}}(\vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{B}})$, it follows that

$$\vec{\boldsymbol{E}}\times(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{A}})=E_i\vec{\boldsymbol{\nabla}}A_i-(\vec{\boldsymbol{E}}\cdot\vec{\boldsymbol{\nabla}})\vec{\boldsymbol{A}}\,, \tag{28}$$

where there is an implicit sum over i, and we have been careful with the location of the differential operator $\vec{\boldsymbol{\nabla}}$ which is only acting on the vector $\vec{\boldsymbol{A}}$. It follows that

$$\vec{\boldsymbol{x}}\times[\vec{\boldsymbol{E}}\times(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{A}})]=E_i(\vec{\boldsymbol{x}}\times\vec{\boldsymbol{\nabla}})A_i-\vec{\boldsymbol{x}}\times(\vec{\boldsymbol{E}}\cdot\vec{\boldsymbol{\nabla}})\vec{\boldsymbol{A}}\,. \tag{29}$$

Next, we observe that summing over the repeated index i yields,

$$\begin{aligned}\nabla_i(E_i\,\vec{\boldsymbol{x}}\times\vec{\boldsymbol{A}})&=(\vec{\boldsymbol{x}}\times\vec{\boldsymbol{A}})(\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{E}})+\vec{\boldsymbol{E}}\cdot\vec{\boldsymbol{\nabla}}(\vec{\boldsymbol{x}}\times\vec{\boldsymbol{A}})\\&=\vec{\boldsymbol{E}}\cdot\vec{\boldsymbol{\nabla}}(\vec{\boldsymbol{x}}\times\vec{\boldsymbol{A}})\\&=E_i\nabla_i(\epsilon_{jk\ell}x_jA_k)=E_i\epsilon_{jk\ell}(\delta_{ij}A_k+x_j\nabla_iA_k)\\&=\vec{\boldsymbol{E}}\times\vec{\boldsymbol{A}}+\vec{\boldsymbol{x}}\times(\vec{\boldsymbol{E}}\cdot\vec{\boldsymbol{\nabla}})\vec{\boldsymbol{A}}\,,\end{aligned} \tag{30}$$

after using $\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{E}}=0$ (in vacuum) and $\nabla_i x_j=\delta_{ij}$. Combining eqs. (29) and (30) yields

$$\vec{\boldsymbol{x}}\times[\vec{\boldsymbol{E}}\times(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{A}})]=E_i(\vec{\boldsymbol{x}}\times\vec{\boldsymbol{\nabla}})A_i+\vec{\boldsymbol{E}}\times\vec{\boldsymbol{A}}-\nabla_i(E_i\,\vec{\boldsymbol{x}}\times\vec{\boldsymbol{A}})\,, \tag{31}$$

which coincides with eq. (25).

(b) Consider an expansion of the vector potential in the radiation (Coulomb) gauge in terms of plane waves,

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{x}},t)=\sum_\lambda\int\frac{d^3k}{(2\pi)^3}\left[\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})a_\lambda(\vec{\boldsymbol{k}})e^{i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}-i\omega t)}+\text{c.c.}\right]\,. \tag{32}$$

The vectors $\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})$ are conveniently chosen as the positive and negative helicity polarization vectors[3]

$$\hat{\boldsymbol{\epsilon}}_\pm=\mp\frac{1}{\sqrt{2}}\left(\hat{\boldsymbol{\epsilon}}_1\pm i\hat{\boldsymbol{\epsilon}}_2\right)\,, \tag{33}$$

where $\hat{\boldsymbol{\epsilon}}_1$ and $\hat{\boldsymbol{\epsilon}}_2$ are the real orthogonal vectors in the plane whose positive normal is in the direction of $\vec{\boldsymbol{k}}$.

Show that the time average of the first (spin) term of $\vec{\boldsymbol{L}}$ can be written as

$$\vec{\boldsymbol{L}}_{\text{spin}}=\frac{2}{\mu_0c}\int\frac{d^3k}{(2\pi)^3}\vec{\boldsymbol{k}}\left[|a_+(\vec{\boldsymbol{k}})|^2-|a_-(\vec{\boldsymbol{k}})|^2\right]\,.$$

Can the term "spin" angular momentum be justified from this expression? Calculate the energy of the field in terms of the plane wave expansion of $\vec{\boldsymbol{A}}$ and compare.

[3] Jackson omits the overall factor of $\mp$ in the definition of $\hat{\boldsymbol{\epsilon}}_\pm$. I prefer to maintain this phase convention, but you are free to choose any convention that suits you.

In the Coulomb gauge, the electric field is (in SI units):

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}},t) = -\frac{\partial \vec{\boldsymbol{A}}}{\partial t} = i\sum_\lambda \int \frac{d^3k}{(2\pi)^3}\,\omega\,\left[\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})a_\lambda(\vec{\boldsymbol{k}})e^{i(\vec{\boldsymbol{k}}\boldsymbol{\cdot}\vec{\boldsymbol{x}}-i\omega t)} - \text{c.c.}\right], \tag{34}$$

where $\omega = ck$ and $k \equiv |\vec{\boldsymbol{k}}|$. Note that due to the overall factor of i, we must subtract the complex conjugate inside the square brackets in order to ensure that $\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}},t)$ is a real field. Inserting eqs. (32) and (34) into eq. (27) and expanding out the integrand, we obtain:

$$\begin{aligned}\vec{\boldsymbol{L}}_{\rm spin} = \frac{1}{\mu_0 c^2}\frac{i}{(2\pi)^6}\sum_\lambda\sum_{\lambda'}\int \omega\, d^3k\, d^3k'\, d^3x\Big\{&[\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\times\hat{\boldsymbol{\epsilon}}_{\lambda'}(\vec{\boldsymbol{k}}')]a_\lambda(\vec{\boldsymbol{k}})a_{\lambda'}(\vec{\boldsymbol{k}}')e^{i(\vec{\boldsymbol{k}}+\vec{\boldsymbol{k}}')\boldsymbol{\cdot}\vec{\boldsymbol{x}}}\,e^{-i(\omega+\omega')t}\\
&+[\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\times\hat{\boldsymbol{\epsilon}}^*_{\lambda'}(\vec{\boldsymbol{k}}')]a_\lambda(\vec{\boldsymbol{k}})a^*_{\lambda'}(\vec{\boldsymbol{k}}')e^{i(\vec{\boldsymbol{k}}-\vec{\boldsymbol{k}}')\boldsymbol{\cdot}\vec{\boldsymbol{x}}}\,e^{-i(\omega-\omega')t}\\
&-[\hat{\boldsymbol{\epsilon}}^*_\lambda(\vec{\boldsymbol{k}})\times\hat{\boldsymbol{\epsilon}}_{\lambda'}(\vec{\boldsymbol{k}}')]a^*_\lambda(\vec{\boldsymbol{k}})a_{\lambda'}(\vec{\boldsymbol{k}}')e^{i(\vec{\boldsymbol{k}}-\vec{\boldsymbol{k}}')\boldsymbol{\cdot}\vec{\boldsymbol{x}}}\,e^{-i(\omega-\omega')t}\\
&-[\hat{\boldsymbol{\epsilon}}^*_\lambda(\vec{\boldsymbol{k}})\times\hat{\boldsymbol{\epsilon}}^*_{\lambda'}(\vec{\boldsymbol{k}}')]a^*_\lambda(\vec{\boldsymbol{k}})a^*_{\lambda'}(\vec{\boldsymbol{k}}')e^{i(\vec{\boldsymbol{k}}+\vec{\boldsymbol{k}}')\boldsymbol{\cdot}\vec{\boldsymbol{x}}}\,e^{-i(\omega+\omega')t}\Big\},\end{aligned}$$

where $\omega = kc$ and $\omega' = k'c$.

We may now perform the integral over $\vec{\boldsymbol{x}}$, using eq. (10), and then use the delta function to integrate over $\vec{\boldsymbol{k}}'$. The end result is

$$\begin{aligned}\vec{\boldsymbol{L}}_{\rm spin} = \frac{i}{\mu_0 c^2}\sum_\lambda\sum_{\lambda'}\int\frac{\omega d^3k}{(2\pi)^3}\Big\{&[\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\times\hat{\boldsymbol{\epsilon}}^*_{\lambda'}(\vec{\boldsymbol{k}})]a_\lambda(\vec{\boldsymbol{k}})a^*_{\lambda'}(\vec{\boldsymbol{k}}) - [\hat{\boldsymbol{\epsilon}}^*_\lambda(\vec{\boldsymbol{k}})\times\hat{\boldsymbol{\epsilon}}_{\lambda'}(\vec{\boldsymbol{k}})]a^*_\lambda(\vec{\boldsymbol{k}})a_{\lambda'}(\vec{\boldsymbol{k}})\\
&+[\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\times\hat{\boldsymbol{\epsilon}}_{\lambda'}(-\vec{\boldsymbol{k}})]a_\lambda(\vec{\boldsymbol{k}})a_{\lambda'}(-\vec{\boldsymbol{k}})\,e^{-2i\omega t} - [\hat{\boldsymbol{\epsilon}}^*_\lambda(\vec{\boldsymbol{k}})\times\hat{\boldsymbol{\epsilon}}^*_{\lambda'}(-\vec{\boldsymbol{k}})]a^*_\lambda(\vec{\boldsymbol{k}})a^*_{\lambda'}(-\vec{\boldsymbol{k}})\,e^{2i\omega t}\Big\}.\end{aligned} \tag{35}$$

