1. [Jackson, problem 9.2] A radiating quadrupole consists of a square of side a with charges $\pm q$ at alternate corners. The square rotates with angular velocity ω about an axis normal to the plane of the square and through its center. Calculate the quadrupole moments, the radiation fields, the angular distribution of radiation, and the total radiated power, all in the long-wavelength approximation. What is the frequency of the radiation?

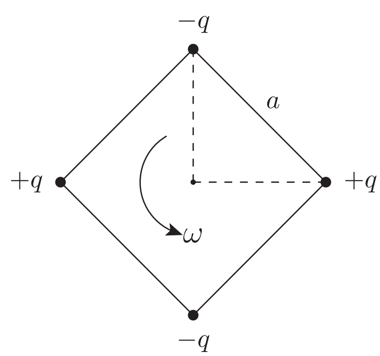

Figure 1: A radiating quadrupole consisting of a square of side a with charges $\pm q$ at alternate corners.

The charge distribution consists of point charges at the four corners of a square in the x–y plane, as depicted in Figure 1. The charges are located at the following positions in Cartesian coordinates,

$$+q\ :\quad \frac{a}{\sqrt{2}}\,(\cos\omega t\,,\,\sin\omega t\,,\,0)\ ,\qquad\qquad +q\ :\quad -\frac{a}{\sqrt{2}}\,(\cos\omega t\,,\,\sin\omega t\,,\,0)\ ,\tag{1}$$

$$-q\ :\quad \frac{a}{\sqrt{2}}\,(\sin\omega t\,,\,-\cos\omega t\,,\,0)\ ,\qquad\qquad -q\ :\quad \frac{a}{\sqrt{2}}\,(-\sin\omega t\,,\,\cos\omega t\,,\,0)\ .\tag{2}$$

The quadrupole moment Cartesian tensor is given by[1]

$$Q_{ij} = \sum_k q_k\left[3x_i^{(k)}x_j^{(k)} - (r^{(k)})^2\delta_{ij}\right]\,,\tag{3}$$

where k labels each charge and i and j label the components of the position vector $\vec{\boldsymbol{x}}$. Note that the charges all lie in the x–y plane (corresponding to $z=0$). Moreover, $r^{(k)}$ is the distance of the kth charge from the origin, located at the center of the square. Hence,

$$(r^{(k)})^2 = (x_1^{(k)})^2 + (x_2^{(k)})^2 + (x_3^{(k)})^2 = \tfrac{1}{2}a^2\,,\qquad \text{for all } k\,.$$

This means that

$$\sum_k q_k(r^{(k)})^2 = \tfrac{1}{2}a^2\sum_k q_k = 0\,,$$

[1]If one uses eq. (4.9) of Jackson, then one should express the charge distribution $\rho(\vec{\boldsymbol{x}},t)$ as a sum of delta functions, whose arguments vanish at the locations of the four charges. Integrating over all space then yields eq. (3).

since there are an equal number of positive and negative charges. Plugging in the location of the four charges in eq. (3), we obtain:

$$Q_{13} = Q_{23} = Q_{33} = 0\,, \qquad Q_{11} = \tfrac{3}{2}\cdot 2\,a^2 q\left[\cos^2\omega t - \sin^2\omega t\right] = 3a^2 q\cos 2\omega t\,,$$
$$Q_{22} = -3a^2 q\cos 2\omega t\,, \qquad Q_{12} = \tfrac{3}{2}\cdot 4\,a^2 q\sin\omega t\cos\omega t = 3a^2 q\sin 2\omega t\,,$$

after employing some well known trigonometric identities. Thus, the electric quadrupole tensor is given by

$$\mathcal{Q}_{ij}(t) = 3a^2 q\begin{pmatrix}\cos 2\omega t & \sin 2\omega t & 0\\ \sin 2\omega t & -\cos 2\omega t & 0\\ 0 & 0 & 0\end{pmatrix}\,. \tag{4}$$

In a Cartesian basis, all the elements of the physical multipole tensors are real. We can introduce the complex time-dependent multipole tensor $Q_{ij}(t)$ by defining

$$Q_{ij}(t) = 3a^2 q\, e^{-2i\omega t}\begin{pmatrix}1 & i & 0\\ i & -1 & 0\\ 0 & 0 & 0\end{pmatrix}\,. \tag{5}$$

One can check that the physical quadrupole Cartesian tensor is given by

$$\mathcal{Q}_{ij}(t) = \operatorname{Re}\left[Q_{ij}(t)\right]\,. \tag{6}$$

To make contact with the convention for harmonic sources employed by eq. (9.1) of Jackson, we note that the complex electric quadrupole tensor is defined to be

$$Q_{ij}(t) = \int (3x_i x_j - r^2\delta_{ij})\rho(\vec{\boldsymbol{x}},t)\,d^3x\,,$$

where $\rho(\vec{\boldsymbol{x}},t) = \rho(\vec{\boldsymbol{x}})\,e^{-i\omega t}$. However, this would not yield the correct time dependence exhibited in eq. (5). However, the solution is simple—we write:

$$\rho(\vec{\boldsymbol{x}},t) = \rho(\vec{\boldsymbol{x}})\,e^{-2i\omega t}\,.$$

That is, one must replace ω with 2ω in the formulae that appear in Chapter 9 of Jackson. Note that this also implies that

$$k = \frac{2\omega}{c}\,. \tag{7}$$

Thus, it follows that

$$Q_{ij}(t) = Q_{ij}\,e^{-2i\omega t}\,, \tag{8}$$

where

$$Q_{ij} = \int (3x_i x_j - r^2\delta_{ij})\rho(\vec{\boldsymbol{x}})\,d^3x = 3a^2 q\begin{pmatrix}1 & i & 0\\ i & -1 & 0\\ 0 & 0 & 0\end{pmatrix}\,. \tag{9}$$

Using eq. (9), the matrix $\vec{\boldsymbol{Q}}$ whose components are defined by

$$Q_i = \sum_{j=1}^{3} Q_{ij}\hat{\boldsymbol{n}}_j\,, \tag{10}$$

are easily evaluated. Using $\hat{\boldsymbol{n}} = (\sin\theta\cos\phi\,,\,\sin\theta\sin\phi\,,\,\cos\theta)$, we find

$$Q_1 = 3a^2 q \sin\theta\, e^{i\phi}\,, \qquad Q_2 = 3a^2 qi \sin\theta\, e^{i\phi}\,, \qquad Q_3 = 0\,.$$

That is,

$$\vec{\boldsymbol{Q}} = 3a^2 q \sin\theta\, e^{i\phi}(\hat{\boldsymbol{x}} + i\hat{\boldsymbol{y}})\,. \tag{11}$$

We can now employ eqs. (7) and (11) in eqs. (9.44), (9.45) and (9.49) of Jackson. First we evaluate

$$\hat{\boldsymbol{n}} \times \vec{\boldsymbol{Q}} = 3a^2 q \sin\theta\, e^{i\phi} \det \begin{pmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ 1 & i & 0 \end{pmatrix}$$

$$= -3a^2 qi \sin\theta\, e^{i\phi}\left[\cos\theta(\hat{\boldsymbol{x}} + i\hat{\boldsymbol{y}}) - \sin\theta\, e^{i\phi}\hat{\boldsymbol{z}}\right]\,.$$

Therefore, eq. (9.44) of Jackson [in SI units] yields

$$\vec{\boldsymbol{H}} = -\frac{ck^3}{8\pi}\,\frac{e^{ikr}}{r}\, a^2 q \sin\theta\, e^{i\phi}\left[\cos\theta(\hat{\boldsymbol{x}} + i\hat{\boldsymbol{y}}) - \sin\theta\, e^{i\phi}\hat{\boldsymbol{z}}\right]\,.$$

The physical magnetic fields are then given by

$$\begin{aligned} \mathrm{Re}(\vec{\boldsymbol{H}}\, e^{-2i\omega t}) = -\frac{ck^3 a^2 q}{8\pi r} \sin\theta \Big\{ & \hat{\boldsymbol{x}} \cos\theta\cos(kr - 2\omega t + \phi) - \hat{\boldsymbol{y}}\cos\theta\sin(kr - 2\omega t + \phi) \\ & -\hat{\boldsymbol{z}}\sin\theta\cos(kr - 2\omega t + 2\phi)\Big\}\,. \end{aligned} \tag{12}$$

The electric fields are obtained by using eq. (9.39) of Jackson. In SI units,

$$\vec{\boldsymbol{E}} = Z_0 \vec{\boldsymbol{H}} \times \hat{\boldsymbol{n}}\,, \tag{13}$$

where $Z_0 = \sqrt{\mu_0/\epsilon_0}$ is the impedance of free space. Thus,

$$\vec{\boldsymbol{E}} = -\frac{Z_0 k^3}{8\pi}\,\frac{e^{ikr}}{r}\, a^2 q \sin\theta\, e^{i\phi} \det \begin{pmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \cos\theta & i\cos\theta & -\sin\theta\, e^{i\phi} \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{pmatrix}$$

$$\begin{aligned} = -\frac{Z_0 k^3}{8\pi}\,\frac{e^{ikr}}{r}\, a^2 q \sin\theta\, e^{i\phi} \Big\{ & \hat{\boldsymbol{x}}(\sin^2\theta\sin\phi\, e^{i\phi} + i\cos^2\theta) \\ & -\hat{\boldsymbol{y}}(\sin^2\theta\cos\phi\, e^{i\phi} + \cos^2\theta) - i\hat{\boldsymbol{z}}\sin\theta\cos\theta\, e^{i\phi}\Big\}\,. \end{aligned}$$

The physical electric fields are then given by

$$\begin{aligned} \mathrm{Re}(\vec{\boldsymbol{E}}\, e^{-2i\omega t}) = -\frac{Z_0 k^3 a^2 q}{8\pi r} \sin\theta \Big\{ & \hat{\boldsymbol{x}}\left[\sin^2\theta\, \sin\phi\, \cos(kr - 2\omega t + 2\phi) - \cos^2\theta\, \sin(kr - 2\omega t + \phi)\right] \\ & -\hat{\boldsymbol{y}}\left[\sin^2\theta\, \cos\phi\, \cos(kr - 2\omega t + 2\phi) + \cos^2\theta\, \cos(kr - 2\omega t + \phi)\right] \\ & +\hat{\boldsymbol{z}}\, \sin\theta\, \cos\theta\, \sin(kr - 2\omega t + 2\phi)\Big\}\,. \end{aligned} \tag{14}$$

As a check, it is easy to verify that eq. (13) is also satisfied by the physical fields given in eqs. (12) and (14).

