1. Consider an oversimplified model of an antenna consisting of a thin wire of length ℓ and negligible cross section, carrying a harmonically varying current density flowing in the z direction. The (complex) current in the wire is given by $Ie^{-i\omega t}$, where I is a constant (independent of position).

(a) Show that the (complex) current density takes the form:

$$\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}},t) = \hat{\boldsymbol{z}} I e^{-i\omega t}\delta(x)\delta(y)[\Theta(z+\tfrac{1}{2}\ell) - \Theta(z-\tfrac{1}{2}\ell)]\,,$$

where the step function $\Theta(x) \equiv 1$ if $x > 0$ and $\Theta(x) \equiv 0$ if $x < 0$. Here, we assume that the point $z = 0$ corresponds to the midpoint of the antenna.

First we note that

$$\Theta(z+\tfrac{1}{2}\ell) - \Theta(z-\tfrac{1}{2}\ell) = \begin{cases} 0\,, & z > \ell/2\,, \\ 1\,, & |z| < \ell/2\,, \\ 0\,, & z < -\ell/2\,. \end{cases}$$

Thus, $\vec{\boldsymbol{J}} = 0$ if $z > \ell/2$ or $z < -\ell/2$. For $|z| < \ell/2$,

$$J_z = Ie^{-i\omega t}\delta(x)\delta(y)\,, \qquad\qquad J_x = J_y = 0\,.$$

The current is obtained by computing

$$\int \vec{\boldsymbol{J}}\boldsymbol{\cdot} d\vec{\boldsymbol{a}} = \int \vec{\boldsymbol{J}}\boldsymbol{\cdot}\hat{\boldsymbol{z}}\, dxdy = \int J_z\, dxdy = Ie^{-i\omega t}\,.$$

(b) Prove that there is an oscillating charge density at $z = \pm\frac{1}{2}\ell$ (*i.e.*, at both ends of the antenna), but the charge density vanishes at any interior point on the antenna.

The continuity equation is

$$\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{J}} + \frac{\partial\rho}{\partial t} = 0\,.$$

For $\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}},t) = \vec{\boldsymbol{J}}(\vec{\boldsymbol{x}})e^{-i\omega t}$ and $\rho(\vec{\boldsymbol{x}},t) = \rho(\vec{\boldsymbol{x}})e^{-i\omega t}$, the continuity equation then reads:

$$\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{J}} = i\omega\rho(\vec{\boldsymbol{x}})\,.$$

Using $\vec{\boldsymbol{J}}$ given in part (a),

$$\begin{aligned}\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{J}} = \frac{\partial J_z}{\partial z} &= I\delta(x)\delta(y)\frac{\partial}{\partial z}\left[\Theta(z+\tfrac{1}{2}\ell) - \Theta(z-\tfrac{1}{2}\ell)\right] \\ &= I\delta(x)\delta(y)\left[\delta(z+\tfrac{1}{2}\ell) - \delta(z-\tfrac{1}{2}\ell)\right]\,.\end{aligned}$$

Setting this result to $i\omega\rho(\vec{\boldsymbol{x}})$, we conclude that:

$$\rho(\vec{\boldsymbol{x}}) = -\frac{iI}{\omega}\delta(x)\delta(y)\left[\delta(z+\tfrac{1}{2}\ell) - \delta(z-\tfrac{1}{2}\ell)\right] ,$$

which corresponds to two point charges located at the two ends of the antenna. Moreover, $\rho(\vec{\boldsymbol{x}},t) = \rho(\vec{\boldsymbol{x}})e^{-i\omega t}$ indicates that the point charges have magnitudes that oscillate in time. (As usual, we take the real part of $\rho(\vec{\boldsymbol{x}},t)$ to find the corresponding physical quantity.)

(c) Show that the antenna acts like an oscillating electric dipole moment, $\vec{\boldsymbol{p}}e^{-i\omega t}$. Evaluate $\vec{\boldsymbol{p}}$ in terms of the current I, the antenna length ℓ and the angular frequency ω.

The electric dipole moment is given by

$$\vec{\boldsymbol{p}}(t) = \int \vec{\boldsymbol{x}}\rho(\vec{\boldsymbol{x}},t)d^3x = e^{-i\omega t}\int \vec{\boldsymbol{x}}\,\rho(\vec{\boldsymbol{x}})\,d^3x = \vec{\boldsymbol{p}}\,e^{-i\omega t}\,, \tag{1}$$

after employing $\rho(\vec{\boldsymbol{x}},t) = \rho(\vec{\boldsymbol{x}})e^{-i\omega t}$ and defining,

$$\vec{\boldsymbol{p}} \equiv \int \vec{\boldsymbol{x}}\,\rho(\vec{\boldsymbol{x}})\,d^3x\,. \tag{2}$$

We therefore compute:

$$\begin{aligned}
\vec{\boldsymbol{p}} &= \int \vec{\boldsymbol{x}}\rho(\vec{\boldsymbol{x}})d^3x \\
&= \int (x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}})\left(\frac{-iI}{\omega}\right)\delta(x)\delta(y)\left[\delta(z+\tfrac{1}{2}\ell) - \delta(z-\tfrac{1}{2}\ell)\right]dxdydz \\
&= \frac{-iI}{\omega}\hat{\boldsymbol{z}}\int zdz\left[\delta(z+\tfrac{1}{2}\ell) - \delta(z-\tfrac{1}{2}\ell)\right] = \frac{iI\ell}{\omega}\hat{\boldsymbol{z}}\,.
\end{aligned}$$

(d) Calculate the angular distribution of the radiated power, $dP/d\Omega$, assuming that $\lambda \gg \ell$, where λ is the wavelength of the emitted radiation. Express your answer in terms of the current I, the antenna length ℓ and the wavelength λ. Integrate over angles to obtain the total radiated power.