However, the last two terms above vanish when integrated over $\vec{\boldsymbol{k}}$, since the corresponding integrands are odd functions of $\vec{\boldsymbol{k}}$. For example, under $\vec{\boldsymbol{k}}\to-\vec{\boldsymbol{k}}$,

$$\begin{aligned}\sum_\lambda\sum_{\lambda'}[\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\times\hat{\boldsymbol{\epsilon}}_{\lambda'}(-\vec{\boldsymbol{k}})]a_\lambda(\vec{\boldsymbol{k}})a_{\lambda'}(-\vec{\boldsymbol{k}})\,e^{-2i\omega t} \longrightarrow &\sum_\lambda\sum_{\lambda'}[\hat{\boldsymbol{\epsilon}}_\lambda(-\vec{\boldsymbol{k}})\times\hat{\boldsymbol{\epsilon}}_{\lambda'}(\vec{\boldsymbol{k}})]a_\lambda(-\vec{\boldsymbol{k}})a_{\lambda'}(\vec{\boldsymbol{k}})\,e^{-2i\omega t},\\
=&\sum_\lambda\sum_{\lambda'}[\hat{\boldsymbol{\epsilon}}_{\lambda'}(-\vec{\boldsymbol{k}})\times\hat{\boldsymbol{\epsilon}}_{\lambda}(\vec{\boldsymbol{k}})]a_{\lambda'}(-\vec{\boldsymbol{k}})a_{\lambda}(\vec{\boldsymbol{k}})\,e^{-2i\omega t},\\
=&-\sum_\lambda\sum_{\lambda'}[\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\times\hat{\boldsymbol{\epsilon}}_{\lambda'}(-\vec{\boldsymbol{k}})]a_\lambda(\vec{\boldsymbol{k}})a_{\lambda'}(-\vec{\boldsymbol{k}})\,e^{-2i\omega t},\end{aligned}$$

where we interchanged λ and λ' in the penultimate step (which is justified since these are dummy labels that are being summed over), and used the antisymmetry of the cross product in the final step. Note that $\omega = |\vec{\boldsymbol{k}}|c$ does not change sign when $\vec{\boldsymbol{k}}\to-\vec{\boldsymbol{k}}$. Hence, eq. (35) simplifies to

$$\vec{\boldsymbol{L}}_{\rm spin} = \frac{i}{\mu_0 c^2}\sum_\lambda\sum_{\lambda'}\int\frac{\omega d^3k}{(2\pi)^3}\Big\{[\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\times\hat{\boldsymbol{\epsilon}}^*_{\lambda'}(\vec{\boldsymbol{k}})]a_\lambda(\vec{\boldsymbol{k}})a^*_{\lambda'}(\vec{\boldsymbol{k}}) - [\hat{\boldsymbol{\epsilon}}^*_\lambda(\vec{\boldsymbol{k}})\times\hat{\boldsymbol{\epsilon}}_{\lambda'}(\vec{\boldsymbol{k}})]a^*_\lambda(\vec{\boldsymbol{k}})a_{\lambda'}(\vec{\boldsymbol{k}})\Big\}. \tag{36}$$

Using the definition of the polarization vectors given in eq. (33), it is straightforward to verify that[4]

$$\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}}) \times \hat{\boldsymbol{\epsilon}}^*_{\lambda'}(\vec{\boldsymbol{k}}) = -i\lambda\,\hat{\boldsymbol{k}}\,\delta_{\lambda\lambda'}\,, \quad \text{for } \lambda, \lambda' = \pm\,. \tag{37}$$

This result allows us to sum over λ' in eq. (36). Both terms in eq. (36) contribute equally and the end result is:

$$\vec{\boldsymbol{L}}_{\rm spin} = \frac{2}{\mu_0 c^2}\int \frac{d^3k}{(2\pi)^3}\,\vec{\boldsymbol{k}}\left\{|a_+(\vec{\boldsymbol{k}})|^2 - |a_-(\vec{\boldsymbol{k}})|^2\right\}, \tag{38}$$

after using $\omega = kc$ and $\vec{\boldsymbol{k}} = k\hat{\boldsymbol{k}}$. Note that $\vec{\boldsymbol{L}}_{\rm spin}$ is time-independent and thus conserved. This is a stronger condition than the conservation of angular momentum, which only requires that the sum $\vec{\boldsymbol{L}} = \vec{\boldsymbol{L}}_{\rm orbital} + \vec{\boldsymbol{L}}_{\rm spin}$ is conserved. Eq. (38) implies that the spin angular momentum of the electromagnetic field is *separately* a constant of the motion.[5] If we interpret each mode $(\vec{\boldsymbol{k}}, \lambda)$ as a photon, then the two possible photon spin states (in a spherical basis) correspond to positive and negative helicity, i.e. states of definite spin angular momentum in which $\vec{\boldsymbol{L}}_{\rm spin}$ points in a direction parallel or antiparallel to the direction of propagation $\hat{\boldsymbol{k}}$, respectively.

It is instructive to consider the energy of the electromagnetic fields, which was obtained in problem 1. In particular, eq. (21) yields

$$U = 2\epsilon_0 \sum_\lambda \int \frac{d^3k}{(2\pi)^3}\,\omega^2 |a_\lambda(\vec{\boldsymbol{k}})|^2\,, \tag{39}$$

where we have used eq. (7) to write $E_0(\vec{\boldsymbol{k}}, \lambda) = i\omega a_\lambda(\vec{\boldsymbol{k}})$. Consider a fixed mode of positive helicity $(\vec{\boldsymbol{k}}_0, \lambda = +1)$. Then, $a_\lambda(\vec{\boldsymbol{k}}) = a_+(\vec{\boldsymbol{k}}_0)\delta^3(\vec{\boldsymbol{k}} - \vec{\boldsymbol{k}}_0)\delta_{\lambda,+1}$, in which case eq. (39) yields

$$U = \frac{2\epsilon_0\omega_0^2}{(2\pi)^3}|a_\lambda(\vec{\boldsymbol{k}}_0)|^2\,,$$

and

$$\vec{\boldsymbol{L}}_{\rm spin} = \frac{2}{\mu_0 c}\cdot\frac{1}{(2\pi)^3}\,\vec{\boldsymbol{k}}_0|a_\lambda(\vec{\boldsymbol{k}}_0)|^2 = \frac{2\epsilon_0\omega_0}{(2\pi)^3}\,\hat{\boldsymbol{k}}_0|a_\lambda(\vec{\boldsymbol{k}}_0)|^2\,,$$

after using $\epsilon_0\mu_0 = 1/c^2$ and $\vec{\boldsymbol{k}}_0 = (\omega_0/c)\hat{\boldsymbol{k}}_0$. That is,

$$\vec{\boldsymbol{L}}_{\rm spin} = \lambda\,\frac{U}{\omega_0}\,\hat{\boldsymbol{k}}_0\,, \qquad \text{for } \lambda = +1\,. \tag{40}$$

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[4]To prove eq. (37), use the fact that $\hat{\boldsymbol{\epsilon}}_1 \times \hat{\boldsymbol{\epsilon}}_2 = -\hat{\boldsymbol{\epsilon}}_2 \times \hat{\boldsymbol{\epsilon}}_1 = \hat{\boldsymbol{k}}$ and $\hat{\boldsymbol{\epsilon}}_1 \times \hat{\boldsymbol{\epsilon}}_1 = \hat{\boldsymbol{\epsilon}}_2 \times \hat{\boldsymbol{\epsilon}}_2 = 0$.

[5]Indeed, Jackson only asks that we show that the time-average of $\vec{\boldsymbol{L}}_{\rm spin}$ is given by eq. (38). In such a calculation, the last two terms in eq. (35) are immediately set to zero when taking the time-average since the time-averaged values

$$\langle e^{\pm 2i\omega t}\rangle = \frac{1}{T}\int_0^T e^{\pm 2i\omega t}\,dt = 0\,, \qquad \text{when } \omega \neq 0\,,$$

where $T = 2\pi/\omega$ is the time for one oscillation cycle. The case of $\omega = 0$ corresponds to $\vec{\boldsymbol{k}} = 0$, in which case the last two terms in eq. (35), when summed over λ and λ', are each manifestly equal to zero, since eq. (33) implies that $\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}}) \times \hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}}) = 0$ for $\lambda = \pm$ (and the cross-terms vanish). However, our result above is more general since no time-averaging is required to obtain eq. (38).

For a fixed mode of negative helicity ($\vec{\boldsymbol{k}}_0, \lambda = -1$), we again obtain eq. (40) with $\lambda = -1$. For a single photon of frequency ω_0, quantum mechanics states that $U = \hbar\omega_0$, and eq. (40) yields

$$\vec{\boldsymbol{L}}_{\rm spin} = \pm\hbar\hat{\boldsymbol{k}}_0\,,$$

corresponding to a spin-one particle of helicity ± 1, with its spin parallel or antiparallel to the direction of propagation $\hat{\boldsymbol{k}}_0$.

3. [Jackson, problem 8.4] Transverse electric and magnetic waves are propagated along a hollow, right circular cylinder with inner radius R and conductivity σ.

(a) Find the cutoff frequencies of the various TE and TM modes. Determine numerically the lowest cutoff frequency (the dominant mode) in terms of the tube radius and the ratio of cutoff frequencies of the next four higher modes to that of the dominant mode. For this part assume that the conductivity of the cylinder is infinite.

For the TM modes, we must solve [cf. eqs. (8.34)–(8.36) of Jackson]:

$$(\vec{\boldsymbol{\nabla}}^2_\perp + \gamma^2)E_z = 0\,, \qquad \text{where } E_z\big|_S = 0\,, \tag{41}$$

and $\gamma^2 = \mu\epsilon\omega^2 - k^2 > 0$ (in SI units). In cylindrical coordinates, $x = r\cos\phi$ and $y = r\sin\phi$, with $r = \sqrt{x^2+y^2}$ and

$$\vec{\boldsymbol{\nabla}}^2_\perp = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}\,.$$

Employing the separation of variables technique,

$$E_z(\vec{\boldsymbol{x}}, t) = R(r)\Omega(\phi)e^{\pm ikz - i\omega t}\,,$$

eq. (41) is re-expressed as

$$\frac{1}{R}\left(\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right) + \frac{1}{r^2\Omega}\frac{d^2\Omega}{d\phi^2} + \gamma^2 = 0\,.$$

Separating variables,

$$\frac{1}{\Omega}\frac{d^2\Omega}{d\phi^2} = -\frac{1}{R}\left(r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr}\right) - \gamma^2r^2 = -m^2\,,$$

where $-m^2$ is the separation constant.

The solution to the equation for Ω is

$$\Omega(\phi) = Ae^{im\phi}\,, \qquad \text{where } m = 0, \pm1, \pm2, \ldots,$$

where n is an integer due to the requirement of single-valueness, $\Omega(\phi + 2\pi) = \Omega(\phi)$, and A is a constant.