Next, we compute the time-averaged power radiated per unit solid angle. Using eqs. (9.45) and (9.46) of Jackson,

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{1152\pi^2}\, k^6 \left[\vec{\boldsymbol{Q}}^{\,*}\cdot\vec{\boldsymbol{Q}} - |\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{Q}}|^2\right]\,, \tag{15}$$

Using eq. (11), we compute:

$$\vec{\boldsymbol{Q}}^{\,*}\cdot\vec{\boldsymbol{Q}} = 18a^4q^2\sin^2\theta\,, \qquad\qquad |\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{Q}}|^2 = |3a^2q\sin^2\theta\, e^{2i\phi}|^2 = 9a^4q^2\sin^2\theta\,.$$

Hence,

$$\vec{\boldsymbol{Q}}^{\,*}\cdot\vec{\boldsymbol{Q}} - |\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{Q}}|^2 = 9a^4q^2\sin^2\theta(2-\sin^2\theta) = 9a^4q^2\sin^2\theta(1+\cos^2\theta)\,.$$

Eq. (15) then yields

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0 a^4 q^2 k^6}{128\pi^2}\sin^2\theta(1+\cos^2\theta)\,.$$

Using $k = 2\omega/c$ [cf. eq. (7)], we obtain

$$\frac{dP}{d\Omega} = \frac{Z_0 a^4 q^2 \omega^6}{2\pi^2 c^4}\sin^2\theta(1+\cos^2\theta)\,. \tag{16}$$

The total radiated power is obtained by integrating over solid angles. Using

$$\int d\Omega\,\sin^2\theta(1+\cos^2\theta) = 2\pi\int_{-1}^{1}(1-\cos^4\theta)\,d\cos\theta = \frac{16\pi}{5}\,,$$

we end up with

$$P = \frac{8Z_0 a^4 q^2\omega^6}{5\pi c^4}\,. \tag{17}$$

As a check, we can use eq. (9.49) of Jackson,

$$P = \frac{c^2 Z_0 k^6}{1440\pi}\sum_{i,j}|Q_{ij}|^2\,. \tag{18}$$

Using eq. (9)

$$\sum_{i,j}|Q_{ij}|^2 = 36a^4q^2\,.$$

Inserting this back into eq. (18) along with $k = 2\omega/c$, we recover eq. (17) as expected.

<u>*ALTERNATIVE SOLUTION:*</u>

In class, I showed that for general time dependent charges and current, the physical (real) magnetic and electric fields of E2 radiation are given by

$$\vec{\boldsymbol{H}}_{\rm E2}(\vec{\boldsymbol{x}},t) = -\frac{1}{24\pi c^2 r}\,\hat{\boldsymbol{n}}\times\frac{\partial^3}{\partial t^3}\vec{\boldsymbol{Q}}\left(t-\frac{r}{c}\right)\,, \tag{19}$$

$$\vec{\boldsymbol{E}}_{\rm E2}(\vec{\boldsymbol{x}},t) = Z_0\vec{\boldsymbol{H}}_{\rm E2}(\vec{\boldsymbol{x}},t)\times\hat{\boldsymbol{n}}\,, \tag{20}$$

after converting the formulae given in class from gaussian to SI units. The time-averaged power radiated per unit solid angle is given by

$$\frac{dP}{d\Omega} = r^2 \vec{\boldsymbol{S}} \boldsymbol{\cdot} \hat{\boldsymbol{n}}\,, \tag{21}$$

where the Poynting vector is given by eq. (6.109) of Jackson, $\vec{\boldsymbol{S}} = \vec{\boldsymbol{E}} \times \vec{\boldsymbol{H}}$. Using eqs. (19)–(21),

$$\begin{aligned}
\frac{dP}{d\Omega} &= \frac{Z_0}{576\pi^2 c^4} \hat{\boldsymbol{n}} \boldsymbol{\cdot} \left\{ \left[\hat{\boldsymbol{n}} \times \frac{\partial^3}{\partial t^3} \vec{\boldsymbol{Q}} \left(t - \frac{r}{c} \right) \right] \times \hat{\boldsymbol{n}} \right\} \times \left[\hat{\boldsymbol{n}} \times \frac{\partial^3}{\partial t^3} \vec{\boldsymbol{Q}} \left(t - \frac{r}{c} \right) \right] \\
&= -\frac{Z_0}{576\pi^2 c^4} \hat{\boldsymbol{n}} \boldsymbol{\cdot} \left\{ \hat{\boldsymbol{n}} \frac{\partial^3}{\partial t^3} \hat{\boldsymbol{n}} \boldsymbol{\cdot} \vec{\boldsymbol{Q}} \left(t - \frac{r}{c} \right) - \frac{\partial^3}{\partial t^3} \vec{\boldsymbol{Q}} \left(t - \frac{r}{c} \right) \right\} \times \left[\hat{\boldsymbol{n}} \times \frac{\partial^3}{\partial t^3} \vec{\boldsymbol{Q}} \left(t - \frac{r}{c} \right) \right] \\
&= \frac{Z_0}{576\pi^2 c^4} \left\{ \left| \frac{\partial^3}{\partial t^3} \vec{\boldsymbol{Q}} \left(t - \frac{r}{c} \right) \right|^2 - \left[\frac{\partial^3}{\partial t^3} \hat{\boldsymbol{n}} \boldsymbol{\cdot} \vec{\boldsymbol{Q}} \left(t - \frac{r}{c} \right) \right]^2 \right\}.
\end{aligned} \tag{22}$$

In the above formula, the components of the real vector $\vec{\boldsymbol{Q}}(t)$ are given by eq. (10), where $Q_{ij}(t)$ is given by eq. (4). Using spherical coordinates, $\hat{\boldsymbol{n}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ and

$$\begin{aligned}
Q_1(t) &= Q_{11} n_1 + Q_{12} n_2 + n_3 Q_{13} n_3 = 3a^2 q \sin\theta \big(\cos\phi \cos 2\omega t + \sin\phi \sin 2\omega t \big) \\
&= 3a^2 q \sin\theta \cos(2\omega t - \phi)\,, \\
Q_2(t) &= Q_{21} n_1 + Q_{22} n_2 + Q_{23} n_3 = 3a^2 q \sin\theta \big(\cos\phi \sin 2\omega t - \sin\phi \cos 2\omega t \big) \\
&= 3a^2 q \sin\theta \sin(2\omega t - \phi)\,, \\
Q_3(t) &= Q_{31} n_1 + Q_{32} n_2 + Q_{33} n_3 = 0\,.
\end{aligned}$$

Thus,

$$\begin{aligned}
\hat{\boldsymbol{n}} \boldsymbol{\cdot} \vec{\boldsymbol{Q}}(t) &= 3a^2 q \sin^2\theta \big[\cos\phi \cos(2\omega t - \phi) + \sin\phi \sin(2\omega t - \phi) \big] = 3a^2 q \sin^2\theta \cos\big(2(\omega t - \phi)\big)\,, \\
\frac{\partial^3}{\partial t^3} \hat{\boldsymbol{n}} \boldsymbol{\cdot} \vec{\boldsymbol{Q}}(t) &= 24\omega^3 a^2 q \sin^2\theta \sin\big(2(\omega t - \phi)\big)\,, \\
\frac{\partial^3}{\partial t^3} \vec{\boldsymbol{Q}}(t) &= 24\omega^3 a^2 q \sin\theta \big(\sin(2\omega t - \phi)\,,\, -\cos(2\omega t - \phi)\,,\, 0 \big)\,.
\end{aligned}$$

It then follows that

$$\left| \frac{\partial^3}{\partial t^3} \vec{\boldsymbol{Q}} \left(t - \frac{r}{c} \right) \right|^2 - \left[\frac{\partial^3}{\partial t^3} \hat{\boldsymbol{n}} \boldsymbol{\cdot} \vec{\boldsymbol{Q}} \left(t - \frac{r}{c} \right) \right]^2 = 576\omega^6 a^4 q^2 \sin^2\theta \left\{ 1 - \sin^2\theta \cos^2 \left(2 \left[\omega \left(t - \frac{r}{c} \right) - \phi \right] \right) \right\}. \tag{23}$$

Averaging over one cycle, we obtain,

$$\left\langle \cos^2 \left(2 \left[\omega \left(t - \frac{r}{c} \right) - \phi \right] \right) \right\rangle = \frac{1}{2}\,. \tag{24}$$

Hence,

$$\frac{d\langle P \rangle}{d\Omega} = \frac{Z_0 \omega^6 a^4 q^2 \sin^2\theta \big(1 - \frac{1}{2}\sin^2\theta \big)}{\pi^2 c^4} = \frac{Z_0 \omega^6 a^4 q^2 \sin^2\theta \big(1 + \cos^2\theta \big)}{2\pi^2 c^4}\,, \tag{25}$$

which reproduces the result previously obtained in eq. (16). Note that in our first derivation above, by using the complex version of $Q_{ij}(t)$ given in eq. (5) along with the complex Poynting vector, one automatically obtains the time-averaged power using eq. (15).

2. [Jackson, problem 9.7]

(a) By means of Fourier superposition of different frequencies or equivalent means, show for a real electric dipole $\vec{\boldsymbol{p}}(t)$ that the instantaneous radiated power per unit solid angle at a distance r from the dipole in a direction $\hat{\boldsymbol{n}}$ is

$$\frac{dP(t)}{d\Omega} = \frac{Z_0}{16\pi^2 c^2}\left|\left[\hat{\boldsymbol{n}} \times \frac{d^2\vec{\boldsymbol{p}}}{dt_0^2}(t_0)\right] \times \hat{\boldsymbol{n}}\right|^2,$$

where $t_0 \equiv t - r/c$ is the retarded time (in the limit of large r).[2] For a magnetic dipole moment $\vec{\boldsymbol{m}}(t)$, substitute $(1/c)\ddot{\vec{\boldsymbol{m}}} \times \hat{\boldsymbol{n}}$ for $(\hat{\boldsymbol{n}} \times \ddot{\vec{\boldsymbol{p}}}) \times \hat{\boldsymbol{n}}$.