For $\lambda \gg \ell$, the electric dipole approximation is very accurate. Hence, we can neglect all other multipole contributions. Using eq. (9.23) of Jackson (in SI units),

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{32\pi^2}k^4|\vec{\boldsymbol{p}}|^2\sin^2\theta = \frac{c^2 Z_0}{32\pi^2}\frac{I^2\ell^2k^4}{\omega^2}\sin^2\theta\,, \tag{3}$$

where $Z_0 \equiv \sqrt{\mu_0/\varepsilon_0}$ is the impedance of free space. Recalling that $\omega = kc$ and $k = 2\pi/\lambda$, the above result can be written as:

$$\frac{dP}{d\Omega} = \frac{Z_0 I^2}{8}\left(\frac{\ell}{\lambda}\right)^2\sin^2\theta\,. \tag{4}$$

Integrating over angles yields:

$$P = \frac{\pi Z_0 I^2}{3}\left(\frac{\ell}{\lambda}\right)^2 . \tag{5}$$

Note that we can also derive eq. (5) by using eq. (9.24) of Jackson,

$$P = \frac{c^2 Z_0 k^4}{12\pi}\,|\vec{\boldsymbol{p}}|^2 = \frac{c^2 Z_0}{12\pi}\left(\frac{2\pi}{\lambda}\right)^4\left(\frac{I^2\ell^2}{c^2}\right)\left(\frac{\lambda}{2\pi}\right)^2 = \frac{\pi Z_0 I^2}{3}\left(\frac{\ell}{\lambda}\right)^2 . \tag{6}$$

The results above have been given in SI units. To convert eqs. (3)–(6) to Gaussian units, one can simply replace $Z_0 \to 4\pi/c$. This can be understood by writing the impedance of free space [cf. eq. (9.5) of Jackson] as,

$$Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = \frac{1}{\epsilon_0 c} . \tag{7}$$

Moreover, using Table 3 on p. 782 of Jackson, we must replace $I \to \sqrt{4\pi\epsilon_0}\, I$, when converting a formula expressed in SI units to gaussian units. Thus,

$$Z_0 I^2 \to \frac{1}{\epsilon_0 c}\, 4\pi\epsilon_0 I^2 = \frac{4\pi I^2}{c} \tag{8}$$

which is consistent with replacing Z_0 with $4\pi/c$ as asserted above.

2. [20] Consider a relativistic particle with charge e moving along a trajectory $\vec{\boldsymbol{r}}(t)$ at velocity $\vec{\boldsymbol{v}} = c\vec{\boldsymbol{\beta}}(t) \equiv d\vec{\boldsymbol{r}}(t)/dt$ and acceleration $\vec{\boldsymbol{a}}(t) \equiv d\vec{\boldsymbol{v}}(t)/dt$. In addition to radiating energy, the particle also radiates angular momentum. Denote the angular momentum radiated per unit time by $\vec{\boldsymbol{\tau}} \equiv d\vec{\boldsymbol{L}}/dt$.

(a) Show that the angular distribution of the angular momentum radiated per unit *retarded* time in gaussian units is given by

$$\frac{d\vec{\boldsymbol{\tau}}\,'}{d\Omega} = \frac{e^2(1-\beta^2)}{4\pi c}\left[\frac{(1-\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\hat{\boldsymbol{n}})\left(\hat{\boldsymbol{n}}\times\dot{\vec{\boldsymbol{\beta}}}\right) + (\hat{\boldsymbol{n}}\times\vec{\boldsymbol{\beta}})\hat{\boldsymbol{n}}\boldsymbol{\cdot}\dot{\vec{\boldsymbol{\beta}}}}{(1-\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\hat{\boldsymbol{n}})^4}\right],$$

where $\vec{\boldsymbol{\tau}}\,' \equiv d\vec{\boldsymbol{L}}/dt_{\rm ret}$ and $\dot{\vec{\boldsymbol{\beta}}} \equiv d\vec{\boldsymbol{\beta}}/dt_{\rm ret}$.

The radiated angular momentum per unit time in gaussian units is given by[1]

$$\vec{\boldsymbol{\tau}} = -\frac{r^3}{4\pi}\int\left[(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{E}})(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{E}}) + (\hat{\boldsymbol{n}}\times\vec{\boldsymbol{B}})(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{B}})\right]d\Omega\,, \tag{9}$$

[1]Eq. (9) should be compared with Eq. (46) of Solution Set 3, which exhibits the radiated angular momentum per unit time in gaussian units in terms of the complex electric and magnetic field vectors (after removing the harmonic $e^{-i\omega t}$ factor). The relation between the two formulae is analogous to the relation between the Poynting vector expressed in terms of real fields, $\vec{\boldsymbol{S}} = c\vec{\boldsymbol{E}}\times\vec{\boldsymbol{B}}/(4\pi)$, and the Poynting vector expressed in terms of complex fields after removing the harmonic $e^{-i\omega t}$ factor, $\vec{\boldsymbol{S}} = c\vec{\boldsymbol{E}}\times\vec{\boldsymbol{B}}^*/(8\pi)$.

where $\vec{\boldsymbol{E}} \equiv \vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}, t)$ and $\vec{\boldsymbol{B}} \equiv \vec{\boldsymbol{B}}(\vec{\boldsymbol{x}}, t)$ are the real (physical) electric and magnetic field vectors and $r \equiv |\vec{\boldsymbol{x}}|$. The $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ fields are given by eqs. (14.13) and (14.14) of Jackson,

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}, t) = e\left[\frac{\hat{\boldsymbol{n}} - \vec{\boldsymbol{\beta}}}{\gamma^2(1 - \vec{\boldsymbol{\beta}} \boldsymbol{\cdot} \hat{\boldsymbol{n}})^3 R^2}\right]_{\rm ret} + \frac{e}{c}\left[\frac{\hat{\boldsymbol{n}} \times \{(\hat{\boldsymbol{n}} - \vec{\boldsymbol{\beta}}) \times \dot{\vec{\boldsymbol{\beta}}}\}}{(1 - \vec{\boldsymbol{\beta}} \boldsymbol{\cdot} \hat{\boldsymbol{n}})^3 R}\right]_{\rm ret}, \tag{10}$$

$$\vec{\boldsymbol{B}}(\vec{\boldsymbol{x}}, t) = \big[\hat{\boldsymbol{n}} \times \vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}, t)\big]_{\rm ret}, \tag{11}$$

where $\gamma \equiv (1-\beta^2)^{-1/2}$, $R \equiv |\vec{\boldsymbol{x}} - \vec{\boldsymbol{r}}(t)|$, $t_{\rm ret} = t - \vec{\boldsymbol{r}}(t)/c$, and $[\cdots]_{\rm ret}$ instructs us to evaluate all time-dependent quantities inside the square brackets at time $t_{\rm ret}$. For an observer located far away from the trajectory of the particle, we may approximate $R \simeq r$.