The equation for R is then given by

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(\gamma^2 - \frac{m^2}{r^2}\right) R = 0\,,$$

which we recognize as Bessel's equation. We reject $N_m(\gamma r)$ as a solution since the Bessel function of the second kind is singular at the origin. Thus, $R(r) = CJ_m(\gamma r)$ where C is a constant. We conclude that

$$E_z(r,\phi,z,t) = E_m J_m(\gamma r) e^{im\phi \pm ikz - i\omega t}\,, \qquad \text{where } m = 0, \pm 1, \pm 2, \ldots, \tag{42}$$

is the most general solution to eq. (41) before imposing the boundary condition, where E_m is a constant. Note that since $J_m(-\gamma r) = (-1)^m J_m(\gamma r)$, eq. (42) can be rewritten as

$$E_z(r,\phi,z,t) = J_m(\gamma r)\left[E_{m1}\sin m\phi + E_{m2}\cos m\phi\right] e^{\pm ikz - i\omega t}\,, \qquad \text{where } m = 0, 1, 2, \ldots, \tag{43}$$

where E_{m1} and E_{m2} are constants.

The boundary condition is given by

$$E_z(R,\phi,z,t) = 0\,.$$

Imposing this condition on eq. (42) yields

$$J_m(\gamma R) = 0\,.$$

Thus, $\gamma R = x_{mn}$, where $x_{mn} > 0$ are the positive zeros of the Bessel function J_m. That is, $J_m(x_{mn}) = 0$, for $n = 1, 2, 3, \ldots$, where n labels the zeros of the Bessel function for fixed m. Hence, the allowed values for the eigenvalues γ are

$$\gamma_{nm} = \frac{x_{mn}}{R}\,, \qquad \text{where } m = 0, 1, 2, \ldots, \text{ and } n = 1, 2, 3, \ldots,$$

which specify the allowed TM modes.

For the TE modes, we must solve:

$$(\vec{\boldsymbol{\nabla}}_\perp^2 + \gamma^2) H_z = 0\,, \qquad \text{where } \left.\frac{\partial H_z}{\partial n}\right|_S = 0\,, \tag{44}$$

and $\gamma^2 = \mu\epsilon\omega^2 - k^2 > 0$ (in SI units). In cylindrical coordinates, the normal is identified as $\hat{\boldsymbol{n}} = \hat{\boldsymbol{r}}$. Hence, the solutions are given by

$$H_z(r,\phi,z,t) = H_m J_m(\gamma r) e^{im\phi \pm ikz - i\omega t}\,, \qquad \text{where } m = 0, \pm 1, \pm 2, \ldots, \tag{45}$$

where the relevant boundary condition is now

$$\left.\frac{\partial H_z}{\partial r}\right|_{r=R} = 0\,.$$

Imposing this condition eq. (45) yields

$$J_m'(\gamma R) = 0\,,$$

where we have introduced the notation $J'_m(x) = \frac{d}{dx}J_m(x)$. Thus, $\gamma R = y_{mn}$, where $y_{mn} > 0$ are the positive zeros of the derivative of the Bessel function J'_m. That is, $J'_m(y_{mn}) = 0$, for $n = 1, 2, 3, \ldots$, where n labels the zeros of the derivative of the Bessel function for fixed m. Hence, the allowed values for the eigenvalue γ are

$$\gamma_{mn} = \frac{y_{mn}}{R}\,, \qquad \text{where } m = 0, 1, 2, \ldots, \text{ and } n = 1, 2, 3, \ldots,$$

which specify the allowed TE modes. Jackson defines the cutoff frequency (in SI units) in eq. (8.38),

$$\omega_{mn} = \frac{\gamma_{mn}}{\sqrt{\mu\epsilon}}\,.$$

The numerical values of the zeros of the Bessel function and the derivative of the Bessel function can be quickly found with a google search for "Bessel function zeros." For example, Wolfram MathWorld (`http://mathworld.wolfram.com/BesselFunctionZeros.html`) provides tables of numerical values for x_{mn} and y_{mn}, which are reproduced in Tables 1 and 2.

Table 1: The zeros of the Bessel function, $J_m(x_{mn}) = 0$.

n	$J_0(x)$	$J_1(x)$	$J_2(x)$	$J_3(x)$	$J_4(x)$	$J_5(x)$
1	2.4048	3.8317	5.1356	6.3802	7.5883	8.7715
2	5.5201	7.0156	8.4172	9.7610	11.0647	12.3386
3	8.6537	10.1735	11.6198	13.0152	14.3725	15.7002
4	11.7915	13.3237	14.7960	16.2235	17.6160	18.9801
5	14.9309	16.4706	17.9598	19.4094	20.8269	22.2178

Table 2: The zeros of the derivative of the Bessel function, $J'_m(y_{mn}) = 0$.

n	$J'_0(x)$	$J'_1(x)$	$J'_2(x)$	$J'_3(x)$	$J'_4(x)$	$J'_5(x)$
1	3.8317	1.8412	3.0542	4.2012	5.3175	6.4156
2	7.0156	5.3314	6.7061	8.0152	9.2824	10.5199
3	10.1735	8.5363	9.9695	11.3459	12.6819	13.9872
4	13.3237	11.7060	13.1704	14.5858	15.9641	17.3128
5	16.4706	14.8636	16.3475	17.7887	19.1960	20.5755

Thus the lowest cutoff frequencies for this problem are:

$$\text{TM}: \qquad \omega_{01} = \frac{1}{\sqrt{\mu\epsilon}}\,\frac{x_{01}}{R}\,, \qquad \text{where } x_{01} = 2.4048\,, \tag{46}$$

$$\text{TE}: \qquad \omega_{11} = \frac{1}{\sqrt{\mu\epsilon}}\,\frac{y_{11}}{R}\,, \qquad \text{where } y_{11} = 1.8412\,. \tag{47}$$

Note that for the TE modes, the $m = n = 1$ corresponds to the lowest cutoff frequency (rather than $m = 0$, $n = 1$). For the TM modes, the first five cutoff frequencies in increasing order correspond to $(m, n) = (0,1)\,,\,(1,1)\,,\,(2,1)\,,\,(0,2)\,,\,(3,1)$. For the TE modes, the first five cutoff frequencies in increasing order correspond to $(m, n) = (1,1)\,,\,(2,1)\,,\,(0,1)\,,\,(3,1)\,,\,(4,1)$. Thus, the lowest cutoff frequency is the TE mode with $(m, n) = (1, 1)$, which we shall denote below by ω_0. The cutoff frequencies, relative to the lowest cutoff frequency ω_0, are given by:

$$\text{TM}: \qquad \frac{\omega_{mn}}{\omega_0} = 1.306, 2.081, 2.789, 2.998, 3.465, \ldots\,,$$

$$\text{TE}: \qquad \frac{\omega_{mn}}{\omega_0} = 1.000, 1.659, 2.081, 2.282, 2.888, \ldots\,.$$

(b) Calculate the attenuation constants of the waveguide as a function of frequency for the lowest two distinct modes and plot them as a function of frequency.

The attenuation constant β is defined by eqs. (8.56) and (8.57) of Jackson,

$$\beta_\lambda = -\frac{1}{2P}\frac{\partial P}{\partial z}\,, \tag{48}$$

where P is the transmitted power.

<u>Case 1</u>: TM_{01}

In part (a), we obtained

$$E_z(r, \phi, z, t) = E_0 J_0\left(\frac{x_{01} r}{R}\right) e^{i(kz-\omega t)}\,, \tag{49}$$

for the longitudinal electric field of the TM mode with $(m, n) = (0, 1)$. Assuming no losses, the power is given by eq. (8.51) of Jackson,

$$P = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\left(\frac{\omega}{\omega_{01}}\right)^2\left(1 - \frac{\omega^2}{\omega_{01}^2}\right)^{1/2}\int_0^R r dr \int_0^{2\pi} d\phi\, |E_z(r, \phi, z, t)|^2\,. \tag{50}$$

We now insert eq. (49) into eq. (50) and evaluate the resulting integrals. The required integral over r is[6]

$$\int_0^R r\, dr\left[J_0\left(\frac{x_{01} r}{R}\right)\right]^2 = \tfrac{1}{2}R^2[J_1(x_{01})]^2\,.$$

The end result is

$$P = \tfrac{1}{2}|E_0|^2\, A\,\sqrt{\frac{\epsilon}{\mu}}\left(\frac{\omega}{\omega_{01}}\right)^2\left(1 - \frac{\omega^2}{\omega_{01}^2}\right)^{1/2}[J_1(x_{01})]^2\,,$$

where $A = \pi R^2$ is the cross-sectional area of the waveguide and $\omega_{01} = \dfrac{1}{\sqrt{\mu\epsilon}}\dfrac{x_{01}}{R}$ [cf. eq. (46)].

[6] See e.g. formula 6.561–5 on p. 684 of I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products* (8th edition), edited by Alan Jeffrey and Daniel Zwillinger (Academic Press, Elsevier, Waltham, MA, 2015).

Next, eq. (8.59) of Jackson yields

$$\frac{dP_{\rm loss}}{dz} = \frac{1}{2\sigma\delta\mu^2\omega_{01}^2}\left(\frac{\omega}{\omega_{01}}\right)^2 \oint_C \left|\frac{\partial E_z}{dn}\right|^2 d\ell\,. \tag{51}$$

In this problem, $\boldsymbol{\hat{n}} = \boldsymbol{\hat{r}}$, so

$$\frac{\partial E_z}{dn} = \boldsymbol{\hat{n}\cdot}\vec{\boldsymbol{\nabla}} E_z = \frac{\partial E_z}{\partial r} = \frac{E_0 x_{01}}{r}\, J_0'\left(\frac{x_{01}r}{R}\right)\, e^{i(kz-\omega t)}\,.$$

Hence, noting that $d\ell = Rd\phi$ and $x_{01} = \omega_{01}R\sqrt{\mu\epsilon}$, it follows that

$$\oint_C \left|\frac{\partial E_z}{dn}\right|^2 d\ell = \frac{|E_0|^2 x_{01}^2}{R^2}\int_0^{2\pi} R\,d\phi\left[J_0'\left(\frac{x_{01}r}{R}\right)\right]^2\Bigg|_{r=R} = 2\pi R\mu\epsilon|E_0|^2\omega_{01}^2[J_1(x_{01})]^2\,,$$

after using the relation $J_0'(x) = -J_1(x)$. Hence, eq. (51) yields:

$$\frac{dP_{\rm loss}}{dz} = \tfrac{1}{2}|E_0|^2\frac{C\epsilon}{\sigma\delta\mu}\left(\frac{\omega}{\omega_{01}}\right)^2[J_1(x_{01})]^2\,,$$

where $C = 2\pi R$ is the circumference of the cross-sectional area of the waveguide.