I will solve this problem by starting with the Jefimenko equation for the magnetic field.[3] The radiation fields behave as $\mathcal{O}(1/r)$ as $r \equiv |\vec{\boldsymbol{x}}| \to \infty$. Thus, eq. (6.56) of Jackson yields

$$\vec{\boldsymbol{H}}(\vec{\boldsymbol{x}}, t) = \frac{1}{4\pi c}\int \frac{d^3x'}{|\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}\,'|}\left[\frac{\partial \vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,', t')}{\partial t'}\right]_{\rm ret} \times \hat{\boldsymbol{R}} + \mathcal{O}\left(\frac{1}{r^2}\right), \tag{26}$$

after using $\vec{\boldsymbol{B}} = \mu_0 \vec{\boldsymbol{H}}$, where $\vec{\boldsymbol{R}} \equiv \vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}\,'$, $\hat{\boldsymbol{R}} \equiv \vec{\boldsymbol{R}}/R$ and $R \equiv |\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}\,'|$. Note that "ret" is an instruction to substitute $t' = t - R/c$ after evaluating the derivative.

In the limit of large r, we have $\hat{\boldsymbol{R}} = \hat{\boldsymbol{n}} + \mathcal{O}(1/r)$, where $\hat{\boldsymbol{n}} \equiv \vec{\boldsymbol{x}}/r$. Hence, we can rewrite eq. (26) as

$$\vec{\boldsymbol{H}}(\vec{\boldsymbol{x}}, t) = -\frac{1}{4\pi c r}\hat{\boldsymbol{n}} \times \int d^3x' \left[\frac{\partial \vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,', t')}{\partial t'}\right]_{\rm ret} + \mathcal{O}\left(\frac{1}{r^2}\right). \tag{27}$$

We can simplify eq. (27) slightly with the help of the chain rule:

$$\frac{\partial \vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,', t - R/c)}{\partial t} = \left[\frac{\partial \vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,', t')}{\partial t'}\right]_{\rm ret}\frac{\partial t'}{\partial t}.$$

All partial derivatives above are to be taken holding $\vec{\boldsymbol{x}}\,'$ fixed. Consequently, $\partial t'/\partial t = 1$, and we can therefore write

$$\vec{\boldsymbol{H}}(\vec{\boldsymbol{x}}, t) = -\frac{1}{4\pi c r}\hat{\boldsymbol{n}} \times \frac{\partial}{\partial t}\int d^3x'\, \vec{\boldsymbol{J}}\left(\vec{\boldsymbol{x}}\,', t - \frac{|\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}\,'|}{c}\right) + \mathcal{O}\left(\frac{1}{r^2}\right). \tag{28}$$

In class, we showed that the electric field is given by eq. (9.19) of Jackson. That is,[4]

$$\vec{\boldsymbol{E}} = Z_0 \vec{\boldsymbol{H}} \times \hat{\boldsymbol{n}} + \mathcal{O}\left(\frac{1}{r^2}\right), \tag{29}$$

[2] I have changed Jackson's notation by defining $t_0 = t - r/c$, since I prefer to reserve $t' = t - |\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}\,'|/c$ for the exact expression for the retarded time.

[3] An alternative approach based on Fourier superposition is given at the end of the solution to this problem.

[4] Eq. (29) is a consequence of the Jefimenko equation for the electric field. In class, we showed that by massaging eq. (6.55) of Jackson we could eliminate $\partial\rho/\partial t$ in favor of $\vec{\boldsymbol{J}}$. The end result (in SI units) is

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}, t) = \frac{1}{4\pi\epsilon_0 c^2}\int \frac{d^3x'}{|\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}\,'|}\left(\left[\frac{\partial \vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,', t')}{\partial t'}\right]_{\rm ret} \times \hat{\boldsymbol{R}}\right) \times \hat{\boldsymbol{R}} + \mathcal{O}\left(\frac{1}{r^2}\right).$$

Using $\hat{\boldsymbol{R}} = \hat{\boldsymbol{n}} + \mathcal{O}(1/r)$ and comparing with eq. (26), we see that eq. (29) follows after noting that $\epsilon_0 c = 1/Z_0$ (which is a consequence of $\epsilon_0\mu_0 = 1/c^2$).

where $Z_0 \equiv \sqrt{\mu_0/\epsilon_0}$ is the impedance of free space.. Note that the $\mathcal{O}(1/r)$ part of eq. (29) is an exact result, independently of the multipole expansion.

For real (time-dependent) fields, the equivalent of eq. (9.21) of Jackson is:[5]

$$\frac{dP(t)}{d\Omega} = \lim_{r\to\infty}\left(r^2\,\boldsymbol{\hat{n}}\cdot\vec{\boldsymbol{E}}\times\vec{\boldsymbol{H}}\right). \tag{30}$$

All we need to do is to keep the leading $\mathcal{O}(1/r)$ terms of the electric and magnetic fields. In particular, note that to leading order in $1/r$,

$$\boldsymbol{\hat{n}}\cdot\left[(\vec{\boldsymbol{H}}\times\boldsymbol{\hat{n}})\times\vec{\boldsymbol{H}}\right] = -\boldsymbol{\hat{n}}\cdot\left[\vec{\boldsymbol{H}}(\vec{\boldsymbol{H}}\cdot\boldsymbol{\hat{n}}) - \boldsymbol{\hat{n}}|\vec{\boldsymbol{H}}|^2\right] = |\vec{\boldsymbol{H}}|^2 - (\vec{\boldsymbol{H}}\cdot\boldsymbol{\hat{n}})^2 = |\vec{\boldsymbol{H}}\times\boldsymbol{\hat{n}}|^2\,.$$

Hence, eq. (30) yields

$$\frac{dP(t)}{d\Omega} = \lim_{r\to\infty} Z_0 r^2|\vec{\boldsymbol{H}}\times\boldsymbol{\hat{n}}|^2\,. \tag{31}$$

Note that only the $\mathcal{O}(1/r)$ terms of $\vec{\boldsymbol{H}}$ contribute in the $r\to\infty$ limit.

We evaluate $\vec{\boldsymbol{H}}$ given in eq. (28) by performing a Taylor series of $\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,', t - |\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}\,'|/c)$ in the limit of $r\to\infty$. Using eq. (9.7) of Jackson, $|\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}\,'| = r - \boldsymbol{\hat{n}}\cdot\vec{\boldsymbol{x}}\,' + \mathcal{O}(1/r)$. Therefore, it is convenient to define $t_0 \equiv t - r/c$ (which is the retarded time in the limit of large r), in which case,

$$t - \frac{|\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}\,'|}{c} = t_0 + \frac{\boldsymbol{\hat{n}}\cdot\vec{\boldsymbol{x}}\,'}{c} + \mathcal{O}\left(\frac{1}{r}\right).$$

In the limit of $r\to\infty$, we can expand t around t_0. Hence

$$\vec{\boldsymbol{J}}\left(\vec{\boldsymbol{x}}, t - \frac{|\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}\,'|}{c}\right) = \vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,', t_0) + \frac{\boldsymbol{\hat{n}}\cdot\vec{\boldsymbol{x}}\,'}{c}\frac{\partial\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,', t')}{\partial t'}\bigg|_{t'=t_0} + \ldots\,.$$

Inserting this result into eq. (28),

$$\vec{\boldsymbol{H}}(\vec{\boldsymbol{x}}, t) = -\frac{1}{4\pi cr}\boldsymbol{\hat{n}}\times\frac{\partial}{\partial t_0}\int d^3x'\left[\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,', t_0) + \frac{\boldsymbol{\hat{n}}\cdot\vec{\boldsymbol{x}}\,'}{c}\frac{\partial\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,', t')}{\partial t'}\bigg|_{t'=t_0} + \ldots\right]. \tag{32}$$

Note that we have replaced the partial derivative with respect to t with a partial derivative with respect to t_0. This is permissible since the partial derivatives are taken at fixed r, in which case,

$$\frac{\partial}{\partial t_0} = \frac{\partial}{\partial(t - r/c)} = \frac{\partial}{\partial t}\,. \tag{33}$$

The series exhibited in eq. (32) is the multipole expansion. The first term yields the electric dipole moment fields and the second term contains the magnetic dipole and electric quadrupole fields.

First, we focus on the electric dipole fields.

$$\vec{\boldsymbol{H}}_{\rm E1}(\vec{\boldsymbol{x}}, t) = -\frac{1}{4\pi cr}\boldsymbol{\hat{n}}\times\frac{\partial}{\partial t_0}\int d^3x'\,\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,', t_0)\,. \tag{34}$$

[5]Here, we use eq. (6.109) of Jackson, which gives $\vec{\boldsymbol{S}} = \vec{\boldsymbol{E}}\times\vec{\boldsymbol{H}}$.

We employ the trick similar to the one used in eqs. (9.14) and (9.15) of Jackson. In the present case, we employ the identity,

$$J_i = \partial'_k(J_k x'_i) - x'_i \vec{\boldsymbol{\nabla}}' \boldsymbol{\cdot} \vec{\boldsymbol{J}} = \partial'_k(J_k x'_i) + x'_i \frac{\partial \rho}{\partial t}\,,$$

after using the continuity equation,

$$\vec{\boldsymbol{\nabla}}' \boldsymbol{\cdot} \vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,', t_0) + \frac{\partial \rho(\vec{\boldsymbol{x}}\,', t_0)}{\partial t_0} = 0\,, \tag{35}$$

where $\partial'_k \equiv \partial/\partial x'_k$. Thus,

$$\int d^3x'\, \vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,', t_0) = \int d^3x'\, \partial'_k(J_k x'_i) + \frac{\partial}{\partial t_0}\int d^3x' \vec{\boldsymbol{x}}\,' \rho(\vec{\boldsymbol{x}}\,', t_0)\,.$$

The first term on the right-hand side above integrates to zero since the volume integral of a total divergence can be converted into a surface integral at infinity, which vanishes under the assumption of localized currents.

