

## Clifford Algebras

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### Exterior Algebra $\Lambda\mathbb{R}^n$

The exterior algebra  $\Lambda\mathbb{R}^n$  consists of the direct sum from 1 to n of the linear vector spaces of differential k-forms on the vector space  $\mathbb{R}^n$

$$\Lambda\mathbb{R}^n = \Lambda^0\mathbb{R} \oplus \Lambda^1\mathbb{R}^n \oplus \dots \oplus \Lambda^n\mathbb{R}^n$$

with the product of the basis vectors  $\{\mathbf{e}_i\}$  of  $\mathbb{R}^n$  defined as the antisymmetric wedge product

$$\begin{aligned}\mathbf{e}_i \wedge \mathbf{e}_j &= -\mathbf{e}_j \wedge \mathbf{e}_i, \quad i \neq j \\ \mathbf{e}_i \wedge \mathbf{e}_i &= 0\end{aligned}$$

The basis for the subspace  $\Lambda\mathbb{R}^k$  in the exterior algebra is

$$\{\mathbf{e}_{i_1} \wedge \mathbf{e}_{i_2} \wedge \dots \wedge \mathbf{e}_{i_k} | 1 \leq i_1 < i_2 < \dots < i_k \leq n, \quad 0 \leq k \leq n\}$$

### Clifford Algebra $\mathcal{C}\ell_n$

The Clifford algebra  $\mathcal{C}\ell_n$  is the exterior space  $\Lambda\mathbb{R}^n$

$$\mathcal{C}\ell_n = \Lambda\mathbb{R}^n$$

with the product of basis vectors  $\{\mathbf{e}_i\}$  of  $\mathbb{R}^n$  defined as the Clifford product

$$\begin{aligned}\mathbf{e}_i \mathbf{e}_j &= -\mathbf{e}_j \mathbf{e}_i, \quad i \neq j \\ \mathbf{e}_i \mathbf{e}_i &= 1\end{aligned}$$

The basis for the subspace of differential k-forms  $\Lambda^k\mathbb{R}^n$  in the Clifford algebra is

$$\{\mathbf{e}_{i_1} \mathbf{e}_{i_2} \dots \mathbf{e}_{i_k} | 1 \leq i_1 < i_2 < \dots < i_k \leq n, \quad 0 \leq k \leq n\}$$

## Geometric Product

Given two vectors  $a$  and  $b$ , the Clifford Product can be expressed as

$$ab = a \cdot b + a \wedge b$$

From symmetry and antisymmetry of terms it follows that

$$a \cdot b = \frac{1}{2}(ab + ba)$$
$$a \wedge b = \frac{1}{2}(ab - ba)$$

Vector division can now be defined, with

$$a^{-1} = a/a^2$$

## Clifford Algebra $\mathcal{Cl}_2$

The basis vectors spanning the Clifford Algebra  $\mathcal{Cl}_2$  are

$$\begin{array}{ll} 1 & 1 \text{ scalar} \\ \{\mathbf{e}_1, \mathbf{e}_2\} & 2 \text{ vectors} \\ \mathbf{e}_1\mathbf{e}_2 & 1 \text{ bivector} \end{array}$$

and obey the geometric product

$$\mathbf{e}_1\mathbf{e}_1 = \mathbf{e}_2\mathbf{e}_2 = 1$$
$$\mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_2\mathbf{e}_1$$

Elements of this algebra are 4-dimensional multivectors of the form

$$a = a_0 + a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_{12}\mathbf{e}_1\mathbf{e}_2$$

The above system can compactly handle algebra and rotations for complex numbers in  $\mathbb{C}$ , and vectors  $\mathbb{R}^2$ .

### Complex numbers and Vectors in $\mathcal{Cl}_2$

We can combine a scalar and a bivector of  $\mathcal{Cl}_2$  to get

$$Z = u + v\mathbf{e}_1\mathbf{e}_2$$

Recognizing that

$$(\mathbf{e}_1\mathbf{e}_2)^2 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_1\mathbf{e}_1\mathbf{e}_2\mathbf{e}_2 = -1$$

we can identify the bivector  $\mathbf{e}_1\mathbf{e}_2$  with what we think of as  $i$  (however unlike  $i$ ,  $\mathbf{e}_1\mathbf{e}_2$  does not commute with everything). We usually think of a complex number in  $\mathbb{C}$  as representing a vector in  $\mathbb{R}^2$

$$x = u\mathbf{e}_1 + v\mathbf{e}_2$$

The mapping from  $\mathbb{C} \rightarrow \mathbb{R}^2$  can be accomplished within the Clifford algebra by

$$x = \mathbf{e}_1 Z$$

### Rotations in $\mathcal{Cl}_2$

Remember that we can think of complex numbers written in polar form  $z = e^{i\theta}$  as representing rotations in the Argand diagram. We quickly verify that

$$R = e^{\mathbf{e}_1\mathbf{e}_2\theta} = \cos \theta + \mathbf{e}_1\mathbf{e}_2 \sin \theta$$