The attenuation constant defined in eq. (48) is therefore given by:

$$\beta_{01} = \frac{1}{2\sigma\delta}\sqrt{\frac{\epsilon}{\mu}}\,\frac{C}{A}\left(1-\frac{\omega_{01}^2}{\omega^2}\right)^{-1/2}\,. \tag{52}$$

The skin depth, which is given by eq. (8.8) of Jackson, can be rewritten as:

$$\delta = \left(\frac{2}{\mu_c\omega\sigma}\right)^{1/2} = \left(\frac{\omega_{01}}{\omega}\right)^{1/2}\delta_{01}\,, \tag{53}$$

where

$$\delta_\lambda \equiv \left(\frac{2}{\mu_c\omega_\lambda\sigma}\right)^{1/2} \tag{54}$$

is the skin depth corresponding to the mode frequency ω_λ. Then, we can rewrite eq. (52) as:

$$\beta_{01} = \frac{1}{2\sigma\delta_{01}}\sqrt{\frac{\epsilon}{\mu}}\,\frac{C}{A}\left(\frac{\omega}{\omega_{01}}\right)^{1/2}\left(1-\frac{\omega^2}{\omega_{01}^2}\right)^{-1/2}\,.$$

Comparing this result with eq. (8.63) of Jackson,

$$\beta_\lambda = \sqrt{\frac{\epsilon}{\mu}}\,\frac{1}{\sigma\delta_\lambda}\left(\frac{C}{2A}\right)\left(\frac{\omega}{\omega_\lambda}\right)^{1/2}\left(1-\frac{\omega_\lambda^2}{\omega^2}\right)^{-1/2}\left[\xi_\lambda + \eta_\lambda\left(\frac{\omega_\lambda}{\omega}\right)^2\right]\,, \tag{55}$$

we see that for the TM_{01} mode of this problem, we have $\xi_{01} = 1$ and $\eta_{01} = 0$.

Case 2: TE_{11}

In part (a), we obtained

$$H_z(r,\phi,z,t) = H_0 J_1\left(\frac{y_{11}r}{R}\right) e^{\pm i\phi}\, e^{i(kz-\omega t)}\,, \tag{56}$$

for the longitudinal magnetic field of the TE mode with $(m,n) = (1,1)$. Assuming no losses, the power is given by eq. (8.51) of Jackson,

$$P = \frac{1}{2}\sqrt{\frac{\mu}{\epsilon}}\left(\frac{\omega}{\omega_{11}}\right)^2\left(1-\frac{\omega^2}{\omega_{11}^2}\right)^{1/2}\int_0^R r\,dr\int_0^{2\pi} d\phi\,|H_z(r,\phi,z,t)|^2\,. \tag{57}$$

We now insert eq. (56) into eq. (57) and evaluate the resulting integrals. The required integral over r is[7]

$$\int_0^R r\,dr\left[J_1\left(\frac{y_{11}r}{R}\right)\right]^2 = \tfrac{1}{2}R^2\left(1-\frac{1}{y_{11}^2}\right)[J_1(y_{11})]^2\,.$$

The end result is

$$P = \tfrac{1}{2}|H_0|^2\,A\sqrt{\frac{\mu}{\epsilon}}\left(\frac{\omega}{\omega_{11}}\right)^2\left(1-\frac{\omega^2}{\omega_{11}^2}\right)^{1/2}\left(1-\frac{1}{y_{11}^2}\right)[J_1(y_{11})]^2\,,$$

where $A=\pi R^2$ is the cross-sectional area of the waveguide and $\omega_{11} = \dfrac{1}{\sqrt{\mu\epsilon}}\dfrac{y_{11}}{R}$ [cf. eq. (47)].

Next, eq. (8.59) of Jackson yields

$$\frac{dP_{\rm loss}}{dz} = \frac{1}{2\sigma\delta}\left(\frac{\omega}{\omega_{11}}\right)^2\oint_C\left[\frac{1}{\mu\epsilon\omega_{11}^2}\left(1-\frac{\omega_{11}^2}{\omega^2}\right)|\boldsymbol{\hat{n}}\times\vec{\boldsymbol{\nabla}}_\perp H_z|^2 + \frac{\omega_{11}^2}{\omega^2}|H_z|^2\right]d\ell\,.$$

In this problem, $\boldsymbol{\hat{n}}=\boldsymbol{\hat{r}}$, so

$$\begin{aligned}
\boldsymbol{\hat{n}}\times\vec{\boldsymbol{\nabla}}_\perp H_z = \boldsymbol{\hat{r}}\times\left(\boldsymbol{\hat{x}}\frac{\partial}{\partial x}+\boldsymbol{\hat{y}}\frac{\partial}{\partial y}\right)H_z &= (\boldsymbol{\hat{x}}\cos\phi+\boldsymbol{\hat{y}}\sin\phi)\times\left(\boldsymbol{\hat{x}}\frac{\partial}{\partial x}+\boldsymbol{\hat{y}}\frac{\partial}{\partial y}\right)H_z \\
&= \boldsymbol{\hat{z}}\left(\cos\phi\frac{\partial}{\partial y}-\sin\phi\frac{\partial}{\partial x}\right)H_z = \boldsymbol{\hat{z}}\,\frac{1}{r}\,\frac{\partial H_z}{\partial\phi} \\
&= \pm\boldsymbol{\hat{z}}\frac{iH_0}{r}\,J_1\left(\frac{y_{11}r}{R}\right)e^{\pm i\phi}\,e^{i(kz-\omega t)}\,.
\end{aligned}\tag{58}$$

Hence, noting that $d\ell = Rd\phi$ and $\oint d\ell = 2\pi R = C$, it follows that

$$\frac{dP_{\rm loss}}{dz} = \tfrac{1}{2}|H_0|^2\,\frac{C}{\sigma\delta}\left(\frac{\omega}{\omega_{11}}\right)^2[J_1(y_{11})]^2\left[\frac{1}{R^2\mu\epsilon\omega_{11}^2}\left(1-\frac{\omega_{11}^2}{\omega^2}\right)+\frac{\omega_{11}^2}{\omega^2}\right]\,.$$

We can simplify this expression using $y_{11} = R\omega_{11}\sqrt{\mu\epsilon}$. The expression in brackets above is:

$$\frac{1}{R^2\mu\epsilon\,\omega_{11}^2}\left(1-\frac{\omega_{11}^2}{\omega^2}\right)+\frac{\omega_{11}^2}{\omega^2} = \frac{1}{y_{11}^2}\left(1-\frac{\omega_{11}^2}{\omega^2}\right)+\frac{\omega_{11}^2}{\omega^2} = \frac{1}{y_{11}^2}+\frac{\omega_{11}^2}{\omega^2}\left(1-\frac{1}{y_{11}^2}\right)\,.$$

[7] See e.g. eq. (5.14.9) on p. 130 of N.N. Lebedev, *Special Functions and Their Applications* (Dover Publications, Inc., Mineola, NY, 1972).

Hence,

$$\frac{dP_{\rm loss}}{dz} = \tfrac{1}{2}|H_0|^2\,\frac{C}{\sigma\delta}\left(\frac{\omega}{\omega_{11}}\right)^2[J_1(y_{11})]^2\left[\frac{1}{y_{11}^2}+\frac{\omega_{11}^2}{\omega^2}\left(1-\frac{1}{y_{11}^2}\right)\right].$$

The attenuation constant defined in eq. (48) is therefore given by:

$$\beta_{11} = \frac{1}{2\sigma\delta}\sqrt{\frac{\epsilon}{\mu}}\,\frac{C}{A}\left(1-\frac{\omega_{11}^2}{\omega^2}\right)^{-1/2}\left[\frac{1}{y_{11}^2-1}+\frac{\omega_{11}^2}{\omega^2}\right]. \tag{59}$$

The skin depth can be rewritten as [cf. eq. (53)]

$$\delta = \left(\frac{2}{\mu_c\omega\sigma}\right)^{1/2} = \left(\frac{\omega_{11}}{\omega}\right)^{1/2}\delta_{11}\,,$$

where δ_λ is defined in eq. (54). Then, we can rewrite eq. (59) as:

$$\beta_{11} = \frac{1}{2\sigma\delta_{11}}\sqrt{\frac{\epsilon}{\mu}}\,\frac{C}{A}\left(\frac{\omega}{\omega_{11}}\right)^{1/2}\left(1-\frac{\omega_{11}^2}{\omega^2}\right)^{-1/2}\left[\frac{1}{y_{11}^2-1}+\frac{\omega_{11}^2}{\omega^2}\right].$$

Comparing this result with eq. (8.63) of Jackson [cf. eq. (55)], we see that for the TE_{11} mode of this problem, we have

$$\xi_{11} = \frac{1}{y_{11}^2-1} \simeq 0.419\,, \qquad \eta_{11} = 1\,.$$

The graphs of the attenuation constants, β_{01} for the TM_{01} mode and β_{11} for the TE_{11} mode, are very similar to Figure 8.6 on p. 366 of Jackson, so we will not elaborate further here.

4. [Jackson, problem 8.5] A waveguide is constructed so that the cross section of the guide forms a right triangle with sides of length a, a and $\sqrt{2}a$, as shown in Figure 1. The medium inside has $\mu_r = \epsilon_r = 1$.

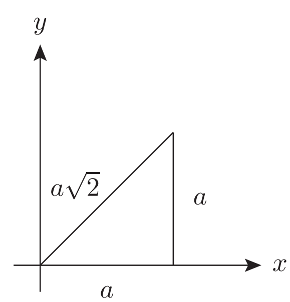

Figure 1: The cross-section of a triangular waveguide, projected onto the x–y plane.

(a) Assuming infinite conductivity for the walls, determine the possible modes of propagation and their cutoff frequencies.

Since the waveguide consists of a single hollow conductor of infinite conductivity, there are no TEM waves. Hence, we consider separately the cases of TM waves and TE waves.