We also recognize the time-dependent electric dipole moment [cf. eq.(9.17) of Jackson],

$$\vec{\boldsymbol{p}}(t_0) = \int d^3x' \vec{\boldsymbol{x}}\,' \rho(\vec{\boldsymbol{x}}\,', t_0)\,.$$

Hence, we end up with

$$\vec{\boldsymbol{H}}_{\rm E1}(\vec{\boldsymbol{x}}, t) = -\frac{1}{4\pi c r}\hat{\boldsymbol{n}} \times \frac{d^2\vec{\boldsymbol{p}}}{dt_0^2}(t_0)\,. \tag{36}$$

Inserting this result into eq. (31), we find

$$\frac{dP_{\rm E1}(t)}{d\Omega} = \frac{Z_0}{16\pi^2 c^2}\left|\left[\hat{\boldsymbol{n}} \times \frac{d^2\vec{\boldsymbol{p}}}{dt_0^2}(t_0)\right] \times \hat{\boldsymbol{n}}\right|^2\,. \tag{37}$$

Next, we examine the second term on the right hand side of eq. (32). Using the identity given in eq. (9.31) of Jackson,

$$(\hat{\boldsymbol{n}} \boldsymbol{\cdot} \vec{\boldsymbol{x}}\,')\vec{\boldsymbol{J}} = \tfrac{1}{2}\left[(\hat{\boldsymbol{n}} \boldsymbol{\cdot} \vec{\boldsymbol{x}}\,')\vec{\boldsymbol{J}} + (\hat{\boldsymbol{n}} \boldsymbol{\cdot} \vec{\boldsymbol{J}})\vec{\boldsymbol{x}}\,'\right] - \tfrac{1}{2}\hat{\boldsymbol{n}} \times (\vec{\boldsymbol{x}}\,' \times \vec{\boldsymbol{J}})\,,$$

we can write eq. (32) as:

$$\vec{\boldsymbol{H}}(\vec{\boldsymbol{x}}, t) = \vec{\boldsymbol{H}}_{\rm E1}(\vec{\boldsymbol{x}}, t) + \vec{\boldsymbol{H}}_{\rm M1}(\vec{\boldsymbol{x}}, t) + \vec{\boldsymbol{H}}_{\rm E2}(\vec{\boldsymbol{x}}, t) + \ldots\,,$$

where $\vec{\boldsymbol{H}}_{\rm E1}(\vec{\boldsymbol{x}}, t)$ is given by eq. (34) and $\vec{\boldsymbol{H}}_{\rm M1}(\vec{\boldsymbol{x}}, t)$ and $\vec{\boldsymbol{H}}_{\rm E2}(\vec{\boldsymbol{x}}, t)$ are respectively given by:

$$\vec{\boldsymbol{H}}_{\rm M1}(\vec{\boldsymbol{x}}, t) = \frac{1}{8\pi c^2 r}\hat{\boldsymbol{n}} \times \left\{\hat{\boldsymbol{n}} \times \frac{\partial^2}{\partial t_0^2}\int d^3x' \left[\vec{\boldsymbol{x}}\,' \times \vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,', t_0)\right]\right\}\,, \tag{38}$$

$$\vec{\boldsymbol{H}}_{\rm E2}(\vec{\boldsymbol{x}}, t) = -\frac{1}{8\pi c^2 r}\hat{\boldsymbol{n}} \times \frac{\partial^2}{\partial t_0^2}\int d^3x' \left\{(\hat{\boldsymbol{n}} \boldsymbol{\cdot} \vec{\boldsymbol{x}}\,')\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,', t_0) + \left[\hat{\boldsymbol{n}} \boldsymbol{\cdot} \vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,', t_0)\right]\vec{\boldsymbol{x}}\,'\right\}\,. \tag{39}$$

Focusing on the magnetic dipole fields, we recognize the time-dependent magnetic dipole moment [cf. eq. (9.34) of Jackson]:

$$\vec{\boldsymbol{m}}(t_0)=\tfrac{1}{2}\int d^3x'\,\left[\vec{\boldsymbol{x}}\,'\times\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,',t_0)\right].$$

Hence, eq. (38) can be rewritten as

$$\vec{\boldsymbol{H}}_{\rm M1}(\vec{\boldsymbol{x}},t)=\frac{1}{4\pi c^2 r}\hat{\boldsymbol{n}}\times\left[\hat{\boldsymbol{n}}\times\frac{d^2\vec{\boldsymbol{m}}}{dt_0^2}(t_0)\right]. \tag{40}$$

To evaluate the time-dependent power, we need to evaluate

$$\left[\hat{\boldsymbol{n}}\times\left(\hat{\boldsymbol{n}}\times\frac{d^2\vec{\boldsymbol{m}}}{dt_0^2}\right)\right]\times\hat{\boldsymbol{n}}=\left[\hat{\boldsymbol{n}}\left(\hat{\boldsymbol{n}}\cdot\frac{d^2\vec{\boldsymbol{m}}}{dt_0^2}\right)-\frac{d^2\vec{\boldsymbol{m}}}{dt_0^2}\right]\times\hat{\boldsymbol{n}}=-\frac{d^2\vec{\boldsymbol{m}}}{dt_0^2}\times\hat{\boldsymbol{n}}\,.$$

Eq. (31) then yields

$$\frac{dP_{\rm M1}(t)}{d\Omega}=\frac{Z_0}{16\pi^2c^2}\left|\frac{d^2\vec{\boldsymbol{m}}}{dt_0^2}(t_0)\times\hat{\boldsymbol{n}}\right|^2. \tag{41}$$

It follows that one can obtain eq. (41) by substituting $(1/c)\ddot{\vec{\boldsymbol{m}}}\times\hat{\boldsymbol{n}}$ for $(\hat{\boldsymbol{n}}\times\ddot{\vec{\boldsymbol{p}}})\times\hat{\boldsymbol{n}}$ in eq. (37), where the double dots refers to twice differentiation with respect to $t_0=t-r/c$.

(b) Show similarly for a real quadrupole tensor $Q_{\alpha\beta}(t)$ given by eq. (9.41) of Jackson with a real charge density $\rho(\vec{\boldsymbol{x}},t)$ that the instantaneous radiated power per unit solid angle is

$$\frac{dP(t)}{d\Omega}=\frac{Z_0}{576\pi^2c^4}\left|\left[\hat{\boldsymbol{n}}\times\frac{d^3\vec{\boldsymbol{Q}}}{dt_0^3}(\hat{\boldsymbol{n}},t_0)\right]\times\hat{\boldsymbol{n}}\right|^2,$$

where $\vec{\boldsymbol{Q}}(\hat{\boldsymbol{n}},t)$ is defined by eq. (9.43) of Jackson.

To obtain the power for electric quadrupole radiation, we employ eq. (39). Here, we employ a similar trick used in eq. (9.37) of Jackson, by writing

$$\begin{aligned}
&\epsilon_{ijk}n_jn_\ell\int d^3x'\,\left[x'_\ell J_k(\vec{\boldsymbol{x}}\,',t_0)+x'_kJ_\ell(\vec{\boldsymbol{x}}\,',t_0)\right]\\
&\qquad=\epsilon_{ijk}n_jn_\ell\int d^3x'\,\left\{\partial'_m\left[x'_kx'_\ell J_m(\vec{\boldsymbol{x}}\,',t_0)\right]-x'_kx'_\ell\vec{\boldsymbol{\nabla}}'\boldsymbol{\cdot}\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,',t_0)\right\}\\
&\qquad=\epsilon_{ijk}n_jn_\ell\int d^3x'\,\left(x'_kx'_\ell-\tfrac{1}{3}|\vec{\boldsymbol{x}}\,'|^2\delta_{k\ell}\right)\frac{\partial\rho(\vec{\boldsymbol{x}}\,',t_0)}{\partial t_0}\\
&\qquad=\tfrac{1}{3}\epsilon_{ijk}n_jn_\ell\frac{dQ_{k\ell}}{dt_0}(t_0)=\tfrac{1}{3}\left[\hat{\boldsymbol{n}}\times\frac{d\vec{\boldsymbol{Q}}}{dt_0}(\hat{\boldsymbol{n}},t_0)\right]_i,
\end{aligned} \tag{42}$$

where [cf. eq. (9.43) of Jackson]:

$$Q_k(\hat{\boldsymbol{n}},t_0)\equiv Q_{k\ell}(t_0)n_\ell\,,$$

after identifying the time-dependent electric quadrupole moment [cf. eq. (9.41) of Jackson]:

$$\frac{dQ_{k\ell}}{dt}(t_0) = \int d^3x' \,\left(3x'_k x'_\ell - |\vec{\boldsymbol{x}}\,'|^2\delta_{k\ell}\right)\rho(\vec{\boldsymbol{x}}\,', t_0)\,.$$

In obtaining eq. (42), we used the continuity equation [cf. eq. (35)], and we converted the volume integral of a total divergence into a surface integral that vanishes under the assumption of localized currents. In the penultimate step of eq. (42), we added the term $-\frac{1}{3}|\vec{\boldsymbol{x}}\,'|^2\delta_{k\ell}$. This step is permissible, since $\epsilon_{ijk}n_j n_\ell \delta_{k\ell} = \epsilon_{ijk}n_j n_k = 0$, due to the asymmetry of the Levi-Civita tensor under the interchange of the indices j and k.