To compute $\vec{\boldsymbol{\tau}}\,' \equiv d\vec{\boldsymbol{L}}/dt_{\rm ret}$, we employ the analogue of eq. (14.37) of Jackson,

$$\vec{\boldsymbol{\tau}}\,' \equiv \frac{d\vec{\boldsymbol{L}}}{dt_{\rm ret}} = \frac{d\vec{\boldsymbol{L}}}{dt}\,\frac{dt}{dt_{\rm ret}} = \vec{\boldsymbol{\tau}}(1 - \vec{\boldsymbol{\beta}} \boldsymbol{\cdot} \hat{\boldsymbol{n}})\,. \tag{12}$$

Using eqs. (10) and (11) and noting that $R = r\big[1 + \mathcal{O}(1/r)\big]$, it follows that $\hat{\boldsymbol{n}} \boldsymbol{\cdot} \vec{\boldsymbol{B}} = 0$ and

$$\hat{\boldsymbol{n}} \boldsymbol{\cdot} \vec{\boldsymbol{E}} = e\left[\frac{1}{\gamma^2(1 - \vec{\boldsymbol{\beta}} \boldsymbol{\cdot} \hat{\boldsymbol{n}})^2 r^2}\right]_{\rm ret} + \mathcal{O}\left(\frac{1}{r^3}\right), \tag{13}$$

$$\hat{\boldsymbol{n}} \times \vec{\boldsymbol{E}} = \frac{e}{c}\left[\frac{\hat{\boldsymbol{n}} \times \big[\hat{\boldsymbol{n}} \times \{(\hat{\boldsymbol{n}} - \vec{\boldsymbol{\beta}}) \times \dot{\vec{\boldsymbol{\beta}}}\}\big]}{(1 - \vec{\boldsymbol{\beta}} \boldsymbol{\cdot} \hat{\boldsymbol{n}})^3 r}\right]_{\rm ret} + \mathcal{O}\left(\frac{1}{r^2}\right). \tag{14}$$

Plugging the results of eqs. (13) and (14) into eqs. (9) and (12), it follows that in the limit of $r \to \infty$,

$$\frac{d\vec{\boldsymbol{\tau}}\,'}{d\Omega} = -\,\frac{e^2(1-\beta^2)}{4\pi c}\,\frac{\hat{\boldsymbol{n}} \times \big[\hat{\boldsymbol{n}} \times \{(\hat{\boldsymbol{n}} - \vec{\boldsymbol{\beta}}) \times \dot{\vec{\boldsymbol{\beta}}}\}\big]}{(1 - \vec{\boldsymbol{\beta}} \boldsymbol{\cdot} \hat{\boldsymbol{n}})^4}\,. \tag{15}$$

Using vector identities,

$$\begin{aligned}\hat{\boldsymbol{n}} \times \big[\hat{\boldsymbol{n}} \times \{(\hat{\boldsymbol{n}} - \vec{\boldsymbol{\beta}}) \times \dot{\vec{\boldsymbol{\beta}}}\big] &= \hat{\boldsymbol{n}} \times \big[(\hat{\boldsymbol{n}} - \vec{\boldsymbol{\beta}})\hat{\boldsymbol{n}} \boldsymbol{\cdot} \dot{\vec{\boldsymbol{\beta}}} - (1 - \vec{\boldsymbol{\beta}} \boldsymbol{\cdot} \hat{\boldsymbol{n}})\dot{\vec{\boldsymbol{\beta}}}\big] \\ &= -(\hat{\boldsymbol{n}} \times \vec{\boldsymbol{\beta}})\hat{\boldsymbol{n}} \boldsymbol{\cdot} \dot{\vec{\boldsymbol{\beta}}} - (1 - \vec{\boldsymbol{\beta}} \boldsymbol{\cdot} \hat{\boldsymbol{n}})(\hat{\boldsymbol{n}} \times \dot{\vec{\boldsymbol{\beta}}})\,. \end{aligned} \tag{16}$$

Hence, we end up with

$$\frac{d\vec{\boldsymbol{\tau}}\,'}{d\Omega} = \frac{e^2(1-\beta^2)}{4\pi c}\left[\frac{(1 - \vec{\boldsymbol{\beta}} \boldsymbol{\cdot} \hat{\boldsymbol{n}})\left(\hat{\boldsymbol{n}} \times \dot{\vec{\boldsymbol{\beta}}}\right) + (\hat{\boldsymbol{n}} \times \vec{\boldsymbol{\beta}})\hat{\boldsymbol{n}} \boldsymbol{\cdot} \dot{\vec{\boldsymbol{\beta}}}}{(1 - \vec{\boldsymbol{\beta}} \boldsymbol{\cdot} \hat{\boldsymbol{n}})^4}\right]. \tag{17}$$

<u>*REMARK:*</u>

As noted at the end of the solution to problem 3 in Solution Set 3, the radiated angular momentum can also be computed by employing the following formula (using gaussian units):

$$\frac{d\vec{\boldsymbol{\tau}}}{d\Omega} = \frac{r^2}{4\pi}\,\vec{\boldsymbol{x}} \times (\vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}})\,. \tag{18}$$

Using $\vec{\boldsymbol{B}}(\vec{\boldsymbol{x}}, t) = \hat{\boldsymbol{n}} \times \vec{\boldsymbol{E}}$ [cf. eq. (11)] and $\vec{\boldsymbol{x}} = r\hat{\boldsymbol{n}}$, it follows that

$$\begin{aligned}\vec{\boldsymbol{x}} \times (\vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}}) &= r\hat{\boldsymbol{n}} \times \left[\vec{\boldsymbol{E}} \times (\hat{\boldsymbol{n}} \times \vec{\boldsymbol{E}})\right] = r\hat{\boldsymbol{n}} \times \left[\hat{\boldsymbol{n}}|\vec{\boldsymbol{E}}|^2 - \vec{\boldsymbol{E}}(\hat{\boldsymbol{n}} \boldsymbol{\cdot} \vec{\boldsymbol{E}})\right] \\ &= -r(\hat{\boldsymbol{n}} \times \vec{\boldsymbol{E}})(\hat{\boldsymbol{n}} \boldsymbol{\cdot} \vec{\boldsymbol{E}})\,. \end{aligned} \tag{19}$$

Hence, eq. (18) yields

$$\frac{d\vec{\boldsymbol{\tau}}}{d\Omega} = -\frac{r^3}{4\pi}\,(\hat{\boldsymbol{n}} \times \vec{\boldsymbol{E}})(\hat{\boldsymbol{n}} \boldsymbol{\cdot} \vec{\boldsymbol{E}})\,. \tag{20}$$

Eq. (20) matches the result quoted in eq. (9) after noting that $\hat{\boldsymbol{n}} \boldsymbol{\cdot} \vec{\boldsymbol{B}} = 0$ in light of eq. (11).

(b) Using the result of part (a), integrate over angles to obtain the total angular momentum radiated per unit retarded time.