TM waves

Following the lecture notes, the TM waves are solutions to

$$\left(\vec{\boldsymbol{\nabla}}_{\perp}^{2}+\frac{\omega^2}{c^2}-k^2\right)E_z=0\,,\qquad \text{where } E_z\big|_S=0\,, \tag{60}$$

after putting $\mu_r = \epsilon_r = 1$. Thus, we must solve

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\gamma^2\right)E_z=0\,,\qquad \text{where } E_z\big|_S=0\,, \tag{61}$$

where

$$\gamma^2\equiv\frac{\omega^2}{c^2}-k^2\,.$$

The general solution to eq. (61) is of the form,

$$E_z(\vec{\boldsymbol{x}},t)=E_z(x,y)\,e^{\pm ikz-i\omega t}\,,$$

subject to the boundary conditions,

$$E_z(x,0)=E_z(a,y)=E_z(x,x)=0\,,\qquad \text{for } 0\leq x\,,\ y\leq a\,.$$

In class, we showed that the solutions for TM waves in the case of a square cross-section (where a is the length of a side of the square), which differs only in the boundary conditions, is

$$E_z(x,y)=E_0\,\sin\left(\frac{m\pi x}{a}\right)\,\sin\left(\frac{n\pi y}{a}\right)\,,$$

where

$$\gamma^2_{mn}=\frac{\pi^2}{a^2}\,\left(m^2+n^2\right)\,,\qquad \text{for } n,m=1,2,3,\ldots.$$

These solutions satisfy two of the three boundary conditions for the triangular waveguide. But the condition $E_z(x,x)=0$ is not satisfied. However, a simple linear combination of two solutions does satisfy this third boundary condition,

$$E_z(x,y)=E_0\left[\sin\left(\frac{m\pi x}{a}\right)\,\sin\left(\frac{n\pi y}{a}\right)-\sin\left(\frac{n\pi x}{a}\right)\,\sin\left(\frac{m\pi y}{a}\right)\right]\,. \tag{62}$$

Having found the solutions that satisfy the boundary conditions, we can invoke the uniqueness theorem for solutions to Laplace's equation in two-dimensions to argue that the most general solution for TM waves consists of arbitrary linear combinations of solutions of the form given in eq. (62) for any two positive integer m and n, assuming that $m\neq n$. (The case of $m=n$ is rejected since in this case $E_z(x,y)=0$.)

The cutoff frequencies (after putting $\mu_r = \epsilon_r = 1$) are given by:

$$\omega_{mn}=c\gamma_{mn}=\frac{c\pi}{a}\,(n^2+m^2)^{1/2}\,,\qquad \text{for } n\neq m \text{ and } n,m=1,2,3,\ldots.$$

TE waves

Following the lecture notes, the TE waves are solutions to

$$\left(\vec{\nabla}_\perp^2 + \frac{\omega^2}{c^2} - k^2\right) B_z = 0\,, \qquad \text{where } \frac{\partial B_z}{\partial n}\bigg|_S = 0\,, \tag{63}$$

after putting $\mu_r = \epsilon_r = 1$. Thus, we must solve

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \gamma^2\right) B_z = 0\,, \qquad \text{where } \frac{\partial B_z}{\partial n}\bigg|_S = 0\,, \tag{64}$$

where once again,

$$\gamma^2 \equiv \frac{\omega^2}{c^2} - k^2\,.$$

The general solution to eq. (64) is of the form,

$$B_z(\vec{\boldsymbol{x}}, t) = B_z(x, y)\, e^{\pm ikz - i\omega t}\,,$$

subject to the boundary conditions,

$$\frac{\partial B_z}{\partial y}(x, 0) = \frac{\partial B_z}{\partial x}(a, y) = 0\,,$$

and along the diagonal $y = x$, where

$$\hat{\boldsymbol{n}} = \frac{1}{\sqrt{2}}(-\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}}) \quad \text{and} \quad \frac{\partial}{\partial n} = \hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}} = \frac{1}{\sqrt{2}}\left(-\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\,, \tag{65}$$

we have

$$-\frac{\partial B_z}{\partial x}(x, x) + \frac{\partial B_z}{\partial y}(x, x) = 0\,. \tag{66}$$

In class, we showed that the solutions for TM waves in the case of a square cross-section (where a is the length of a side of the square), which differs only in the boundary conditions, is

$$B_z(x, y) = B_0 \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{a}\right)\,, \tag{67}$$

where

$$\gamma_{mn}^2 = \frac{\pi^2}{a^2}\left(m^2 + n^2\right)\,, \qquad \text{for } n, m = 0, 1, 2, 3, \ldots,$$

where the case of $n = m = 0$ is not allowed.[9] These solutions satisfy two of the three boundary conditions for the triangular waveguide. But the boundary condition given by eq. (66) is not satisfied. However, a simple linear combination of two solutions does satisfy this third boundary condition,

$$B_z(x, y) = E_0\left[\cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{a}\right) + \cos\left(\frac{n\pi x}{a}\right) \cos\left(\frac{m\pi y}{a}\right)\right]\,. \tag{68}$$

Having found the solutions that satisfy the boundary conditions, we can invoke the uniqueness theorem for solutions to Laplace's equation in two-dimensions to argue that the most general solution for TE waves consists of arbitrary linear combinations of solutions of the form given in eq. (62) for any two non-negative integer m and n, unless $m = n = 0$ which must be rejected.[8] Note that if n and m are positive integers, the case of $n = m$ is a valid solution for the TE case, in contrast to the TM case treated above.

The cutoff frequencies (after putting $\mu_r = \epsilon_r = 1$) are given by:

$$\omega_{mn} = c\gamma_{mn} = \frac{c\pi}{a}\,(n^2+m^2)^{1/2}\,, \qquad \text{for } n, m = 0, 1, 2, 3, \ldots \text{ (the case } n = m = 0 \text{ is not allowed)}\,.$$

(b) For the lowest modes of each type, calculate the attenuation constant, assuming that the walls have large, but finite, conductivity. Compare the result with that for a square guide of side a made from the same material.

The attenuation constant β is defined in eq. (8.57) of Jackson,

$$\beta_\lambda = -\frac{1}{2P}\frac{dP}{dz}\,,$$

where λ labels the mode. To evaluate this, we employ eq. (8.51) of Jackson for P and eq. (8.59) of Jackson for dP/dz. It is convenient to define

$$\delta_\lambda \equiv \left(\frac{2}{\mu_c\omega_\lambda\sigma}\right)^{1/2} = \delta\left(\frac{\omega}{\omega_\lambda}\right)^{1/2}\,, \tag{69}$$

using the frequency dependence of the skin depth given in eq. (8.8) of Jackson.

Since the medium inside has $\mu_r = \epsilon_r = 1$, we may put $\epsilon = \epsilon_0$, $\mu = \mu_0$ and $\epsilon_0\mu_0 = 1/c^2$. Then, using eqs. (8.51), (8.57) and (8.59) of Jackson [along with eq. (69) above], one obtains the attenuation constant for TM and TE modes respectively.

For TM modes,

$$\beta_\lambda = \sqrt{\frac{\epsilon_0}{\mu_0}}\,\frac{1}{2\sigma\delta_\lambda}\,\frac{(\omega/\omega_\lambda)^{1/2}}{(1-\omega_\lambda^2/\omega^2)^{1/2}}\,\frac{\displaystyle\oint_C \frac{c^2}{\omega_\lambda^2}\left|\frac{\partial E_z}{\partial n}\right|^2 d\ell}{\displaystyle\int_A |E_z|^2\,da}\,. \tag{70}$$

[8]It appears that B_z is a constant (independent of position) in the case of $m = n = 0$ [cf. eq. (67)]. But, this constant must be zero due to Faraday's law,

$$\oint_C \vec{\boldsymbol{E}}\boldsymbol{\cdot} d\vec{\boldsymbol{\ell}} = i\omega\int_S \vec{\boldsymbol{B}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}\,da\,,$$

for harmonic fields (where $d/dt \to -i\omega$ and the factors $e^{-i\omega t}$ have been stripped off). Choose a surface S that lies in the x-y plane (in which case $\hat{\boldsymbol{n}} = \hat{\boldsymbol{z}}$), and whose boundary C lies *inside* the metallic walls of the conductor. Since $\vec{\boldsymbol{E}} = 0$ inside the conductor and B_z is a constant inside the waveguide, it follows that

$$0 = \int_S \vec{\boldsymbol{B}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}\,da = B_z\int_S da = AB_z\,.$$

where A is the cross-sectional area of the waveguide. Hence, $B_z = 0$ as claimed. Thus, any mode that corresponds to $m = n = 0$ must be purely transverse, i.e. a TEM mode. However, TEM waves cannot be supported by a single hollow conductor of infinite conductivity, so we conclude that there are no non-trivial solutions in the case of $n = m = 0$.