Hence, eq. (39) yields

$$\vec{\boldsymbol{H}}_{\rm E2}(\vec{\boldsymbol{x}}, t) = -\frac{1}{24\pi c^2 r}\hat{\boldsymbol{n}} \times \frac{d^3\vec{\boldsymbol{Q}}}{dt_0^3}(\hat{\boldsymbol{n}}, t_0)\,. \tag{43}$$

Plugging this result into eq. (31), we end up with:

$$\frac{dP_{\rm E2}(t)}{d\Omega} = \frac{Z_0}{576\pi^2 c^4}\left|\left[\hat{\boldsymbol{n}} \times \frac{d^3\vec{\boldsymbol{Q}}}{dt_0^3}(\hat{\boldsymbol{n}}, t_0)\right] \times \hat{\boldsymbol{n}}\right|^2\,. \tag{44}$$

An alternative method for solving Jackson, problem 9.7

In the analysis of sections 2 and 3 in Chapter 9 of Jackson, the physical electric and magnetic fields are given by

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}, t) = \mathrm{Re}\big[\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}, \omega)e^{-i\omega t}\big]\,, \qquad \vec{\boldsymbol{H}}(\vec{\boldsymbol{x}}, t) = \mathrm{Re}\big[\vec{\boldsymbol{H}}(\vec{\boldsymbol{x}}, \omega)e^{-i\omega t}\big]\,.$$

The complex fields $\vec{\boldsymbol{E}} \equiv \vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}, \omega)$ and $\vec{\boldsymbol{H}} \equiv \vec{\boldsymbol{H}}(\vec{\boldsymbol{x}}, \omega)$ are then employed to compute the differential power distribution via eq. (9.21) of Jackson.

To generalize to the case of arbitrary time dependence, we introduce the Fourier transforms,

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}, t) = \int_{-\infty}^{\infty} \vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}, \omega)e^{-i\omega t}\,d\omega\,, \qquad \vec{\boldsymbol{H}}(\vec{\boldsymbol{x}}, t) = \int_{-\infty}^{\infty} \vec{\boldsymbol{H}}(\vec{\boldsymbol{x}}, \omega)e^{-i\omega t}\,d\omega\,.$$

Since $\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}, t)$ and $\vec{\boldsymbol{H}}(\vec{\boldsymbol{x}}, t)$ are the physical fields, they must be real fields. This requirement imposes reality conditions on the Fourier coefficients,

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}, -\omega) = \vec{\boldsymbol{E}}^{\,*}(\vec{\boldsymbol{x}}, \omega)\,, \qquad \vec{\boldsymbol{H}}(\vec{\boldsymbol{x}}, -\omega) = \vec{\boldsymbol{H}}^{\,*}(\vec{\boldsymbol{x}}, \omega)\,.$$

Consider first the case of electric dipole radiation. The $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{H}}$ fields given in eq. (9.19) of Jackson are in fact the Fourier coefficients. Since $k = \omega/c$, it follows that

$$\vec{\boldsymbol{H}}_{\rm E1}(\vec{\boldsymbol{x}}, \omega) = \frac{\omega^2}{4\pi c}\big[\hat{\boldsymbol{n}} \times \vec{\boldsymbol{p}}(\omega)\big]\frac{e^{i\omega r/c}}{r}\,, \qquad \vec{\boldsymbol{E}}_{\rm E1}(\vec{\boldsymbol{x}}, \omega) = Z_0\,\vec{\boldsymbol{H}}(\vec{\boldsymbol{x}}, \omega) \times \hat{\boldsymbol{n}}\,,$$

where the time-dependent electric dipole moment is obtained via the Fourier transform,

$$\vec{\boldsymbol{p}}(t) = \int_{-\infty}^{\infty} \vec{\boldsymbol{p}}(\omega)\,e^{-i\omega t}\,d\omega\,. \tag{45}$$

Hence,

$$\vec{\boldsymbol{H}}_{\rm E1}(\vec{\boldsymbol{x}},t) = \frac{1}{4\pi cr}\,\hat{\boldsymbol{n}}\times\int_{-\infty}^{\infty}\omega^2\,\vec{\boldsymbol{p}}\,(\omega)e^{-i\omega(t-r/c)}\,d\omega$$
$$= -\frac{1}{4\pi cr}\,\hat{\boldsymbol{n}}\times\frac{d^2}{dt^2}\int_{-\infty}^{\infty}\vec{\boldsymbol{p}}(\omega)\,e^{-i\omega(t-r/c)}\,d\omega$$
$$= -\frac{1}{4\pi cr}\,\hat{\boldsymbol{n}}\times\frac{d^2\vec{\boldsymbol{p}}}{dt_0^2}(t_0)\,,$$

where $t_0 \equiv t - r/c$. Thus, we have established eq. (36). From this, we may obtain the power distribution given in eq. (37) as before. The case of M1 radiation may be treated following the prescription given at the top of p. 414 of Jackson. That is, we interchange $\vec{\boldsymbol{E}}_{\rm E1} \to Z_0\vec{\boldsymbol{H}}_{\rm M1}\,, Z_0\vec{\boldsymbol{H}}_{E1} \to -\vec{\boldsymbol{E}}_{\rm M1}$ and $\vec{\boldsymbol{p}} \to \vec{\boldsymbol{m}}/c$, thereby establishing eq. (40). From this, we may obtain the power distribution given in eq. (41) as before.

Finally, in the case of electric quadrupole radiation, eq. (9.44) of Jackson (with $k = \omega/c$) yields

$$\vec{\boldsymbol{H}}_{\rm E2}(\vec{\boldsymbol{x}},\omega) = -\frac{i\omega^3}{24\pi c^2}\,\hat{\boldsymbol{n}}\times\vec{\boldsymbol{Q}}(\hat{\boldsymbol{n}},\omega)\,\frac{e^{i\omega r/c}}{r}\,,\qquad \vec{\boldsymbol{E}}_{\rm E2}(\vec{\boldsymbol{x}},\omega) = Z_0\,\vec{\boldsymbol{H}}_{\rm E2}(\vec{\boldsymbol{x}},\omega)\times\hat{\boldsymbol{n}}\,,$$

where the time-dependent electric quadrupole vector [cf. eq. (9.43) of Jackson] is obtained via the Fourier transform,

$$\vec{\boldsymbol{Q}}(\hat{\boldsymbol{n}},t) = \int_{-\infty}^{\infty}\vec{\boldsymbol{Q}}(\hat{\boldsymbol{n}},\omega)\,e^{-i\omega t}\,d\omega\,.$$

Hence,

$$\vec{\boldsymbol{H}}_{\rm E2}(\vec{\boldsymbol{x}},t) = -\frac{i}{24\pi c^2 r}\,\hat{\boldsymbol{n}}\times\int_{-\infty}^{\infty}\omega^3\,\vec{\boldsymbol{Q}}\,(\hat{\boldsymbol{n}},\omega)e^{-i\omega(t-r/c)}\,d\omega$$
$$= -\frac{1}{24\pi c^2 r}\,\hat{\boldsymbol{n}}\times\frac{d^3}{dt^3}\int_{-\infty}^{\infty}\vec{\boldsymbol{Q}}(\hat{\boldsymbol{n}},\omega)\,e^{-i\omega(t-r/c)}\,d\omega$$
$$= -\frac{1}{24\pi c^2 r}\,\hat{\boldsymbol{n}}\times\frac{d^3\vec{\boldsymbol{Q}}}{dt_0^3}(\hat{\boldsymbol{n}},t_0)\,,$$

Thus, we have established eq. (43). From this, we may obtain the power distribution given in eq. (44) as before.

3. [Jackson, problem 9.8]

(a) Show that a classical oscillating electric dipole $\vec{\boldsymbol{p}}$ with fields given by eq. (9.18) of Jackson radiates electromagnetic angular momentum to infinity at the rate

$$\frac{d\vec{\boldsymbol{L}}}{dt} = \frac{k^3}{12\pi\epsilon_0}\,{\rm Im}\,[\vec{\boldsymbol{p}}^*\times\vec{\boldsymbol{p}}]\,.$$

In class, we derived the following result for the radiated angular momentum per unit time in gaussian units (which was denoted by $\vec{\boldsymbol{\tau}}$):

$$\vec{\boldsymbol{\tau}} = -\frac{r^3}{8\pi}\,\mathrm{Re}\int\bigl[(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{E}}^*)(\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{E}}) + (\hat{\boldsymbol{n}}\times\vec{\boldsymbol{B}})(\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{B}}^*)\bigr]\,d\Omega\,, \tag{46}$$

where $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ are the complex electric and magnetic field vectors (after removing the harmonic $e^{-i\omega t}$ factor). To rewrite this in SI units, we must replace $\vec{\boldsymbol{E}}\to\sqrt{4\pi\epsilon_0}\,\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}\to\sqrt{4\pi\mu_0}\,\vec{\boldsymbol{H}}$, where $c=1/\sqrt{\epsilon_0\mu_0}$. Note that we must also replace $\rho\to\rho/\sqrt{4\pi\epsilon_0}$ and $\vec{\boldsymbol{J}}\to\vec{\boldsymbol{J}}/\sqrt{4\pi\epsilon_0}$, which means that $\vec{\boldsymbol{p}}\to\vec{\boldsymbol{p}}/\sqrt{4\pi\epsilon_0}$ [cf. Table 3 on p. 782 of Jackson]. Thus, in SI units, we have

$$\vec{\boldsymbol{\tau}} = -\tfrac{1}{2}r^3\,\mathrm{Re}\int\bigl[\epsilon_0(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{E}}^*)(\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{E}}) + \mu_0(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{H}})(\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{H}}^*)\bigr]\,d\Omega\,, \tag{47}$$

We now make use of the electric dipole fields given by eq. (9.18) of Jackson,

$$\begin{aligned}
\vec{\boldsymbol{H}} &= \frac{ck^2}{4\pi}(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{p}})\,\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)\,,\\
\vec{\boldsymbol{E}} &= \frac{1}{4\pi\epsilon_0}\left\{k^2(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{p}})\times\hat{\boldsymbol{n}}\,\frac{e^{ikr}}{r} + [3\hat{\boldsymbol{n}}(\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{p}})-\vec{\boldsymbol{p}}]\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)e^{ikr}\right\}\,.
\end{aligned}$$

Note that $\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{H}}=0$ and

$$\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{E}} = \frac{1}{4\pi\epsilon_0}\left\{2\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{p}}\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)e^{ikr}\right\} = -\frac{ik}{2\pi\epsilon_0 r^2}\,\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{p}}\,e^{ikr} + \mathcal{O}\left(\frac{1}{r^3}\right)\,.$$