We can see how to describe rotations of vectors in  $\mathbb{R}^2$  within  $\mathcal{Cl}_2$ . Consider rotating the vector  $x \in \mathbb{R}^2$  by an angle  $\theta$ .

$$x \xrightarrow{\theta} x'$$

Convert the vector  $x$  in  $\mathbb{R}^2$  to a complex number in  $\mathbb{C}$ , multiply by the rotation, then convert back to  $\mathbb{R}^2$ . In  $\mathcal{Cl}_2$  this is done simply as

$$\begin{aligned} x' &= xR \\ &= (u\mathbf{e}_1 + v\mathbf{e}_2)(\cos \theta + \mathbf{e}_1\mathbf{e}_2 \sin \theta) \\ &= (u \cos \theta - v \sin \theta)\mathbf{e}_1 + (u \sin \theta + v \cos \theta)\mathbf{e}_2 \end{aligned}$$

## Clifford Algebra $\mathcal{Cl}_3$

The basis vectors spanning the Clifford Algebra  $\mathcal{Cl}_3$  are

1	1 scalar
$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$	3 vectors
$\{\mathbf{e}_1\mathbf{e}_2, \mathbf{e}_2\mathbf{e}_3, \mathbf{e}_1\mathbf{e}_3\}$	3 bivectors
$\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$	1 trivector

and obey the geometric product

$$\begin{aligned}\mathbf{e}_i\mathbf{e}_i &= 1 \\ \mathbf{e}_i\mathbf{e}_j &= -\mathbf{e}_j\mathbf{e}_i, \quad i \neq j\end{aligned}$$

Elements of this algebra are the 8-dimensional multivectors of the form

$$a = a_0 + a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 + a_{12}\mathbf{e}_1\mathbf{e}_2 + a_{23}\mathbf{e}_2\mathbf{e}_3 + a_{13}\mathbf{e}_1\mathbf{e}_3 + a_{123}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$$

### Quaternions and Vectors in $\mathcal{Cl}_3$

The quaternions are the next higher dimensional analogue of complex numbers. If we combine scalars and the subspace of bivectors we get the elements

$$H = a + b\mathbf{e}_1\mathbf{e}_2 + c\mathbf{e}_2\mathbf{e}_3 + d\mathbf{e}_1\mathbf{e}_3$$

Recognizing that

$$(\mathbf{e}_1\mathbf{e}_2)^2 = (\mathbf{e}_2\mathbf{e}_3)^2 = (\mathbf{e}_1\mathbf{e}_3)^2 = -1$$

we can associate with the bivectors  $\mathbf{e}_1\mathbf{e}_2, \mathbf{e}_2\mathbf{e}_3, \mathbf{e}_1\mathbf{e}_3$  Hamilton's quaternions  $i, j, k$ . (actually, to get it right we would have to associate  $j$  with  $-\mathbf{e}_2\mathbf{e}_3$ . This blemish in our analogy arises from the fact that Hamilton erroneously treated his quaternions as vectors (but really they are bivectors)).

### Rotations in $\mathcal{Cl}_3$

As we saw in the case of  $\mathcal{Cl}_2$ , and will see now, bivectors are the generators of rotations. The product rule between the basis vectors  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$  can be

written more fancily as

$$\mathbf{e}_i \mathbf{e}_j = \delta_{ij} + \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3 \epsilon_{ijk} \mathbf{e}_k$$

The physicist should immediately recognize that this is the same product rule for the pauli spin matrices

$$\sigma^i \sigma^j = \delta_{ij} + i \epsilon_{ijk} \sigma^k$$

We see that the sigma matrices are a matrix representation of the 3-dimensional vector space.

We recall that rotations in 3-dimensions can be described by the unitary transformations of matrices  $e^{i\hat{n}\cdot\vec{\sigma}\theta/2}$ . In our Clifford algebra a rotor in 3-dimensions is

$$R = e^{-(n_1 \mathbf{e}_1 \mathbf{e}_2 + n_2 \mathbf{e}_2 \mathbf{e}_3 + n_3 \mathbf{e}_1 \mathbf{e}_3)\theta/2} = \cos(\theta/2) - (n_1 \mathbf{e}_1 \mathbf{e}_2 + n_2 \mathbf{e}_2 \mathbf{e}_3 + n_3 \mathbf{e}_1 \mathbf{e}_3) \sin(\theta/2)$$