For TE modes,

$$\beta_\lambda = \sqrt{\frac{\epsilon_0}{\mu_0}}\,\frac{1}{2\sigma\delta_\lambda}\,\frac{(\omega/\omega_\lambda)^{1/2}}{(1-\omega_\lambda^2/\omega^2)^{1/2}}\,\frac{\displaystyle\oint_C \left\{\frac{c^2}{\omega_\lambda^2}(1-\omega_\lambda^2/\omega^2)|\hat{\boldsymbol{n}}\times\vec{\boldsymbol{\nabla}}_t B_z|^2 + (\omega_\lambda^2/\omega^2)|B_z|^2\right\} d\ell}{\displaystyle\int_A |B_z|^2\, da}\,. \tag{71}$$

Attenuation of the lowest TM mode

The lowest TM modes correspond to $(m,n) = (1,2)$ and $(2,1)$. Due to the symmetry of the problem under the interchange of the x and y coordinates, the attenuation constant is the same for both modes. For definiteness, we focus on the case of $(m,n) = (1,2)$. The corresponding frequency is

$$\omega_\lambda = \frac{\sqrt{5}c\pi}{a}\,. \tag{72}$$

Using eq. (62), the z-component of the electric field is given by:

$$E_z(x,y) = E_0\left[\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi y}{a}\right) - \sin\left(\frac{2\pi x}{a}\right)\sin\left(\frac{\pi y}{a}\right)\right]\,. \tag{73}$$

Hence, it follows that

$$\int_A |E_z|^2\, da = E_0^2\int_0^a dx\int_0^x dy\left[\sin^2\left(\frac{\pi x}{a}\right)\sin^2\left(\frac{2\pi y}{a}\right) + \sin^2\left(\frac{2\pi x}{a}\right)\sin^2\left(\frac{\pi y}{a}\right)\right.$$
$$\left. -2\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi y}{a}\right)\sin\left(\frac{2\pi x}{a}\right)\sin\left(\frac{\pi y}{a}\right)\right]. \tag{74}$$

If we interchange the order of integration, it follows that

$$\int_A |E_z|^2\, da = E_0^2\int_0^a dy\int_y^a dx\left[\sin^2\left(\frac{\pi x}{a}\right)\sin^2\left(\frac{2\pi y}{a}\right) + \sin^2\left(\frac{2\pi x}{a}\right)\sin^2\left(\frac{\pi y}{a}\right)\right.$$
$$\left. -2\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi y}{a}\right)\sin\left(\frac{2\pi x}{a}\right)\sin\left(\frac{\pi y}{a}\right)\right]. \tag{75}$$

We now relabel variables by interchanging x and y in eq. (75). Since the integrand is invariant under the interchange of x and y, we see that we can add the results of eqs. (74) and (75) and divide by two to obtain

$$\int_A |E_z|^2\, da = \frac{E_0^2}{2}\int_0^a dx\int_0^a dy\left[\sin^2\left(\frac{\pi x}{a}\right)\sin^2\left(\frac{2\pi y}{a}\right) + \sin^2\left(\frac{2\pi x}{a}\right)\sin^2\left(\frac{\pi y}{a}\right)\right.$$
$$\left. -2\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi y}{a}\right)\sin\left(\frac{2\pi x}{a}\right)\sin\left(\frac{\pi y}{a}\right)\right]. \tag{76}$$

The integrals are straightforward. In particular,

$$\int_0^a \sin^2\left(\frac{\pi x}{a}\right)\,dx = \int_0^a \sin^2\left(\frac{2\pi x}{a}\right)\,dx = \frac{a}{2}\,,$$

and

$$\int_0^a \sin\left(\frac{\pi x}{a}\right)\,\sin\left(\frac{2\pi x}{a}\right) = 0\,.$$

Hence, we conclude that

$$\int_A |E_z|^2\,da = \frac{a^2 E_0^2}{4}\,.$$

Next, we integrate counterclockwise around the perimeter of the triangle (cf. Fig. 1). When traversing along the diagonal side of the triangle from the point (a,a) to the origin, we have

$$d\ell = \sqrt{(dx)^2+(dy)^2} = -dx\sqrt{1+\left(\frac{dy}{dx}\right)^2} = -\sqrt{2}\,dx\,.$$

since the diagonal corresponds to $y=x$ so that $dy/dx=1$. The minus sign arises since $d\ell$ is positive whereas dx is negative when traversing from (a,a) to the origin. Hence,

$$\begin{aligned}
\oint_C \left|\frac{\partial E_z}{\partial n}\right|^2 d\ell &= \int_0^a \left|\frac{\partial E_z(x,0)}{\partial y}\right|^2 dx + \int_0^a \left|\frac{\partial E_z(a,y)}{\partial x}\right|^2 dy - \sqrt{2}\int_a^0 dx\,\frac{1}{2}\left(-\frac{\partial E_z}{\partial x}+\frac{\partial E_z}{\partial y}\right)^2\Bigg|_{y=x} \\
&= E_0^2\left\{\int_0^a dx\left[\frac{2\pi}{a}\sin\left(\frac{\pi x}{a}\right)-\frac{\pi}{a}\sin\left(\frac{2\pi x}{a}\right)\right]^2 + \int_0^a dy\left[\frac{\pi}{a}\sin\left(\frac{2\pi y}{a}\right)+\frac{2\pi}{a}\sin\left(\frac{\pi y}{a}\right)\right]^2\right. \\
&\qquad\left. +2\sqrt{2}\int_0^a dx\left[-\frac{\pi}{a}\cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)+\frac{2\pi}{a}\cos\left(\frac{2\pi x}{a}\right)\sin\left(\frac{\pi x}{a}\right)\right]^2\right\}.
\end{aligned}$$

Note that the last integrand simplifies:

$$\begin{aligned}
-\frac{\pi}{a}\cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)+\frac{2\pi}{a}\cos\left(\frac{2\pi x}{a}\right)\sin\left(\frac{\pi x}{a}\right) &= \frac{2\pi}{a}\sin\left(\frac{\pi x}{a}\right)\left[\cos\left(\frac{2\pi x}{a}\right)-\cos^2\left(\frac{\pi x}{a}\right)\right] \\
&= -\frac{2\pi}{a}\sin^3\left(\frac{\pi x}{a}\right)\,.
\end{aligned}$$

Hence,

$$\begin{aligned}
\oint_C \left|\frac{\partial E_z}{\partial n}\right|^2 d\ell &= 2E_0^2\left(\frac{4\pi^2}{a^2}\cdot\frac{a}{2}+\frac{\pi^2}{a^2}\cdot\frac{a}{2}\right)+2\sqrt{2}\cdot\frac{4\pi^2}{a^2}\int_0^a \sin^6\left(\frac{\pi x}{a}\right)dx \\
&= \frac{5\pi^2 E_0^2}{a}\left(1+\frac{1}{\sqrt{2}}\right)\,.
\end{aligned}$$

Using eqs. (70) and (72), we end up with

$$\beta_\lambda = \sqrt{\frac{\epsilon_0}{\mu_0}}\,\frac{2+\sqrt{2}}{a\sigma\delta_\lambda}\,\frac{(\omega/\omega_\lambda)^{1/2}}{(1-\omega_\lambda^2/\omega^2)^{1/2}}\,, \qquad (77)$$

where $\omega_\lambda = \sqrt{5}c\pi/a$. Comparing with eq. (8.63) of Jackson, we see that $C = a(2+\sqrt{2})$ and $A = \frac{1}{2}a^2$ so that eq. (77) corresponds to $\xi_\lambda = 1$ and $\eta_\lambda = 0$.

We can repeat the above analysis for the square guide. Note that the mode of the triangular guide given in eq. (73) is also a mode of the square guide.[9] Thus, we only need to make minor modifications of the above computations. First, we note that

$$\int_A |E_z|^2\, da = \frac{a^2 E_0^2}{2}\,,$$

since the area of integration is twice as large for the square guide. Next, we obtain

$$\oint_C \left|\frac{\partial E_z}{\partial n}\right|^2 d\ell = 4E_0^2\left(\frac{4\pi^2}{a^2}\cdot\frac{a}{2} + \frac{\pi^2}{a^2}\cdot\frac{a}{2}\right) = \frac{10\pi^2 E_0^2}{a}\,,$$

since we replace the integration over the diagonal with the integration over the other two sides of the square. Thus,

$$\beta_\lambda = \sqrt{\frac{\epsilon_0}{\mu_0}}\,\frac{2}{a\sigma\delta_\lambda}\,\frac{(\omega/\omega_\lambda)^{1/2}}{(1-\omega_\lambda^2/\omega^2)^{1/2}}\,.$$

Since $C = 4a$ and $A = a^2$ for the square, we again recover eq. (8.63) of Jackson with $\xi_\lambda = 1$ and $\eta_\lambda = 0$. Consequently, the relative attenuation of the triangular guide and the square guide is simply governed by the quantity C/A. Thus, the attenuation of the TM mode with $(m,n) = (1,2)$ of the triangular guide is $1 + \frac{1}{2}\sqrt{2} \simeq 1.71$ times larger than that of the square guide.

Attenuation of the lowest TE mode

The lowest TE modes correspond to $(m,n) = (1,0)$ and $(0,1)$. Due to the symmetry of the problem under the interchange of the x and y coordinates, the attenuation constant is the same for both modes. For definiteness, we focus on the case of $(m,n) = (1,0)$. The corresponding frequency is

$$\omega_\lambda = \frac{c\pi}{a}\,. \tag{78}$$

Using eq. (68), the z-component of the magnetic field is given by:

$$B_z(x,y) = E_0\left[\cos\left(\frac{\pi x}{a}\right) + \cos\left(\frac{\pi y}{a}\right)\right]\,. \tag{79}$$

Following the analysis of the lowest TM mode above, we first compute

$$\int_A |B_z|^2\, da = \frac{E_0^2}{2}\int_0^a dx\int_0^a dy\left[\cos^2\left(\frac{\pi x}{a}\right) + \cos^2\left(\frac{\pi y}{a}\right) + 2\cos\left(\frac{\pi x}{a}\right)\,\cos\left(\frac{\pi y}{a}\right)\right] = \tfrac{1}{2}E_0^2 a^2\,.$$

[9]Note that this mode is not the lowest mode of the square guide, since $(m,n) = (1,1)$ yields a lower frequency for the square guide. This latter mode is not permitted for the triangular guide as noted in part (a) of this problem.