Thus, we only need to keep the $\mathcal{O}(1/r)$ terms in $\hat{\boldsymbol{n}}\times\vec{\boldsymbol{E}}^*$. Using the vector identity,

$$\hat{\boldsymbol{n}}\times\bigl\{(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{p}})\times\hat{\boldsymbol{n}}\bigr\} = \hat{\boldsymbol{n}}\times\vec{\boldsymbol{p}}\,,$$

for a unit vector $\hat{\boldsymbol{n}}$, it follows that,

$$\hat{\boldsymbol{n}}\times\vec{\boldsymbol{E}}^* = \frac{k^2}{4\pi\epsilon_0}\,\hat{\boldsymbol{n}}\times\vec{\boldsymbol{p}}^*\,\frac{e^{-ikr}}{r} + \mathcal{O}\left(\frac{1}{r}\right)\,.$$

Hence, it follows that

$$\vec{\boldsymbol{\tau}} = \mathrm{Re}\,\frac{ik^3}{16\pi^2\epsilon_0}\int d\Omega\,\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{p}}\,(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{p}}^*)\,,$$

where we have dropped terms that vanish in the limit of $r\to\infty$. In component form,

$$\tau_i = \mathrm{Re}\,\frac{ik^3}{16\pi^2\epsilon_0}\epsilon_{ijk}p_\ell p_k^*\int d\Omega\,n_j n_\ell\,, \tag{48}$$

where there is an implicit sum over the repeated indices j, k, and ℓ. Using eq. (9.47) of Jackson,

$$\int d\Omega\,n_j n_\ell = \frac{4\pi}{3}\,\delta_{j\ell}\,.$$

Inserting this result into eq. (48) yields

$$\tau_i = \mathrm{Re}\,\frac{ik^3}{12\pi\epsilon_0}(\vec{\boldsymbol{p}}\times\vec{\boldsymbol{p}}^*)_i\,.$$

Finally, noting that $\mathrm{Re}(iz) = -\,\mathrm{Im}\,z$ for any complex number z, and $\vec{\boldsymbol{p}}\times\vec{\boldsymbol{p}}^* = -\vec{\boldsymbol{p}}^*\times\vec{\boldsymbol{p}}$, we end up with[6]

$$\vec{\boldsymbol{\tau}} = \frac{d\vec{\boldsymbol{L}}}{dt} = \frac{k^3}{12\pi\epsilon_0}\,\mathrm{Im}(\vec{\boldsymbol{p}}^*\times\vec{\boldsymbol{p}})\,. \tag{49}$$

(b) What is the ratio of angular momentum radiated to energy radiated? Interpret.

Eq. (9.24) of Jackson states that the total power radiated is given by

$$P = \frac{c^2 Z_0 k^4}{12\pi}\,|\vec{\boldsymbol{p}}|^2\,, \tag{50}$$

where $Z_0 = \sqrt{\mu_0/\epsilon_0}$ is the impedance of free space. In particular, note that $c\,\epsilon_0 Z_0 = 1$. Thus, using $\omega = kc$ it follows that

$$\frac{\vec{\boldsymbol{\tau}}}{P} = \frac{\mathrm{Im}(\vec{\boldsymbol{p}}^*\times\vec{\boldsymbol{p}})}{\omega|\vec{\boldsymbol{p}}|^2}\,. \tag{51}$$

To interpret eq. (51), consider the case where the electric dipole moment possesses a definite value of m in the spherical basis. Recall that

$$q_{1,\pm 1} = \mp\sqrt{\frac{3}{8\pi}}(p_x \mp i p_y)\,, \qquad\qquad q_{10} = \sqrt{\frac{3}{4\pi}}\,p_z\,.$$

Consider three cases:

1. If $q_{11}\neq 0$ and $q_{10} = q_{1,-1} = 0$, then
$$\vec{\boldsymbol{p}} = \frac{p}{\sqrt{2}}(-1\,,\,-i\,,\,0) \quad\Longrightarrow\quad \vec{\boldsymbol{p}}^*\times\vec{\boldsymbol{p}} = i|\vec{\boldsymbol{p}}|^2\hat{\boldsymbol{z}}\,;$$

2. If $q_{10}\neq 0$ and $q_{11} = q_{1,-1} = 0$, then
$$\vec{\boldsymbol{p}} = p(0\,,\,0\,,\,1) \qquad\Longrightarrow\qquad \vec{\boldsymbol{p}}^*\times\vec{\boldsymbol{p}} = 0\,;$$

3. If $q_{1,-1}\neq 0$ and $q_{11} = q_{10} = 0$, then
$$\vec{\boldsymbol{p}} = \frac{p}{\sqrt{2}}(1\,,\,-i\,,\,0) \qquad\Longrightarrow\qquad \vec{\boldsymbol{p}}^*\times\vec{\boldsymbol{p}} = -i|\vec{\boldsymbol{p}}|^2\hat{\boldsymbol{z}}\,,$$

[6] In class, I wrote $\vec{\boldsymbol{\tau}} = -d\vec{\boldsymbol{L}}/dt$, where $-d\vec{\boldsymbol{L}}/dt$ denotes the rate of angular momentum *lost* by the radiating sources, which is equal to the rate of angular momentum transported to infinity. Jackson denotes this quantity $d\vec{\boldsymbol{L}}/dt$ without the explicit minus sign. Thus, I have adopted Jackson's convention in obtaining eq. (49).

where $p \equiv |\vec{\boldsymbol{p}}|$. That is,

$$\vec{\boldsymbol{p}}^* \times \vec{\boldsymbol{p}} = im|\vec{\boldsymbol{p}}|^2 \hat{\boldsymbol{z}}\,, \qquad \text{for } m = -1, 0, +1\,.$$

Inserting this result into eq. (51) yields

$$\frac{\tau_z}{P} = \frac{dL_z/dt}{dU/dt} = \frac{m}{\omega}\,.$$

In the quantum mechanics of electromagnetic radiation, photons possess an energy $U = \hbar\omega$ and a spin angular momentum $S_z = m\hbar$, so that $S_z/U = m/\omega$. The analogy is quite striking!

(c) For a charge e rotating in the x–y plane at radius a and angular speed ω, show that there is only a z component of radiated angular momentum with magnitude $dL_z/dt = e^2k^3a^2/(6\pi\epsilon_0)$. What about a charge oscillating along the z axis?

For a charge e rotating in the x–y plane at radius a and angular speed ω, the components of the electric dipole vector are given by, $\vec{\boldsymbol{p}} = ea(\cos\omega t\,, \sin\omega t\,, 0)$. This result can be rewritten as

$$\vec{\boldsymbol{p}} = \mathrm{Re}\biggl\{ea\, e^{-i\omega t}\,(1\,,\, i\,,\, 0)\biggr\}\,.$$

Thus, we may define a *complex* electric dipole vector,

$$\vec{\boldsymbol{p}}(t) = \vec{\boldsymbol{p}}\, e^{-i\omega t}\,, \qquad \text{where } \vec{\boldsymbol{p}} = ea\,(1\,,\, i\,,\, 0)\,.$$

It then follows that

$$\vec{\boldsymbol{p}}^* \times \vec{\boldsymbol{p}} = \det\begin{pmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ ea & -iea & 0 \\ ea & iea & 0 \end{pmatrix} = 2ie^2a^2\,\hat{\boldsymbol{z}}\,.$$

Hence, $\mathrm{Im}(\vec{\boldsymbol{p}}^* \times \vec{\boldsymbol{p}}) = 2e^2a^2\,\hat{\boldsymbol{z}}$. Inserting this result into eq. (49) yields

$$\tau_z = \frac{dL_z}{dt} = \frac{e^2k^3a^2}{6\pi\epsilon_0}\,.$$

For a charge oscillating along the z-axis, the *real* physical charge density is

$$\rho(\vec{\boldsymbol{x}}, t) = q\delta(x)\delta(y)\delta(z - z_0\cos\omega t)\,.$$

Hence, $\vec{\boldsymbol{p}} = \hat{\boldsymbol{z}}\, qz_0\cos\omega t = \hat{\boldsymbol{z}}\, qz_0\,\mathrm{Re}\, e^{-i\omega t}$. Thus, we identify the corresponding complex electric dipole moment vector (with the harmonic factor stripped off) as

$$\vec{\boldsymbol{p}} = \hat{\boldsymbol{z}}\, qz_0\,.$$

Note that this is in fact a *real* vector, in which case $\vec{\boldsymbol{p}}^* \times \vec{\boldsymbol{p}} = \vec{\boldsymbol{p}} \times \vec{\boldsymbol{p}} = 0$. Hence, for this case, $\vec{\boldsymbol{\tau}} = 0$ and no angular momentum is radiated.

The above two cases correspond to $m = 1$ and $m = 0$, respectively, which were treated explicitly at the end of part (b).

(d) What are the results corresponding to parts (a) and (b) for magnetic dipole radiation?