<u>*HINT:*</u> The following integrals will be useful:

$$\int \frac{\hat{\boldsymbol{n}}\, d\Omega}{(1 - \vec{\boldsymbol{\beta}} \boldsymbol{\cdot} \hat{\boldsymbol{n}})^3} = \frac{4\pi\vec{\boldsymbol{\beta}}}{(1 - \beta^2)^2}\,, \tag{21}$$

$$\int \frac{\hat{n}_i \hat{n}_j\, d\Omega}{(1 - \vec{\boldsymbol{\beta}} \boldsymbol{\cdot} \hat{\boldsymbol{n}})^4} = \frac{4\pi}{3(1 - \beta^2)^2}\left[\delta_{ij} + \frac{4\beta_i\beta_j}{1 - \beta^2}\right]\,. \tag{22}$$

Derivations of these integrals are given in an Appendix to this Solution Set.

Integrating eq. (17) over angles and making use of eqs. (21) and (22),

$$\vec{\boldsymbol{\tau}}\,' = \frac{e^2}{c(1 - \beta^2)}\left[\vec{\boldsymbol{\beta}} \times \dot{\vec{\boldsymbol{\beta}}} + \tfrac{1}{3}\dot{\vec{\boldsymbol{\beta}}} \times \vec{\boldsymbol{\beta}}\right]\,, \tag{23}$$

where the second term on the right hand side of eq. (23) arises from the following calculation:

$$\epsilon_{ijk}\beta_k\dot{\beta}_\ell \int \frac{\hat{n}_j \hat{n}_\ell\, d\Omega}{(1 - \vec{\boldsymbol{\beta}} \boldsymbol{\cdot} \hat{\boldsymbol{n}})^4} = \epsilon_{ijk}\beta_k\dot{\beta}_\ell \frac{4\pi}{3(1 - \beta^2)^2}\left[\delta_{j\ell} + \frac{4\beta_j\beta_\ell}{1 - \beta^2}\right] = \frac{4\pi}{3(1 - \beta^2)^2}(\dot{\vec{\boldsymbol{\beta}}} \times \vec{\boldsymbol{\beta}})_i\,, \tag{24}$$

after noting that $\epsilon_{ijk}\beta_j\beta_k = 0$ (using Einstein's summation convention where there is an implicit sum over repeated index pairs).

Finally, after using $\dot{\vec{\boldsymbol{\beta}}} \times \vec{\boldsymbol{\beta}} = -\vec{\boldsymbol{\beta}} \times \dot{\vec{\boldsymbol{\beta}}}$ and $\gamma^2 = (1 - \beta^2)^{-1}$, eq. (23) simplifies to

$$\boxed{\vec{\boldsymbol{\tau}}\,' = \frac{2e^2\gamma^2\,\vec{\boldsymbol{\beta}} \times \dot{\vec{\boldsymbol{\beta}}}}{3c}\,.} \tag{25}$$

(c) The energy radiated per unit retarded time is denoted by P'. Consider the case where the particle moves in a circle of radius R. Compute $P'/|\vec{\boldsymbol{\tau}}'|$ in the nonrelativistic limit and interpret your result.

For circular motion. $\dot{\vec{\boldsymbol{\beta}}}$ is perpendicular to $\vec{\boldsymbol{\beta}}$. In the nonrelativistic limit, $\gamma \simeq 1$ and eq. (25) yields

$$|\vec{\boldsymbol{\tau}}'| = \frac{2e^2\beta\dot{\beta}}{3c}\,. \tag{26}$$

Using eq. (14.46) of Jackson and setting $\gamma = 1$,

$$P' = \frac{2e^2\dot{\beta}^2}{3c}\,. \tag{27}$$

Hence,

$$\frac{P'}{|\vec{\boldsymbol{\tau}}'|} = \frac{\dot{\beta}}{\beta}\,. \tag{28}$$

Finally, we note that for circular motion, $dv/dt = v^2/R$. Hence

$$\frac{\dot{\beta}}{\beta} = \frac{v}{R} = \omega\,, \tag{29}$$

where ω is the angular velocity of the particle. Hence, we conclude that in the nonrelativistic limit,

$$\frac{P'}{|\vec{\boldsymbol{\tau}}'|} = \omega\,. \tag{30}$$

This is not surprising in light of the remarks noted at the end of the solution to part (b) of problem 3 in Solution Set 3. Namely, in the quantum mechanics of electromagnetic radiation, photons posses an energy $U = \hbar\omega$ and a spin angular momentum of magnitude $|S_z| = \hbar$. Anticipating from the solution to problem 2 on Problem Set 5 that the fundamental frequency of the radiation emitted by the particle in circular motion is ω, one is not surprised to find that

$$\frac{P'}{|\vec{\boldsymbol{\tau}}'|} = \frac{d|S_z|/dt_{\rm ret}}{dU/dt_{\rm ret}} = \frac{\hbar\omega}{\hbar} = \omega\,. \tag{31}$$

3. [20] A particle of charge e initially moves along a straight line at constant velocity v_0. Its velocity then decreases uniformly from v_0 to zero in a time interval T. At the end of this time interval, the particle remains at rest.

(a) Find the angular distribution of the radiated *energy* E emitted during the time interval of constant deceleration. Do *not* assume that v_0 is small compared to the speed of light.

If the initial velocity vector is denoted by $\vec{\boldsymbol{v}}_0$, then at the *retarded* time $t_{\rm ret}$,

$$\vec{\boldsymbol{v}}(t_{\rm ret}) = \vec{\boldsymbol{v}}_0 \left(1 - \frac{t_{\rm ret}}{T}\right) , \tag{32}$$

which describes a particle whose velocity decreases uniformly from v_0 to zero in a time interval T. Defining $\vec{\boldsymbol{v}} \equiv c\vec{\boldsymbol{\beta}}$, it follows that

$$\vec{\boldsymbol{\beta}}(t_{\rm ret}) = \vec{\boldsymbol{\beta}}_0 \left(1 - \frac{t_{\rm ret}}{T}\right) . \tag{33}$$

Since $\vec{\boldsymbol{\beta}}$ and $\dot{\vec{\boldsymbol{\beta}}} \equiv d\vec{\boldsymbol{\beta}}/dt_{\rm ret}$ are parallel, eq. (14.39) of Jackson applies,

$$\frac{dP(t_{\rm ret})}{d\Omega} = \frac{e^2\dot{v}^2}{4\pi c^3}\,\frac{\sin^2\theta}{(1-\beta\cos\theta)^5} , \tag{34}$$

where $\dot{v} \equiv d|\vec{\boldsymbol{v}}|/dt_{\rm ret}$ and θ is the angle between $\vec{\boldsymbol{\beta}}$ and the direction of the observer.