Next, we evaluate the relevant integrals over the closed triangular contour C. Our analysis above showed that the integration of an arbitrary function $f(x,y)$ is given by

$$\oint_C f(x,y)\,d\ell = \int_0^a f(x,0)\,dx + \int_0^a f(0,y)\,dy + \sqrt{2}\int_0^a f(x,x)\,dx\,.$$

Applying this result to integrate $|B_z|^2$ using eq. (79), we have

$$\begin{aligned}\oint_C |B_z|^2\,d\ell = E_0^2\biggl\{&\int_0^a \left[\cos^2\left(\frac{\pi x}{a}\right) + 2\cos\left(\frac{\pi x}{a}\right) + 1\right]dx \\ &+ \int_0^a \left[\cos^2\left(\frac{\pi y}{a}\right) + 2\cos\left(\frac{\pi y}{a}\right) + 1\right]dy + \sqrt{2}\int_0^a 4\cos^2\left(\frac{\pi x}{a}\right)\,dx\biggr\} \\ = (3+2\sqrt{2})E_0^2 a\,.&\end{aligned}$$

Likewise, using

$$\hat{\boldsymbol{\nabla}}_t = \hat{\boldsymbol{x}}\frac{\partial}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial}{\partial y}\,,$$

it follows that

$$\oint_C |\hat{\boldsymbol{n}}\times\vec{\boldsymbol{\nabla}}_t B_z|^2\,dx = \int_0^a dx\,\left|\frac{\partial B_z}{\partial x}\right|^2_{y=0} + \int_0^a dy\,\left|\frac{\partial B_z}{\partial y}\right|^2_{x=a} + \frac{1}{\sqrt{2}}\int_0^a dx\,\left|\frac{\partial B_z}{dx} + \frac{\partial B_z}{dy}\right|^2_{x=y}\,,$$

since $\hat{\boldsymbol{n}} = -\hat{\boldsymbol{y}}$ in the first integral, $\hat{\boldsymbol{n}} = \hat{\boldsymbol{x}}$ in the second integral and $\hat{\boldsymbol{n}} = (-\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}})/\sqrt{2}$ in the third integral. Plugging in eq. (79) then yields

$$\begin{aligned}\oint_C |\hat{\boldsymbol{n}}\times\vec{\boldsymbol{\nabla}}_t B_z|^2\,dx &= \frac{\pi^2 E_0^2}{a^2}\left\{\int_0^a \sin^2\left(\frac{\pi x}{a}\right)\,dx + \int_0^a \sin^2\left(\frac{\pi y}{a}\right)\,dy\right\} + \frac{2\sqrt{2}E_0^2\pi^2}{a^2}\int_0^a \sin^2\left(\frac{\pi x}{a}\right)\,dx \\ &= \frac{\pi^2 E_0^2}{a}\left(1+\sqrt{2}\right)\,.\end{aligned}$$

Hence eq. (71) yields

$$\begin{aligned}\beta_\lambda &= \sqrt{\frac{\epsilon_0}{\mu_0}}\left(\frac{1}{a\sigma\delta_\lambda}\right)\frac{(\omega/\omega_\lambda)^{1/2}}{(1-\omega_\lambda^2/\omega^2)^{1/2}}\left[\left(1-\frac{\omega_\lambda^2}{\omega^2}\right)(1+\sqrt{2}) + \frac{\omega_\lambda^2}{\omega^2}(3+2\sqrt{2})\right] \\ &= \sqrt{\frac{\epsilon_0}{\mu_0}}\left(\frac{2+\sqrt{2}}{a\sigma\delta_\lambda}\right)\frac{(\omega/\omega_\lambda)^{1/2}}{(1-\omega_\lambda^2/\omega^2)^{1/2}}\left[\frac{1}{\sqrt{2}} + \frac{\omega_\lambda^2}{\omega^2}\right]\,. \qquad (80)\end{aligned}$$

Comparing with eq. (8.63) of Jackson, we see that $C = a(2+\sqrt{2})$ and $A = \frac{1}{2}a^2$ so that eq. (80) corresponds to $\xi_\lambda = 1/\sqrt{2}$ and $\eta_\lambda = 1$.

We can again repeat the above analysis for the square guide. Note that the mode of the triangular guide given in eq. (79) is also a mode of the square guide. Thus, we only need to make minor modifications of the above computations. First, we note that

$$\int_A |B_z|^2\,da = a^2 E_0^2\,,$$

since the area of integration is twice as large for the square guide. Next, we obtain

$$\oint_C |B_z|^2\, d\ell = 2E_0^2 \left\{ \int_0^a \left[\cos^2\left(\frac{\pi x}{a}\right) + 2\cos\left(\frac{\pi x}{a}\right) + 1\right] dx + \int_0^a \left[\cos^2\left(\frac{\pi y}{a}\right) + 2\cos\left(\frac{\pi y}{a}\right) + 1\right] dy \right\} = 6E_0^2 a\,,$$

and

$$\oint_C |\hat{\boldsymbol{n}} \times \vec{\boldsymbol{\nabla}}_t B_z|^2\, dx = \int_0^a dx \left|\frac{\partial B_z}{\partial x}\right|^2_{y=0} + \int_0^a dy \left|\frac{\partial B_z}{\partial y}\right|^2_{x=a} + \int_0^a dx \left|\frac{\partial B_z}{\partial x}\right|^2_{y=a} + \int_0^a dy \left|\frac{\partial B_z}{\partial y}\right|^2_{x=0} = \frac{2\pi^2 E_0^2}{a}\,.$$

Hence eq. (71) yields

$$\begin{aligned} \beta_\lambda &= \sqrt{\frac{\epsilon_0}{\mu_0}} \left(\frac{1}{2a\sigma\delta_\lambda}\right) \frac{(\omega/\omega_\lambda)^{1/2}}{(1-\omega_\lambda^2/\omega^2)^{1/2}} \left[2\left(1 - \frac{\omega_\lambda^2}{\omega^2}\right) + \frac{6\omega_\lambda^2}{\omega^2}\right] \\ &= \sqrt{\frac{\epsilon_0}{\mu_0}} \left(\frac{1}{a\sigma\delta_\lambda}\right) \frac{(\omega/\omega_\lambda)^{1/2}}{(1-\omega_\lambda^2/\omega^2)^{1/2}} \left[1 + \frac{2\omega_\lambda^2}{\omega^2}\right]. \qquad (81) \end{aligned}$$

Comparing with eq. (8.63) of Jackson, we see that $C = 4a$ and $A = a^2$ so that eq. (81) corresponds to $\xi_\lambda = \frac{1}{2}$ and $\eta_\lambda = 1$. Thus, the attenuation of the TE mode with $(m, n) = (1, 0)$ of the triangular guide is larger than that of the square guide for all values of the frequency.

5. [Jackson, problem 8.6] A resonant cavity of copper consists of a hollow, right circular cylinder of inner radius R and length L, with flat end faces.

(a) Determine the resonant frequencies of the cavity for all types of waves. With $(1/\sqrt{\mu\epsilon}\,R)$ as a unit of frequency, plot the lowest four resonant frequencies of each type as a function of R/L for $0 < R/L < 2$. Does the same mode have the lowest frequency for all R/L?

The resonant frequencies for TM modes are given by eq. (8.81) of Jackson. Defining the unit of frequency by

$$\omega_0 \equiv \frac{1}{\sqrt{\mu\epsilon}\,R}\,, \qquad (82)$$

the resonant frequencies are given by:

$$\omega_{mnp} = \omega_0 \sqrt{x_{mn}^2 + \frac{p^2\pi^2 R^2}{L^2}}\,, \qquad \text{where } m, p = 0, 1, 2, 3, \ldots \text{ and } n = 1, 2, 3, \ldots,$$

and x_{mn} is the nth zero of J_m. A plot of the lowest lying resonant TM mode frequencies as a function of R/L is shown in Figure 2.

For $p = 0$, we have (independent of the value of R/L),

$$\frac{\omega_{mn0}}{\omega_0} = x_{mn}\,; \qquad x_{01} = 2.405\,,\; x_{11} = 3.832\,,\; x_{21} = 5.136\,,\; x_{02} = 5.520\,,\; \ldots\,.$$

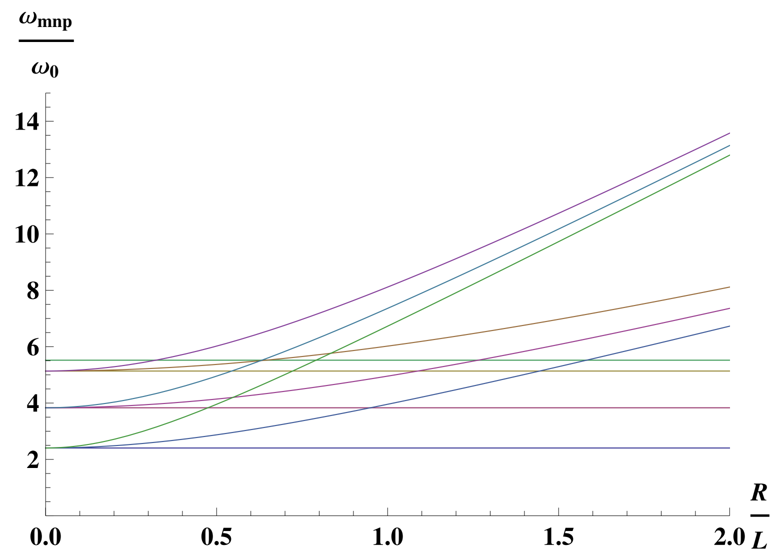

Figure 2: The lowest lying resonant frequencies for $p = 0$ (TM_{010}, TM_{110}, TM_{210}, TM_{020}), $p = 1$ (TM_{011}, TM_{111}, TM_{211}) and $p = 2$ (TM_{012}, TM_{112}, TM_{212}).

For $p = 1$, we have

$$\begin{aligned}
\frac{\omega_{011}}{\omega_0} &= \sqrt{(2.405)^2 + \pi^2 R^2/L^2}\,; & 2.405 < \frac{\omega_{011}}{\omega_0} &< 6.728 \text{ for } 0 < R/L < 2\,,\\
\frac{\omega_{111}}{\omega_0} &= \sqrt{(3.832)^2 + \pi^2 R^2/L^2}\,; & 3.832 < \frac{\omega_{111}}{\omega_0} &< 7.360 \text{ for } 0 < R/L < 2\,,\\
\frac{\omega_{211}}{\omega_0} &= \sqrt{(5.136)^2 + \pi^2 R^2/L^2}\,; & 5.136 < \frac{\omega_{211}}{\omega_0} &< 8.115 \text{ for } 0 < R/L < 2\,.
\end{aligned}$$

For $p = 2$, we have

$$\begin{aligned}
\frac{\omega_{012}}{\omega_0} &= \sqrt{(2.405)^2 + 4\pi^2 R^2/L^2}\,; & 2.405 < \frac{\omega_{012}}{\omega_0} &< 12.795 \text{ for } 0 < R/L < 2\,,\\
\frac{\omega_{112}}{\omega_0} &= \sqrt{(3.832)^2 + 4\pi^2 R^2/L^2}\,; & 3.832 < \frac{\omega_{112}}{\omega_0} &< 13.138 \text{ for } 0 < R/L < 2\,,\\
\frac{\omega_{212}}{\omega_0} &= \sqrt{(5.136)^2 + 4\pi^2 R^2/L^2}\,; & 5.136 < \frac{\omega_{212}}{\omega_0} &< 13.576 \text{ for } 0 < R/L < 2\,.
\end{aligned}$$

The TM_{010} mode with frequency $\omega_{010} \simeq 2.405\,\omega_0$ is the lowest frequency for all values of $0 < R/L < 2$.