For magnetic dipole radiation, we use eqs. (9.35) and (9.36) of Jackson,

$$\vec{\boldsymbol{E}} = -\frac{Z_0 k^2}{4\pi}(\hat{\boldsymbol{n}} \times \vec{\boldsymbol{m}}\,)\,\frac{e^{ikr}}{r}\left(1 - \frac{1}{ikr}\right)\,,$$

$$\vec{\boldsymbol{H}} = \frac{1}{4\pi}\left\{k^2(\hat{\boldsymbol{n}} \times \vec{\boldsymbol{m}}\,) \times \hat{\boldsymbol{n}}\,\frac{e^{ikr}}{r} + [3\hat{\boldsymbol{n}}(\hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{m}}\,) - \vec{\boldsymbol{m}}\,]\left(\frac{1}{r^3} - \frac{ik}{r^2}\right)e^{ikr}\right\}\,.$$

As noted by Jackson at the top of p. 414, one can obtain results for magnetic dipole radiation from that of electric dipole radiation by the following set of interchanges,

$$\vec{\boldsymbol{E}} \to Z_0\,\vec{\boldsymbol{H}}\,, \qquad Z_0\vec{\boldsymbol{H}} \to -\vec{\boldsymbol{E}}\,, \qquad \vec{\boldsymbol{p}} \to \vec{\boldsymbol{m}}/c\,.$$

Applying these interchanges on the results obtained in eqs. (49) and (50) yields

$$\vec{\boldsymbol{\tau}} = \frac{\mu_0 k^3}{12\pi}\,\mathrm{Im}(\vec{\boldsymbol{m}}^{\,*} \times \vec{\boldsymbol{m}})\,, \tag{52}$$

$$P = \frac{\mu_0 c k^4}{12\pi}\,|\vec{\boldsymbol{m}}|^2\,. \tag{53}$$

The corresponding ratio of these two quantities is:

$$\frac{\vec{\boldsymbol{\tau}}}{P} = \frac{\mathrm{Im}(\vec{\boldsymbol{m}}^{\,*} \times \vec{\boldsymbol{m}})}{\omega|\vec{\boldsymbol{m}}|^2}\,. \tag{54}$$

The analysis of part (b) is nearly identical, with the magnetic dipole moment in the spherical basis replacing the electric dipole moment. Thus, again, we conclude that

$$\vec{\boldsymbol{m}}^{\,*} \times \vec{\boldsymbol{m}} = im|\vec{\boldsymbol{m}}|^2\,\hat{\boldsymbol{z}}\,, \qquad \text{for } m = -1, 0, +1\,.$$

which again yields

$$\frac{\tau_z}{P} = \frac{dL_z/dt}{dU/dt} = \frac{m}{\omega}\,.$$

<u>*ADDED NOTE*</u>:

In the literature, you will sometimes see another expression for the rate of angular momentum transport by radiation, in place of eq. (47). To derive this expression, we first note that in the radiation zone, the Jefimenko equations imply that $\hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{E}} = \hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{H}} = 0$ at $\mathcal{O}(1/r)$. Thus,

$$\hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{E}} = \mathcal{O}\left(\frac{1}{r^2}\right)\,, \qquad \hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{H}} = \mathcal{O}\left(\frac{1}{r^2}\right)\,. \tag{55}$$

It follows that we only need to keep expressions for $\hat{\boldsymbol{n}} \times \vec{\boldsymbol{E}}$ and $\hat{\boldsymbol{n}} \times \vec{\boldsymbol{H}}$ at $\mathcal{O}(1/r)$ in eq. (47). For harmonic fields, eq. (9.5) of Jackson is

$$\vec{\boldsymbol{E}} = \frac{iZ_0}{k}\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{H}} = -Z_0\,\hat{\boldsymbol{n}} \times \vec{\boldsymbol{H}} + \mathcal{O}\left(\frac{1}{r^2}\right)\,. \tag{56}$$

where we have used the leading behavior of $\vec{\boldsymbol{H}} \propto (1/r)e^{ikr-i\omega t}$. Likewise,

$$\hat{\boldsymbol{n}} \times \vec{\boldsymbol{E}} = -Z_0\, \hat{\boldsymbol{n}} \times \left(\hat{\boldsymbol{n}} \times \vec{\boldsymbol{H}}\right) = Z_0\, \vec{\boldsymbol{H}} + \mathcal{O}\left(\frac{1}{r^2}\right), \tag{57}$$

after expanding out the triple product and using eq. (55).

Inserting eqs. (56) and (57) into eq. (47), and using $\epsilon_0 Z_0 = \mu_0/Z_0 = \sqrt{\epsilon_0\mu_0} = 1/c$, it follows that

$$\vec{\boldsymbol{\tau}} = -\frac{r^3}{2c}\,\mathrm{Re}\int \left[\vec{\boldsymbol{H}}^*(\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{E}}) - \vec{\boldsymbol{E}}(\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{H}}^*)\right] d\Omega\,.$$

Writing $\hat{\boldsymbol{n}} = \vec{\boldsymbol{x}}/r$ and noting the vector identity,

$$\vec{\boldsymbol{H}}^*(\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{E}}) - \vec{\boldsymbol{E}}(\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{H}}^*) = -\frac{1}{r}\,\vec{\boldsymbol{x}}\times(\vec{\boldsymbol{E}}\times\vec{\boldsymbol{H}}^*)\,,$$

we end up with[7]

$$\frac{d\vec{\boldsymbol{\tau}}}{d\Omega} = \frac{r^2}{2c}\,\mathrm{Re}\,\vec{\boldsymbol{x}}\times(\vec{\boldsymbol{E}}\times\vec{\boldsymbol{H}}^*)\,. \tag{58}$$

Eq. (58) provides an alternative expression for the rate of angular momentum transport and is equivalent to eq. (47) in the limit of $r\to\infty$.[8]

The infinitesimal area element is $da = r^2\, d\Omega$, so eq. (58) can be rewritten as

$$\frac{d\vec{\boldsymbol{\tau}}}{da} = \frac{1}{2c}\,\mathrm{Re}\,\vec{\boldsymbol{x}}\times(\vec{\boldsymbol{E}}\times\vec{\boldsymbol{H}}^*)\,. \tag{59}$$

Since $\vec{\boldsymbol{\tau}} = d\vec{\boldsymbol{L}}/dt$, we interpret $d\vec{\boldsymbol{\tau}}/da$ as the angular momentum flux that is transported from the sources out to the observer located a long distance away. Eq. (59) should be compared with the expression for the angular momentum density [cf. problem 7.27 of Jackson],

$$\frac{1}{\mu_0 c^2}\,\vec{\boldsymbol{x}}\times(\vec{\boldsymbol{E}}\times\vec{\boldsymbol{B}}) = \frac{1}{c^2}\,\vec{\boldsymbol{x}}\times(\vec{\boldsymbol{E}}\times\vec{\boldsymbol{H}})\,.$$

The above result is applicable to the real fields. The corresponding result for the time-averaged angular momentum density of a distribution of harmonic electromagnetic fields is given by

$$\vec{\boldsymbol{\mathscr{L}}} = \frac{1}{2c^2}\,\mathrm{Re}\,\vec{\boldsymbol{x}}\times(\vec{\boldsymbol{E}}\times\vec{\boldsymbol{H}}^*)\,.$$

We conclude that

$$\frac{d\vec{\boldsymbol{\tau}}}{da} = \frac{1}{2c}\,\mathrm{Re}\,\vec{\boldsymbol{x}}\times(\vec{\boldsymbol{E}}\times\vec{\boldsymbol{H}}^*) = c\,\vec{\boldsymbol{\mathscr{L}}} + \mathcal{O}\left(\frac{1}{r^3}\right).$$

That is, the angular momentum flux in the radiation zone is equal to c times the angular momentum density, although this identification is correct only at the lowest nontrivial order in the inverse distance expansion.[9]

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[7]For the record, the corresponding result in gaussian units is obtained by replacing $\vec{\boldsymbol{H}}$ with $\vec{\boldsymbol{B}}$ and multiplying Eq. (58) by $c/(4\pi)$.

[8]In this limit, $\vec{\boldsymbol{\tau}}$ approaches a constant value which is equal to the rate of angular momentum transported to the surface at infinity by the radiation.

[9]This added note was inspired by a treatment in Emil Jan Konopinski, *Electromagnetic Fields and Relativistic Particles* (McGraw Hill Inc., New York, 1981). In particular, see the discussion on p. 226, including the very enlightening footnote at the bottom of that page.

4. [Jackson, problem 9.16] A thin linear antenna of length d is excited in such a way that a sinusoidal current makes a full wavelength of oscillation as shown in the figure below.

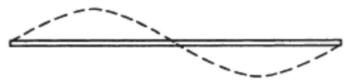

Figure 2: A thin linear antenna with a sinusoidal current that makes a full wavelength of oscillation.

(a) Calculate exactly the power radiated per unit solid angle and plot the angular distribution of radiation.

Choose the z-axis to lie along the antenna, and let $z = 0$ correspond to the center of the antenna. Then, $\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}, t) = \vec{\boldsymbol{J}}(\vec{\boldsymbol{x}})\, e^{-i\omega t}$, where

$$\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}, t) = I \sin\left(\frac{2\pi z}{d}\right) \delta(x)\, \delta(y)\, \hat{\boldsymbol{z}}\,, \qquad \text{for } |z| \leq \tfrac{1}{2}d\,, \tag{60}$$

where d is the length of the antenna. In class, we derived the following results for the complex magnetic and electric fields (assumed to be harmonic) in gaussian units,

$$\vec{\boldsymbol{B}}(\vec{\boldsymbol{x}}, t) = \frac{i\omega}{c^2 r}\, e^{i(kr-\omega t)}\, \hat{\boldsymbol{n}} \times \int d^3x'\, \vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,')\, e^{-ik\vec{\boldsymbol{x}}\,'\boldsymbol{\cdot}\hat{\boldsymbol{n}}} + \mathcal{O}\left(\frac{1}{r^2}\right),$$

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}, t) = \boldsymbol{B}(\vec{\boldsymbol{x}}, t) \times \hat{\boldsymbol{n}} + \mathcal{O}\left(\frac{1}{r^2}\right),$$

where $\hat{\boldsymbol{n}} \equiv \vec{\boldsymbol{x}}/r$ and $r \equiv |\vec{\boldsymbol{x}}|$. In SI units,[10] the above results take the following form,

$$\vec{\boldsymbol{H}}(\vec{\boldsymbol{x}}, t) = \frac{i\omega}{4\pi c\, r}\, e^{i(kr-\omega t)}\, \hat{\boldsymbol{n}} \times \int d^3x'\, \vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,')\, e^{-ik\vec{\boldsymbol{x}}\,'\boldsymbol{\cdot}\hat{\boldsymbol{n}}} + \mathcal{O}\left(\frac{1}{r^2}\right), \tag{61}$$

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}, t) = Z_0 \vec{\boldsymbol{H}}(\vec{\boldsymbol{x}}, t) \times \hat{\boldsymbol{n}} + \mathcal{O}\left(\frac{1}{r^2}\right). \tag{62}$$

Using eq. (9.21) of Jackson, the time-averaged power radiated per unit solid angle is given by

$$\frac{dP}{d\Omega} = \tfrac{1}{2}\,\mathrm{Re}\left[r^2\, \hat{\boldsymbol{n}} \boldsymbol{\cdot} \vec{\boldsymbol{E}} \times \vec{\boldsymbol{H}}^{\,*}\right].$$