Using eqs. (32) and (33), it follows that $\dot{v} = -v_0/T$, and $\beta \equiv |\vec{\boldsymbol{\beta}}| = \beta_0(1 - t_{\rm ret}/T)$. Inserting these results into eq. (34) yields

$$\frac{dP(t_{\rm ret})}{d\Omega} = \frac{e^2 v_0^2}{4\pi c^3 T^2}\,\frac{\sin^2\theta}{\left[1 - \beta_0\cos\theta\left(1 - \dfrac{t_{\rm ret}}{T}\right)\right]^5} .$$

The observer who measures the angular distribution of the radiation will detect the radiation that has been emitted over the entire time interval $0 \leq t_{\rm ret} \leq T$. Hence, in light of eqs.(14.36) and (14.37) of Jackson,

$$\frac{dE}{d\Omega} = \int_0^T \frac{dP(t_{\rm ret})}{d\Omega}\,dt_{\rm ret} = \frac{e^2 v_0^2\sin^2\theta}{4\pi c^3 T^2}\int_0^T \frac{dt}{\left[1 - \beta_0\cos\theta\left(1 - \dfrac{t}{T}\right)\right]^5} , \tag{35}$$

after relabeling the integration variable by replacing $t_{\rm ret}$ with t.

The integral in eq. (35) is elementary. Define a new variable,

$$x = 1 - \beta_0\cos\theta\left(1 - \frac{t}{T}\right) , \qquad\qquad dx = \frac{\beta_0\cos\theta}{T}\,dt .$$

In terms of x, eq. (35) becomes,

$$\frac{dE}{d\Omega} = \frac{e^2 v_0^2\sin^2\theta}{4\pi c^3 T^2}\cdot\frac{T}{\beta_0\cos\theta}\int_{1-\beta_0\cos\theta}^1 \frac{dx}{x^5} = -\frac{e^2 v_0^2\sin^2\theta}{16\pi c^3\beta_0 T\cos\theta}\left[1 - \frac{1}{(1-\beta_0\cos\theta)^4}\right] .$$

Putting $v_0 = c\beta_0$ then yields

$$\frac{dE}{d\Omega} = \frac{e^2\beta_0}{16\pi c T}\cdot\frac{\sin^2\theta}{\cos\theta}\left[\frac{1}{(1-\beta_0\cos\theta)^4} - 1\right] . \tag{36}$$

(b) Suppose $v_0 \ll c$. Calculate $dE/d\Omega$ (to leading order in v_0/c). Integrate over angles to determine the total radiated energy emitted during the deceleration.

If $v_0 \ll c$ then $\beta_0 \ll 1$, and

$$\frac{1}{(1-\beta_0\cos\theta)^4} - 1 \simeq (1+4\beta_0\cos\theta) - 1 = 4\beta_0\cos\theta\,.$$

In this case, eq. (36) reduces to

$$\frac{dE}{d\Omega} = \frac{e^2\beta_0^2}{4\pi cT}\,\sin^2\theta\,.$$

Integrating over solid angles using

$$\int d\Omega\,\sin^2\theta = 2\pi\int_{-1}^{1}(1-\cos^2\theta)\,d\cos\theta = \frac{8\pi}{3}\,,$$

we end up with

$$E = \frac{2e^2\beta_0^2}{3cT}\,.$$

4. Consider the scattering of an electromagnetic wave of (angular) frequency ω and polarization $\hat{\boldsymbol{\varepsilon}}_0$ off of an electron bound in an atom. The wavelength of the incoming wave is assumed to be significantly larger than the size of the atom. You may also assume that the non-relativistic limit is a good approximation.

One can model the electron by assuming that it is bound by a damped harmonic oscillator force with oscillation frequency ω_0 and damping coefficient η. The electron, with mass m and charge $-e$, also experiences a force due to the electric field of the incoming wave. The response of the electron to the initial wave is an induced time-dependent electric dipole moment, $\vec{\boldsymbol{p}}(t) = \mathrm{Re}(\vec{\boldsymbol{p}}\,e^{-i\omega t})$, where the complex vector $\vec{\boldsymbol{p}}$ is given by

$$\vec{\boldsymbol{p}} = \frac{e^2}{m}\left(\frac{E_0\,\hat{\boldsymbol{\epsilon}}_0}{\omega_0^2-\omega^2-i\eta\omega}\right)\,. \tag{37}$$

(a) Assuming that the incoming wave is left circularly polarized and propagates in the positive z-direction, compute the angular distribution of the scattering cross section, under the assumption that the final state polarization $\hat{\boldsymbol{\varepsilon}}$ is not observed. Express the coefficient of the angular factor as a function of the frequency of the incoming wave.

We may use eq. (10.4) of Jackson. Since there is no magnetic moment, we shall set $\vec{\boldsymbol{m}} = 0$. (The higher ℓ moments can be neglected in the long wavelength approximation.) In this problem I prefer to employ gaussian units, so I will remove the factor of $4\pi\epsilon_0$ from eq. (10.4) of Jackson. Thus,[2]

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{|E_0|^2}\,|\hat{\boldsymbol{\epsilon}}^* \cdot \vec{\boldsymbol{p}}|^2\,. \tag{38}$$

Inserting the expression for $\vec{\boldsymbol{p}}$ given in eq. (37), and noting that $k = \omega/c$, we find[3]

$$\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{mc^2}\right)^2 |\hat{\boldsymbol{\epsilon}}^* \cdot \hat{\boldsymbol{\epsilon}}_0|^2\,\frac{\omega^4}{(\omega_0^2-\omega^2)^2+\eta^2\omega^2}\,. \tag{39}$$

Since the final state polarization is not observed, we must sum over the two possible polarization states. We shall employ the polarization sum formula derived in class,

$$\sum_\lambda \epsilon_i^{(\lambda)\,*}\epsilon_j^{(\lambda)} = \delta_{ij} - \hat{n}_i\hat{n}_j\,, \tag{40}$$

where the $\hat{n}_i$ are the components of the unit vector $\hat{\boldsymbol{n}} \equiv \vec{\boldsymbol{k}}/k$. Thus,

$$\sum_\lambda |\hat{\boldsymbol{\epsilon}}^{(\lambda)\,*} \cdot \hat{\boldsymbol{\epsilon}}_0|^2 = \sum_\lambda \epsilon_i^{(\lambda)\,*}\epsilon_j^{(\lambda)}(\epsilon_0)_i(\epsilon_0^*)_j = (\epsilon_0)_i(\epsilon_0^*)_j\,[\delta_{ij}-\hat{n}_i\hat{n}_j] = 1 - |\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{\epsilon}}_0|^2\,, \tag{41}$$

after using $\hat{\boldsymbol{\epsilon}}_0 \cdot \hat{\boldsymbol{\epsilon}}_0^* = 1$ in the final step.