Consider rotating the vector  $x \in \mathbb{R}^3$  by an angle  $\theta$  in the plane and direction defined by the unit bivector  $B = n_1 \mathbf{e}_1 \mathbf{e}_2 + n_2 \mathbf{e}_2 \mathbf{e}_3 + n_3 \mathbf{e}_1 \mathbf{e}_3$

$$x \xrightarrow{B, \theta} x'$$

The desired rotation of  $x$  is achieved in the Clifford algebra by

$$\begin{aligned} x' &= R x R^\dagger \\ &= (\cos(\theta/2) - B \sin(\theta/2))(u \mathbf{e}_1 + v \mathbf{e}_2 + w \mathbf{e}_3)(\cos(\theta/2) + B \sin(\theta/2)) \end{aligned}$$

## Motivation for all this

What's the intuition behind these spaces and algebras? We will see how we can 'derive' the product rules of our Clifford algebra by choosing product rules that enforce the norm we want on our vector space. For euclidean space, we want an L2-norm. We'll see how this approach will let us create a Clifford algebra that can describe a 4-dimensional Minkowski space and the associated rotations.

**L2-norm in  $\mathbb{R}^n$** 

In euclidean space the length of a vector satisfies the L2 norm and is given by

$$L = \sqrt{x^2 + y^2 + z^2}$$

Consider a linear vector space with basis  $\mathbf{e}_i$ . If we define the norm squared  $L^2$  of a vector  $x = x^i \mathbf{e}_i$  by the product

$$L^2 = xx = x_i^2 \mathbf{e}_i^2 + \frac{1}{2} x_i x_j (\mathbf{e}_i \mathbf{e}_j + \mathbf{e}_j \mathbf{e}_i)$$

then  $xx$  will have an L2-norm if we choose the following algebra

$$\mathbf{e}_i \mathbf{e}_j + \mathbf{e}_j \mathbf{e}_i = 2\delta_{ij}$$

**Minkowski norm in  $\mathbb{R}^{1,3}$** 

Special relativity requires that the radius of a ball of light in different frames obey (setting  $c = 1$ )

$$t^2 - r^2 = t'^2 - r'^2 = 0$$

If we take a vector space with basis  $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$  and define the norm squared  $S^2$  of our four vector  $x = x^i \gamma_i$  by the product

$$\begin{aligned} S^2 &= (x^0 \gamma_0 + x^i \gamma_i)(x^0 \gamma_0 + x^j \gamma_j) \\ &= (x^0)^2 \gamma_0^2 + (x^i)^2 \gamma_i^2 + \frac{1}{2} x^i x^j (\gamma_i \gamma_j + \gamma_j \gamma_i) + x^0 x^i (\gamma_0 \gamma_i + \gamma_i \gamma_0) \end{aligned}$$

then  $xx$  will have the Minkowski norm if we choose

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu} = 2\mathbf{diag}(+ - - -);$$

## Clifford Algebra $\mathcal{Cl}_{1,3}$

The basis vectors spanning the Clifford Algebra  $\mathcal{Cl}_{1,3}$  are

1	1 scalar
$\{\gamma_\mu\}$	4 vectors
$\{\gamma_\mu\gamma_\nu\}$	6 bivectors
$\{\gamma_\mu\gamma_\nu\gamma_\alpha\}$	4 trivectors
$\gamma_0\gamma_1\gamma_2\gamma_3$	1 pseudoscalar

and obey the product rule

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu} = 2\mathbf{diag}(+ - - -);$$

Elements of this algebra are 16-dimensional multivectors made of linear combinations of elements in each subspace

$$a = \sum_{r=0}^4 \langle a \rangle_r$$

### Rotations in $\mathcal{Cl}_{1,3}$

Bivectors are the generators of rotations in our 4-dimensional vector space, and recognizing that

$$(\gamma_0\gamma_i)^2 = -\gamma_0^2\gamma_i^2 = +1$$

we see that a rotation in the x-t plane is accomplished through

$$R = e^{\gamma_0\gamma_i\alpha/2} = \cosh(\alpha/2) + \gamma_1\gamma_0 \sinh(\alpha/2)$$

Consider rotating the vector  $x = t\gamma_0 + x\gamma_1 \in \mathbb{R}^{1,3}$  by an angle  $\alpha$  in the plane described by  $\gamma_0 \wedge \gamma_1$

$$x \xrightarrow{\gamma_0 \wedge \gamma_1, \alpha} x'$$

This is achieved through

$$\begin{aligned}x' &= RxR^\dagger = RR^\dagger x \\ &= (\cosh \alpha + \gamma_1 \gamma_0 \sinh \alpha)(t\gamma_0 + x\gamma_1) \\ &= (x \cosh \alpha + t \sinh \alpha)\gamma_0 + (x \sinh \alpha + t \cosh \alpha)\gamma_1\end{aligned}$$

## References

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