In light of eq. (62), we compute

$$(\vec{\boldsymbol{H}} \times \hat{\boldsymbol{n}}) \times \vec{\boldsymbol{H}}^{\,*} = -\vec{\boldsymbol{H}}^{\,*} \times (\vec{\boldsymbol{H}} \times \hat{\boldsymbol{n}}) = \hat{\boldsymbol{n}}(\vec{\boldsymbol{H}} \boldsymbol{\cdot} \vec{\boldsymbol{H}}^{\,*}) - \vec{\boldsymbol{H}}(\hat{\boldsymbol{n}} \boldsymbol{\cdot} \vec{\boldsymbol{H}}^{\,*}) = \hat{\boldsymbol{n}}|\vec{\boldsymbol{H}}|^2\,,$$

where at the last step we $\hat{\boldsymbol{n}} \boldsymbol{\cdot} \vec{\boldsymbol{H}}^{\,*} = 0$, which is a consequence of eq. (61). Hence,

$$\frac{dP}{d\Omega} = \tfrac{1}{2} Z_0 r^2 |\vec{\boldsymbol{H}}|^2\,. \tag{63}$$

[10] To convert from gaussian to SI units, we must replace the fields $\vec{\boldsymbol{E}} \to \sqrt{4\pi\epsilon_0}\, \vec{\boldsymbol{E}}$, and $\vec{\boldsymbol{B}} \to \sqrt{4\pi\mu_0}\, \vec{\boldsymbol{H}}$, where $c = 1/\sqrt{\epsilon_0\mu_0}$ and the current $\vec{\boldsymbol{J}} \to \vec{\boldsymbol{J}}/\sqrt{4\pi\epsilon_0}$ [cf. Table 3 on p. 782 of Jackson].

Thus, our task is to compute the integral,

$$\int d^3x'\, \vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}')\, e^{-ik\vec{\boldsymbol{x}}'\boldsymbol{\cdot}\hat{\boldsymbol{n}}}\,.$$

By assumption, the sinusoidal current makes a full wavelength, which implies that

$$k = \frac{2\pi}{d}\,. \tag{64}$$

Inserting eq. (60) into the integral above and employing rectangular coordinates,

$$\int d^3x'\, \vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}')\, e^{-ik\vec{\boldsymbol{x}}'\boldsymbol{\cdot}\hat{\boldsymbol{n}}} = \hat{\boldsymbol{z}}\, I \int_{-d/2}^{d/2} \sin kz\, e^{-ikz\cos\theta}\, dz\,,$$

where θ is the angle between $\vec{\boldsymbol{n}}$ and the positive z-axis (which corresponds to the usual polar angle of spherical coordinates). The following indefinite integral appears in many integration tables,

$$\int e^{az} \sin kz\, dz = \frac{e^{az}(a\sin kz - k\cos kz)}{a^2+k^2}\,.$$

Using eq. (64), the limits of integration are $|z| \leq \pi/k$,

$$\begin{aligned}
\int_{-\pi/k}^{\pi/k} \sin kz\, e^{-ikz\cos\theta}\, dz &= \frac{e^{-ikz\cos\theta}(-ik\cos\theta\sin kz - k\cos kz)}{(-ik\cos\theta)^2+k^2}\Bigg|_{-\pi/k}^{\pi/k} \\
&= \frac{e^{-i\pi\cos\theta} - e^{i\pi\cos\theta}}{k\sin^2\theta} \\
&= -\frac{2i\sin(\pi\cos\theta)}{k\sin^2\theta}\,.
\end{aligned}$$

Using $\omega = kc$ and $\hat{\boldsymbol{n}} \times \hat{\boldsymbol{z}} = -\sin\theta\,\hat{\boldsymbol{\phi}}$, it follows that

$$\vec{\boldsymbol{H}}(\vec{\boldsymbol{x}}, t) = \frac{I\omega}{2\pi kc\, r}\, e^{i(kr-\omega t)}\, \frac{\sin(\pi\cos\theta)}{\sin^2\theta}\, \hat{\boldsymbol{n}} \times \hat{\boldsymbol{z}} = -\frac{I}{2\pi r}\, e^{i(kr-\omega t)}\, \frac{\sin(\pi\cos\theta)}{\sin\theta}\, \hat{\boldsymbol{\phi}}\,.$$

Plugging this result into eq. (63), we end up with

$$\frac{dP}{d\Omega} = \frac{Z_0 I^2}{8\pi^2}\left[\frac{\sin(\pi\cos\theta)}{\sin\theta}\right]^2\,. \tag{65}$$

A plot of the angular distribution is shown in Figure 3.

(b) Determine the total power radiated and find a numerical value for the radiation resistance.

The total power is

$$P = \int \frac{dP}{d\Omega}\, d\Omega = 2\pi \int_{-1}^{1} \frac{dP}{d\Omega}\, d\cos\theta\,,$$

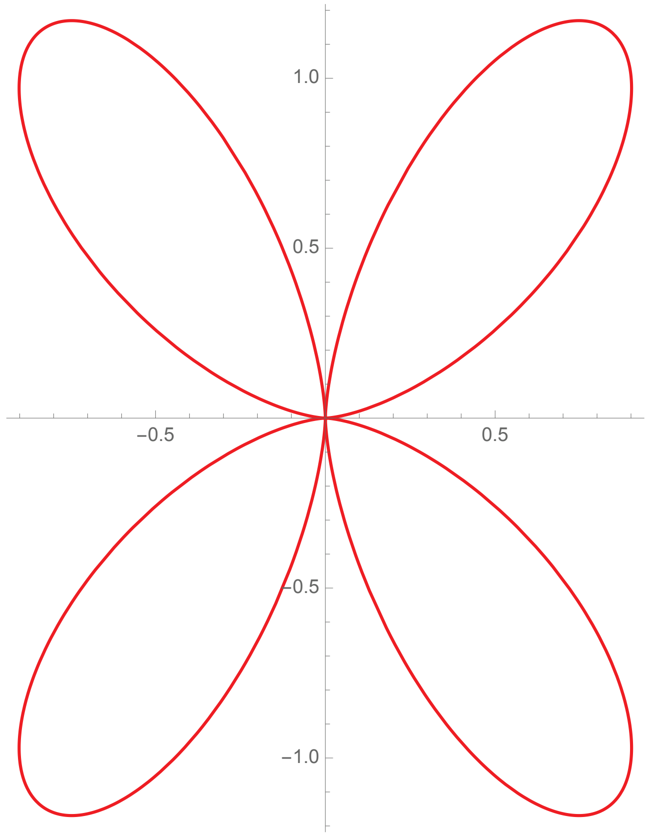

Figure 3: A polar plot of the antenna pattern of a thin linear antenna with a sinusoidal current that makes a full wavelength of oscillation. The angular distribution of the radiated power is given by eq. (65) and is proportional to $\sin^2(\pi\cos\theta)/\sin^2\theta$. Normalization has been chosen such that $Z_0 I^2 = 8\pi^2$. This plot was created with Mathematica software.

since the angular distribution obtained in eq. (65) is independent of the azimuthal angle ϕ. Defining $x \equiv \cos\theta$, and employing $\sin^2\theta = 1 - \cos^2\theta$ in the denominator of eq. (65),

$$P = \frac{Z_0 I^2}{4\pi} \int_{-1}^{1} \frac{\sin^2(\pi x)}{1-x^2}\, dx = \frac{Z_0 I^2}{8\pi} \int_{-1}^{1} \frac{1-\cos(2\pi x)}{1-x^2}\, dx\,,$$

after employing a well-known trigonometric identity. We now apply the method of partial fractions to write

$$\frac{1}{1-x^2} = \frac{1}{2}\left(\frac{1}{1-x} + \frac{1}{1+x}\right).$$

The resulting two integrals are equal after making a variable change $x \to -x$ in the first integral. Thus,

$$P = \frac{Z_0 I^2}{8\pi} \int_{-1}^{1} \frac{1-\cos(2\pi x)}{1+x}\, dx\,.$$

Next, we make a change of variables, $t = 2\pi(1+x)$, which converts the above integral into the following form,

$$P = \frac{Z_0 I^2}{8\pi} \int_{0}^{4\pi} \frac{1-\cos t}{t}\, dt\,.$$

This integral can be evaluated in terms of the cosine integral, which is defined as

$$\mathrm{Ci}(x) = -\int_x^\infty \frac{\cos t}{t}\, dt\,.$$

It then follows that:[11]

$$\int_0^x \frac{1-\cos t}{t}\, dt = \gamma + \ln x - \mathrm{Ci}(x)\,, \tag{66}$$

where $\gamma \simeq 0.5772$ is the Euler constant. Thus,

$$P = \frac{Z_0 I^2}{8\pi}\left[\gamma + \ln(4\pi) - \mathrm{Ci}(4\pi)\right].$$

Using the following numerical value, $\mathrm{Ci}(4\pi) = -0.006$,[12] we obtain

$$P = \frac{Z_0 I^2}{8\pi}(3.114)\,.$$

The corresponding radiative resistance (in ohms) is equal to the coefficient of $\frac{1}{2}I^2$ [cf. the text below eq. (9.29) of Jackson]. Thus, using $Z_0 = 376.7$ ohms [given below eq. (7.11)$'$ of Jackson],

$$R_{\rm rad} = (3.114)\,\frac{Z_0}{4\pi} = 93.3 \text{ ohms}\,. \tag{67}$$

[11] See e.g. formula 8.230 no. 2 on p. 895 of I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products* (8th edition), edited by Daniel Zwillinger and Victor Moll (Academic Press, Elsevier, Inc., Waltham, MA, 2015). Eq. (66) can also be found on p. 41 [cf. problem 3 on this page] of N.N. Lebedev, *Special Functions and Their Applications* (Dover Publications, Inc., Mineola, NY, 1972).

[12] For example, one can consult Milton Abramowitz and Irene A. Stegun, *Handbook of Mathematical Functions* (Dover Publications, Inc., Mineola, NY, 1965), which provides numerical tables of the cosine integral. Alternatively, one can use a mathematical program such as Mathematica or Maple to evaluate the cosine integral directly.