For a left circularly polarized incident wave moving in the $\hat{\boldsymbol{z}}$ direction, $\hat{\boldsymbol{\epsilon}}_0 = \frac{1}{\sqrt{2}}(\hat{\boldsymbol{x}}+i\hat{\boldsymbol{y}})$, following the optics conventions introduced by Jackson on the top of p. 300.[4] We shall define the angles of the scattered wave using spherical coordinates with respect to the fixed z-axis,

$$\hat{\boldsymbol{n}} = \hat{\boldsymbol{x}}\sin\theta\cos\phi + \hat{\boldsymbol{y}}\sin\theta\sin\phi + \hat{\boldsymbol{z}}\cos\theta\,.$$

Consequently,

$$\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{\epsilon}}_0 = \frac{1}{\sqrt{2}}\,\sin\theta\,e^{i\phi}\,.$$

Inserting this result into eq. (41) yields:[5]

$$\sum_\lambda |\hat{\boldsymbol{\epsilon}}^{(\lambda)\,*} \cdot \hat{\boldsymbol{\epsilon}}_0|^2 = 1 - \tfrac{1}{2}\sin^2\theta = 1 - \tfrac{1}{2}(1-\cos^2\theta) = \tfrac{1}{2}(1+\cos^2\theta)\,. \tag{42}$$

Using the above result in eq. (39), we end up with

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{2}\left(\frac{e^2}{mc^2}\right)^2 \frac{\omega^4}{(\omega_0^2-\omega^2)^2+\eta^2\omega^2}\,(1+\cos^2\theta)\,.} \tag{43}$$

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[2]Since E_0 may be complex, one should write $|E_0|^2$ in eq. (38) as I did in class rather than E_0^2 as Jackson does in eq. (10.4).

[3]To convert the cross section formulae of this problem to SI units, simply replace $e^2 \to e^2/(4\pi\epsilon_0)$.

[4]See also the class handout entitled *Polarization Vectors and Polarization Sums*.

[5]Note that $\sum_\lambda |\hat{\boldsymbol{\epsilon}}^{(\lambda)\,*}\cdot\hat{\boldsymbol{\epsilon}}_0|^2 = 1 - |\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{\epsilon}}_0|^2 = |\hat{\boldsymbol{n}}\times\hat{\boldsymbol{\epsilon}}_0|^2$. Thus, for a left circularly polarized incoming wave moving in the $\hat{\boldsymbol{z}}$ direction, $\hat{\boldsymbol{\epsilon}}_0 = \frac{1}{\sqrt{2}}(\hat{\boldsymbol{x}}+i\hat{\boldsymbol{y}})$, and we have $\hat{\boldsymbol{n}}\times\hat{\boldsymbol{\epsilon}}_0 = -\frac{i}{\sqrt{2}}\left[(\hat{\boldsymbol{x}}+i\hat{\boldsymbol{y}})\cos\theta - \hat{\boldsymbol{z}}\sin\theta\,e^{i\phi}\right]$. It follows that $|\hat{\boldsymbol{n}}\times\hat{\boldsymbol{\epsilon}}_0|^2 = \frac{1}{2}\left[2\cos^2\theta+\sin^2\theta\right] = \frac{1}{2}(1+\cos^2\theta)$, and we recover the result of eq. (42).

(b) Integrate the differential cross section over angles to obtain the total scattering cross section. Compare your result to the Thomson cross section in the following three limiting cases: (i) $\omega \gg \omega_0 \sim \eta$; (ii) $\omega = \omega_0 \gg \eta$; and (iii) $\omega \ll \omega_0 \sim \eta$.

Integrating the differential cross section obtained in eq. (43) over angles over solid angles using,

$$\int d\Omega(1+\cos^2\theta) = 2\pi \int_{-1}^{1} d\cos\theta\,(1+\cos^2\theta) = \frac{16\pi}{3}\,,$$

we obtain the total cross section,

$$\sigma = \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2 \frac{\omega^4}{(\omega_0^2-\omega^2)^2+\eta^2\omega^2}\,. \tag{44}$$

We recognize the Thomson cross section [cf. eq. (14.126) of Jackson]:

$$\sigma_T \equiv \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2\,.$$

Hence, one can rewrite eq. (44) as:

$$\sigma = \frac{\omega^4}{(\omega_0^2-\omega^2)^2+\eta^2\omega^2}\,\sigma_T\,.$$

Three limiting cases will now be considered.

Case 1: $\omega \gg \omega_0 \sim \eta$.

In this limit, the factor

$$\frac{\omega^4}{(\omega_0^2-\omega^2)^2+\eta^2\omega^2} \simeq 1 \qquad \longrightarrow \qquad \sigma \simeq \sigma_T\,.$$

Case 2: $\omega = \omega_0 \gg \eta$.

In this limit,

$$\sigma \simeq \frac{\omega_0^2}{\eta^2}\,\sigma_T \gg \sigma_T\,.$$

This is a case of resonant enhancement due to the fact that the frequency of the incoming wave matches the natural frequency of the harmonic potential that binds the electron to the atom.

Case 3: $\omega \ll \omega_0 \sim \eta$.

In this limit,

$$\sigma \simeq \frac{\omega^4}{\omega_0^4}\,\sigma_T\,,$$

which corresponds to Rayleigh scattering in the long wavelength approximation.

REMARK: A more proper analysis would include the effects of radiation reaction. For a discussion of these effects, see section 16.8 of Jackson.

(c) Derive eq. (37).

The electron is assumed to be bound by a damped harmonic oscillator force,

$$\vec{\boldsymbol{F}} = -m\omega_0^2\vec{\boldsymbol{x}}(t) - \eta m \frac{d\vec{\boldsymbol{x}}(t)}{dt}\,, \tag{45}$$

where $\vec{\boldsymbol{x}}(t)$ is the position (at time t) of the electron, ω_0 is the natural frequency of the oscillator, and η is the damping coefficient. Note that the minus signs above indicate that this is a restoring force. Using Newton's second law, it is straightforward to show that the response of the electron to the initial wave can be modeled by a time-dependent electric dipole moment, $\vec{\boldsymbol{p}}(t) = \mathrm{Re}(\vec{\boldsymbol{p}}\,e^{-i\omega t})$, where the complex vector $\vec{\boldsymbol{p}}$ is given by eq. (37).