The resonant frequencies for TE modes is are given by eq. (8.83) of Jackson,

$$\omega_{mnp} = \omega_0 \sqrt{y_{mn}^2 + \frac{p^2\pi^2 R^2}{L^2}}\,, \qquad \text{where } m = 0, 1, 2, 3, \ldots \text{ and } n, p = 1, 2, 3, \ldots,$$

and y_{mn} is the nth zero of J_m'. Note that there are no resonances corresponding to $p = 0$ due to the boundary conditions $(\partial H_z/\partial z)_{z=0} = (\partial H_z/\partial z)_{z=L} = 0$ on the endcaps of the cylinder. A plot of the lowest lying resonant TE mode frequencies as a function of R/L is shown in Figure 3.

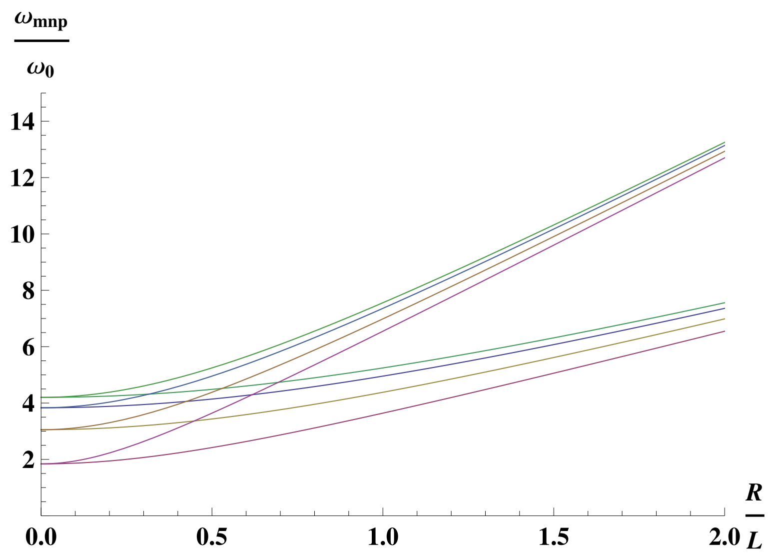

Figure 3: The lowest lying resonant frequencies for $p = 1$ (TE_{111}, TE_{211}, TE_{011}, TE_{311}) and $p = 2$ (TE_{112}, TE_{212}, TE_{012}, TE_{312}).

For $p = 1$, we have

$$\frac{\omega_{011}}{\omega_0} = \sqrt{(3.832)^2 + \pi^2 R^2/L^2}\,; \qquad 3.832 < \frac{\omega_{011}}{\omega_0} < 7.360 \text{ for } 0 < R/L < 2\,,$$
$$\frac{\omega_{111}}{\omega_0} = \sqrt{(1.841)^2 + \pi^2 R^2/L^2}\,; \qquad 1.841 < \frac{\omega_{111}}{\omega_0} < 6.547 \text{ for } 0 < R/L < 2\,,$$
$$\frac{\omega_{211}}{\omega_0} = \sqrt{(3.054)^2 + \pi^2 R^2/L^2}\,; \qquad 3.054 < \frac{\omega_{211}}{\omega_0} < 6.986 \text{ for } 0 < R/L < 2\,.$$
$$\frac{\omega_{311}}{\omega_0} = \sqrt{(4.201)^2 + \pi^2 R^2/L^2}\,; \qquad 4.201 < \frac{\omega_{311}}{\omega_0} < 7.558 \text{ for } 0 < R/L < 2\,.$$

For $p = 2$, we have

$$\frac{\omega_{012}}{\omega_0} = \sqrt{(3.832)^2 + 4\pi^2 R^2/L^2}\,; \qquad 3.832 < \frac{\omega_{012}}{\omega_0} < 13.138 \text{ for } 0 < R/L < 2\,,$$
$$\frac{\omega_{112}}{\omega_0} = \sqrt{(1.841)^2 + 4\pi^2 R^2/L^2}\,; \qquad 1.841 < \frac{\omega_{112}}{\omega_0} < 12.702 \text{ for } 0 < R/L < 2\,,$$
$$\frac{\omega_{212}}{\omega_0} = \sqrt{(3.054)^2 + 4\pi^2 R^2/L^2}\,; \qquad 3.054 < \frac{\omega_{212}}{\omega_0} < 12.932 \text{ for } 0 < R/L < 2\,.$$
$$\frac{\omega_{312}}{\omega_0} = \sqrt{(4.201)^2 + 4\pi^2 R^2/L^2}\,; \qquad 4.201 < \frac{\omega_{312}}{\omega_0} < 13.250 \text{ for } 0 < R/L < 2\,.$$

The TE_{111} mode with frequency $\omega_{111} = \omega_0\sqrt{(1.841)^2 + \pi^2 R^2/L^2}$, is the lowest frequency for all values of $0 < R/L < 2$.

(b) If $R = 2$ cm, $L = 3$ cm, and the cavity is made of pure copper, what is the numerical value of Q for the lowest resonant mode?

For $R = 2$ cm, $L = 3$ cm, we have $R/L = \frac{2}{3}$. The lowest TE mode has a frequency

$$\omega_{111} = \omega_0 \sqrt{(1.841)^2 + 4\pi^2/9} \simeq 4.068\,\omega_0\,,$$

which is larger than the lowest TM mode with frequency $\omega_{010} \simeq 2.405\,\omega_0$. Thus, we focus on the TM_{010} mode, where the longitudinal electric field is given by:

$$E_z(r, \phi, z, t) = E_0\, J_0\left(\frac{x_{01} r}{R}\right) e^{-i\omega t}\,.$$

In order to compute the quality factor, we must first evaluate ξ, which is defined in eq. (8.62) of Jackson. But, we have already computed this in part (b) of problem 4, where we found that $\xi_{01} = 1$ [cf. the text following eq. (55)]. Using eq (8.95) of Jackson and noting the modification for $p = 0$ that is discussed following this equation, the quality factor is given by

$$Q = \frac{\mu L}{\mu_c \delta}\left(\frac{1}{1 + \frac{1}{2}\xi_{01} CL/A}\right)\,. \tag{83}$$

Putting $\xi_{01} = 1$, $A = \pi R^2$ and $C = 2\pi R$ in eq. (83), it follows that

$$Q = \frac{\mu L R}{\mu_c \delta (L + R)}\,. \tag{84}$$

Note that the volume of the cylinder is $V = \pi R^2 L$ and the total surface area is

$$S = 2\pi R L + 2\pi R^2\,.$$

It follows that

$$\frac{V}{S} = \frac{LR}{2(L + R)}\,.$$

Thus, the quality factor can be re-expressed as

$$Q = \frac{2\mu}{\mu_c}\left(\frac{V}{S\delta}\right)\,.$$

Comparing this result with eq. (8.96) of Jackson, we conclude that for the cylindrical cavity, the geometrical factor is 2.

We now plug in the numbers. For copper at room temperature, $\sigma^{-1} = 1.68 \times 10^{-8}\ \Omega \cdot \mathrm{m}$ [cf. Jackson p. 220 below eq. (5.165) or Wikipedia], or equivalently $\sigma = 5.96 \times 10^7$ S/m, where the SI unit siemens is equivalent to an inverse-ohm (which was called a mho when I was a student at MIT!). Since the resonant cavity is hollow, $\mu = \mu_0 = 4\pi \times 10^{-7}$ H/m and $\epsilon = \epsilon_0$. Wikipedia provides the following numbers for copper: $\mu_c = 1.2566290 \times 10^{-6}$ H/m, or when normalized to the vacuum permeability, $\mu_c/\mu_0 = 0.999994$.

The unit of frequency defined in eq. (82) is

$$\omega_0 = \frac{1}{\sqrt{\epsilon_0 \mu_0}\, R} = \frac{c}{R} = \frac{3 \times 10^8\ \mathrm{m \cdot s^{-1}}}{2 \times 10^{-2}\ \mathrm{m}} = 1.5 \times 10^{10}\ \mathrm{s}^{-1}\,.$$

Thus, the frequency of the TM_{010} mode is

$$\omega_{010} = 2.405\,\omega_0 = 3.61 \times 10^{10}\ \mathrm{s}^{-1}\,.$$

The corresponding skin depth is[10]

$$\delta = \left(\frac{2}{\mu_c\,\omega_{010}\,\sigma}\right)^{1/2} = \left(\frac{2(1.68 \times 10^{-8}\ \Omega \cdot \mathrm{m})}{(1.2566290 \times 10^{-6}\ \Omega \cdot \mathrm{s} \cdot \mathrm{m}^{-1})(3.61 \times 10^{10}\ \mathrm{s}^{-1})}\right)^{1/2} = 8.6 \times 10^{-7}\ \mathrm{m}\,.$$

Note that we can obtain the same result for δ by using the result quoted by Jackson below eq. (5.165) on p. 220 for the skin depth of copper at room temperature,

$$\delta = \frac{6.52 \times 10^{-2}\ \mathrm{m}}{\sqrt{\nu\ (\mathrm{Hz})}}\,, \qquad \text{where } \omega = 2\pi\nu\,.$$

Thus, for $\nu = \omega_{010}/(2\pi) = 5.745 \times 10^9$ Hz,

$$\delta = \frac{6.52 \times 10^{-2}\ \mathrm{m}}{\sqrt{5.745 \times 10^9}} = 8.6 \times 10^{-7}\ \mathrm{m}\,,$$

which confirms our previous computation. Finally, we plug into eq. (84) to obtain

$$Q = \frac{\mu L R}{\mu_c \delta (L+R)} = \frac{(2\ \mathrm{cm})(3\ \mathrm{cm})}{(0.999994)(8.6 \times 10^{-5}\ \mathrm{cm})(5\ \mathrm{cm})} = 1.4 \times 10^4\,.$$

[10] Note that the SI unit henry can be equivalently expressed as $1\ \mathrm{H} = 1\ \Omega \cdot \mathrm{s}$.