Consider a charged particle of mass m and charge $-e$ subject to a damped harmonic oscillator force given in eq. (45), which also experiences the force due to the electric field of the incoming wave. Newton's second law then takes the following form,

$$m\frac{d^2\vec{\boldsymbol{x}}(t)}{dt^2} = -e\vec{\boldsymbol{E}}(t) - m\omega_0^2\vec{\boldsymbol{x}}(t) - \eta m \frac{d\vec{\boldsymbol{x}}(t)}{dt}\,, \tag{46}$$

where the time-dependent electric field of the incident wave is given by,

$$\vec{\boldsymbol{E}}(t) = \mathrm{Re}(\vec{\boldsymbol{E}}\,e^{-i\omega t})\,, \qquad\qquad \text{with } \vec{\boldsymbol{E}} = E_0\hat{\boldsymbol{\epsilon}}_0\,.$$

Assuming a solution to eq. (46) of the form $\vec{\boldsymbol{x}}(t) = \mathrm{Re}(\vec{\boldsymbol{x}}\,e^{-i\omega t})$, we insert this ansatz into eq. (46) and obtain,

$$-m\omega^2\vec{\boldsymbol{x}} = -eE_0\hat{\boldsymbol{\epsilon}}_0 - m\omega_0^2\vec{\boldsymbol{x}} + i\eta\omega m\vec{\boldsymbol{x}}\,.$$

Solving for $\vec{\boldsymbol{x}}$, we get

$$\vec{\boldsymbol{x}} = -\frac{e}{m}\left(\frac{E_0\hat{\boldsymbol{\epsilon}}_0}{\omega_0^2 - \omega^2 - i\eta\omega}\right)\,.$$

Thus, the time-dependent electric dipole moment is given by $\vec{\boldsymbol{p}}(t) = \mathrm{Re}(\vec{\boldsymbol{p}}\,e^{-i\omega t})$, where

$$\vec{\boldsymbol{p}} = -e\vec{\boldsymbol{x}} = \frac{e^2}{m}\left(\frac{E_0\hat{\boldsymbol{\epsilon}}_0}{\omega_0^2 - \omega^2 - i\eta\omega}\right)\,,$$

which establishes eq. (37).

APPENDIX: Evaluation of the integrals given in eqs. (21) and (22)

Consider the following two integrals:

$$I_1 \equiv \int \frac{\hat{\boldsymbol{n}}\, d\Omega}{(1-\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\hat{\boldsymbol{n}})^3}\,, \tag{47}$$

$$I_2 \equiv \int \frac{\hat{n}_i \hat{n}_j\, d\Omega}{(1-\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\hat{\boldsymbol{n}})^4}\,. \tag{48}$$

We shall employ the methods described in the class handout entitled *Evaluation of some integrals over solid angles—Part 2* to evaluate eqs. (47) and (48). We begin by evaluated the following integral:

$$I_0 \equiv \int \frac{d\Omega}{(1-\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\hat{\boldsymbol{n}})^2} = 2\pi \int_{-1}^{1} \frac{d\cos\theta}{1-\beta\cos\theta} = \frac{2\pi}{\beta}\int_{1-\beta}^{1+\beta}\frac{dy}{y^2} = -\frac{2\pi}{\beta}\left[\frac{1}{1+\beta}-\frac{1}{1-\beta}\right]\,, \tag{49}$$

after defining $\beta \equiv |\vec{\boldsymbol{\beta}}|$ and changing the integration variable to $y = 1 - \beta w$. Hence,

$$\boxed{I_0 = \frac{4\pi}{1-\beta^2}\,.} \tag{50}$$

Next, we can take the derivative of I_0 with respect to $\vec{\boldsymbol{\beta}}$,

$$\frac{\partial I_0}{\partial \vec{\boldsymbol{\beta}}} = 2\int \frac{\hat{\boldsymbol{n}}\, d\Omega}{(1-\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\hat{\boldsymbol{n}})^3}\,. \tag{51}$$

Thus, we can identify

$$I_1 = \frac{1}{2}\frac{\partial I_0}{\partial \vec{\boldsymbol{\beta}}}\,. \tag{52}$$

Since eq. (50) is a function of β, we can use the chain rule to write

$$2I_1 = \frac{\partial I_0}{\partial \vec{\boldsymbol{\beta}}} = \frac{\partial \beta}{\partial \vec{\boldsymbol{\beta}}}\frac{\partial I_0}{\partial \beta} = \frac{\vec{\boldsymbol{\beta}}}{\beta}\frac{\partial I_0}{\partial \beta} = \frac{8\pi\vec{\boldsymbol{\beta}}}{(1-\beta^2)^2}\,. \tag{53}$$

To obtain the penultimate step above, we noted that $\beta = (\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{\beta}})^{1/2}$. Hence, it follows that

$$\frac{\partial \beta}{\partial \vec{\boldsymbol{\beta}}} = \frac{\partial}{\partial \vec{\boldsymbol{\beta}}}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{\beta}})^{1/2} = \tfrac{1}{2}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{\beta}})^{-1/2}\frac{\partial}{\partial \vec{\boldsymbol{\beta}}}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{\beta}}) = (\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{\beta}})^{-1/2}\vec{\boldsymbol{\beta}} = \frac{\vec{\boldsymbol{\beta}}}{\beta}\,. \tag{54}$$

Hence, eq. (53) yields

$$\boxed{I_1 = \frac{4\pi\vec{\boldsymbol{\beta}}}{(1-\beta^2)^2}} \tag{55}$$

which confirms the result of eq. (21).

Finally, we can take the derivative of eq. (51) to obtain

$$\frac{\partial^2 I_0}{\partial\beta_i\partial\beta_j} = 6\int \frac{\hat{n}_i\hat{n}_j\, d\Omega}{(1-\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\hat{\boldsymbol{n}})^4} = 6I_2\,. \tag{56}$$

The second derivative can now be easily evaluated with the help of eq. (54). Starting from

$$\frac{\partial I_0}{\partial\beta_i} = \frac{\beta_i}{\beta}\,\frac{\partial I_0}{\partial\beta}\,, \tag{57}$$

we take one further derivative to obtain

$$\begin{aligned}
\frac{\partial^2 I_0}{\partial\beta_i\partial\beta_j} &= \delta_{ij}\frac{1}{\beta}\,\frac{\partial I_0}{\partial\beta} + \beta_i\frac{\partial}{\partial\beta}\left(\frac{1}{\beta}\,\frac{\partial I_0}{\partial\beta}\right)\frac{\partial\beta}{\partial\beta_j} \\
&= \frac{1}{\beta}\left[\delta_{ij}\frac{\partial I_0}{\partial\beta} + \beta_i\beta_j\frac{\partial}{\partial\beta}\left(\frac{1}{\beta}\,\frac{\partial I_0}{\partial\beta}\right)\right] \\
&= \frac{8\pi}{(1-\beta^2)^2}\left[\delta_{ij} + \frac{4\beta_i\beta_j}{1-\beta^2}\right]
\end{aligned} \tag{58}$$

Hence, it follows that

$$\boxed{I_2 = \frac{4\pi}{3(1-\beta^2)^3}\left[(1-\beta^2)\delta_{ij} + 4\beta_i\beta_j\right]} \tag{59}$$

which confirms the result of eq. (22).

An alternate method for evaluating eqs. (47) and (48)

There is an alternative technique for evaluating I_1 and I_2. We can define the following two integrals,

$$J_{ij} \equiv \int \frac{\hat{n}_i \hat{n}_j \, d\Omega}{(1-\vec{\boldsymbol{\beta}} \boldsymbol{\cdot} \hat{\boldsymbol{n}})^4}\,, \tag{60}$$

$$K_i \equiv \int \frac{\hat{n}_i \, d\Omega}{(1-\vec{\boldsymbol{\beta}} \boldsymbol{\cdot} \hat{\boldsymbol{n}})^3}\,. \tag{61}$$

By the covariance properties of Euclidean tensors, it follows that

$$J_{ij} = c_1 \delta_{ij} + c_2 \beta_i \beta_j\,, \tag{62}$$

$$K_i = \kappa \beta_i\,. \tag{63}$$

Consider first the evaluation of K_i. Multiplying by β_i and summing over i yields

$$\kappa \beta^2 = \int \frac{\vec{\boldsymbol{\beta}} \boldsymbol{\cdot} \hat{\boldsymbol{n}} \, d\Omega}{(1-\vec{\boldsymbol{\beta}} \boldsymbol{\cdot} \hat{\boldsymbol{n}})^3} = \int d\Omega \left[\frac{1}{(1-\vec{\boldsymbol{\beta}} \boldsymbol{\cdot} \hat{\boldsymbol{n}})^3} - \frac{1}{(1-\vec{\boldsymbol{\beta}} \boldsymbol{\cdot} \hat{\boldsymbol{n}})^2}\right]. \tag{64}$$

In eq. (5) of the class handout entitled *Evaluation of some integrals over solid angles—Part 2*, we obtained

$$\int \frac{d\Omega}{(1-\vec{\boldsymbol{\beta}} \boldsymbol{\cdot} \hat{\boldsymbol{n}})^3} = \frac{4\pi}{(1-\beta^2)^2}\,. \tag{65}$$

Using this result along with eq. (50), we end up with

$$\kappa \beta^2 = 4\pi \left[\frac{1}{(1-\beta^2)^2} - \frac{1}{1-\beta^2}\right] = \frac{4\pi \beta^2}{(1-\beta^2)^2}\,. \tag{66}$$

Hence, we conclude that

$$K_i = \frac{4\pi \beta_i}{(1-\beta^2)^2}\,, \tag{67}$$

in agreement with eq. (55).

Likewise, to evaluate J_{ij}, we first multiply by δ_{ij} and sum over i and j to get one equation. A second equation is obtained by multiplying by $\beta_i \beta_j$ and summing over i and j, Thus, we get two equations for the two unknowns c_1 and c_2,

$$3c_1 + c_2 \beta^2 = \int \frac{d\Omega}{(1-\vec{\boldsymbol{\beta}} \boldsymbol{\cdot} \hat{\boldsymbol{n}})^4}\,, \tag{68}$$

$$c_1 \beta^2 + c_2 \beta^4 = \int \frac{(\vec{\boldsymbol{\beta}} \boldsymbol{\cdot} \hat{\boldsymbol{n}})^2 \, d\Omega}{(1-\vec{\boldsymbol{\beta}} \boldsymbol{\cdot} \hat{\boldsymbol{n}})^4}\,. \tag{69}$$

We first evaluate the following integral:

$$\int \frac{d\Omega}{(1-\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\hat{\boldsymbol{n}})^4} = 2\pi\int_{-1}^{1}\frac{d\cos\theta}{(1-\beta\cos\theta)^4} = \frac{2\pi}{\beta}\int_{1-\beta}^{1+\beta}\frac{dy}{y^4}$$

$$= -\frac{2\pi}{3\beta}\left[\frac{1}{(1+\beta)^3}-\frac{1}{(1-\beta)^3}\right] = \frac{4\pi(\beta^2+3)}{3(1-\beta^2)^3}\,, \tag{70}$$

by employing the same steps previously used in evaluating eq. (49). To evaluate the integral in eq. (69), we write

$$\int \frac{(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\hat{\boldsymbol{n}})^2\,d\Omega}{(1-\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\hat{\boldsymbol{n}})^4} = \int d\Omega\,\left[\frac{1}{(1-\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\hat{\boldsymbol{n}})^2}+\frac{2\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}}{(1-\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\hat{\boldsymbol{n}})^4}-\frac{1}{(1-\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\hat{\boldsymbol{n}})^4}\right]. \tag{71}$$

Using eq. (11) of the class handout entitled *Evaluation of some integrals over solid angles—Part 2*, it follows that

$$\int \frac{\hat{\boldsymbol{n}}\,d\Omega}{(1-\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\hat{\boldsymbol{n}})^4} = \frac{16\pi}{3}\,\frac{\vec{\boldsymbol{\beta}}}{(1-\beta^2)^3}\,. \tag{72}$$

The other two integrals appearing on the right hand side of eq. (71) have been obtained in eqs. (50) and (70). Combining these results, we obtain

$$\int \frac{(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\hat{\boldsymbol{n}})^2\,d\Omega}{(1-\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\hat{\boldsymbol{n}})^4} = \frac{4\pi}{1-\beta^2}+\frac{32\pi}{3}\,\frac{\beta^2}{(1-\beta^2)^3}-\frac{4\pi(\beta^2+3)}{3(1-\beta^2)^3}$$

$$= \frac{4\pi\beta^2(1+3\beta^2)}{3(1-\beta^2)^3}\,. \tag{73}$$

Thus, eqs. (68) and (69) yield

$$3c_1 + c_2\beta^2 = \frac{4\pi(\beta^2+3)}{3(1-\beta^2)^3}\,, \tag{74}$$

$$c_1\beta^2 + c_2\beta^4 = \frac{4\pi\beta^2(1+3\beta^2)}{3(1-\beta^2)^3}\,. \tag{75}$$

One can now solve for c_1 and c_2. It is straightforward to obtain,

$$c_1 = \frac{4\pi}{3}\,\frac{1}{(1-\beta^2)^2}\,, \qquad\qquad c_2 = \frac{16\pi}{3}\,\frac{1}{(1-\beta^2)^3}\,. \tag{76}$$

Hence, we conclude that

$$J_{ij} = \frac{4\pi}{3(1-\beta^2)^3}\left[(1-\beta^2)\delta_{ij}+4\beta_i\beta_j\right], \tag{77}$$

in agreement with eq. (59).