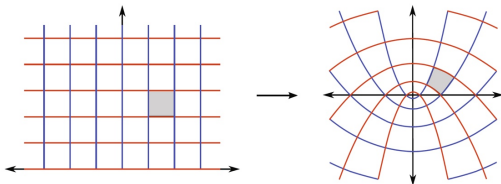


The Conformal Algebra

Dana Faiez

June 14, 2017



Outline ...

- Conformal Transformation/Generators
- 2D Conformal Algebra
- Global Conformal Algebra and Mobius Group
- Conformal Field Theory
- 2D Conformal Field Theory
- Virasoro Algebra
- Application in Statistical Mechanics
- Massless Free Boson Example

Conformal transformations

Distance squared between two points with coordinates x^μ and $x^\mu + dx^\mu$ is

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu$$

Change of coordinates

$$x \Rightarrow x'(x)$$

$$dx^\mu = \frac{\partial x^\mu}{\partial x'^\rho} dx'^\rho$$

This coordinate transformation keeps distance between two nearby points ds invariant.

$$ds^2 = g_{\mu\nu}(x) \frac{\partial x^\mu}{\partial x'^\rho} \frac{\partial x^\nu}{\partial x'^\sigma} dx'^\rho dx'^\sigma = g'_{\rho\sigma}(x') dx'^\rho dx'^\sigma$$

Under a change of coordinates, the metric transforms according to

$$g'_{\rho\sigma}(x') = g_{\mu\nu}(x) \frac{\partial x^\mu}{\partial x'^\rho} \frac{\partial x^\nu}{\partial x'^\sigma}$$

Ex: Scale transformation
(for λ real number)

$$x^\mu \rightarrow x'^\mu = \lambda x^\mu$$

then g transforms like:

$$g'_{\rho\sigma}(x') = g'_{\rho\sigma}(\lambda x) = \Omega^2(x) g_{\mu\nu}$$

such that:

$$\Omega(x) \equiv e^{\frac{wx}{2}}$$

where $w = -2 \ln \lambda \Rightarrow g'_{\rho\sigma}(x') = \frac{1}{\lambda^2} g_{\mu\nu}(x)$

More generally, we could have

$$g'_{\rho\sigma}(x') = \Omega^2(x)g_{\mu\nu}(x)$$

Transformations $x \rightarrow x'(x)$ that satisfy this condition are known as conformal transformations.

Conformal Group

Conformal transformations forms a group:

For two conformal transformations Ω_1 and Ω_2 ,

$$g''_{\rho\sigma}(x) = \Omega_2^2(x)g'_{\rho\sigma}(x)$$

$$= \Omega_2^2(x)\Omega_1^2(x)g_{\rho\sigma}(x) = (\Omega_2(x)\Omega_1(x))^2g_{\rho\sigma}(x).$$

\Rightarrow conformal transformation associated with $\Omega_2\Omega_1$.

Infinitesimal Limit

Consider the infinitesimal coordinate transformation:

$$x'^{\mu} = x^{\mu} + \epsilon \zeta^{\mu}(x)$$

Expand $\Omega^2(x')$ for $\epsilon \rightarrow 0$

$$\Omega^2(x') \simeq 1 + \epsilon \kappa(x') = 1 + \epsilon \kappa(x) + O(\epsilon^2)$$

with κ being some unknown function of the coordinates. Plug this into $g'_{\rho\sigma}(x) = \Omega^2(x)g_{\mu\nu}(x)$

$$g_{\mu\sigma} \partial_{\rho} \zeta^{\mu} + g_{\rho\nu} \partial_{\sigma} \zeta^{\nu} + \zeta^{\lambda} \partial_{\lambda} g_{\rho\sigma} + \kappa g_{\rho\sigma} = 0$$

Conformal Killing Condition.

Let's consider the simple case of flat spacetime where

$$g_{\mu\nu} = \eta_{\mu\nu}$$

with $\eta = \text{diag}(1, -1, -1, \dots, -1)$, in d -dimension. Then the conformal Killing condition reduces to:

$$\partial_\rho \zeta_\sigma + \partial_\sigma \zeta_\rho + \kappa \eta_{\rho\sigma} = 0$$

Contracting with $\eta^{\rho\sigma}$ we get:

$$-2(\partial \cdot \zeta) = d\kappa$$

plugging this into the reduced conformal Killing condition, we get:

$$\partial_\rho \zeta_\sigma + \partial_\sigma \zeta_\rho = \frac{2}{d}(\partial \cdot \zeta)\eta_{\rho\sigma}$$

Act with $\partial^\rho = \eta^{\rho\sigma} \partial_\sigma$ on $\partial_\rho \zeta_\sigma + \partial_\sigma \zeta_\rho = -\frac{2}{d}(\partial \cdot \zeta)\eta_{\rho\sigma}$ to get:

$$d\partial^2\zeta_\sigma = (2-d)\partial_\sigma(\partial \cdot \zeta)$$

We can make two conclusions:

1. The case $d = 2$ is special. Any solutions of the generalized Laplace equation $\partial^2\zeta_\sigma = 0$ yield a conformal transformation. (We'll talk about this case later)

Generators

2. For $d \neq 2 \rightarrow \zeta^\mu$ can depend on x at most quadratically.

\Rightarrow All solutions of the conformal Killing condition of Minkowski spacetime for $d \neq 2$ can be found:

$$\zeta^\mu = \underbrace{a^\mu}_{\text{translation}} + \underbrace{b_\nu^\mu x^\nu}_{\text{Lorentz}} + \underbrace{c x^\mu}_{\text{scale}} + \underbrace{d_\nu (\eta^{\mu\nu} x^2 - 2x^\mu x^\nu)}_{\text{Special conformal}}$$

Transformation	Transformation	Generator
D	$x^\mu \rightarrow \lambda x^\mu$	$x^\mu \partial_\mu$
P_μ	$x^\mu \rightarrow a^\mu + b_\nu^\mu x^\nu$	∂_μ
$J_{\mu\nu}$		$(x_\mu \partial_\nu - x_\nu \partial_\mu)$
K^μ	$x^\mu \rightarrow a_\lambda (\eta^{\mu\lambda} x^2 - 2x^\mu x^\lambda)$	$(\eta^{\mu\nu} x^2 - 2x^\mu x^\nu) \partial_\nu$

$$[D, P^\mu] = -P^\mu$$

$$[D, K^\mu] = +K^\mu$$

$$[J^{\mu\nu}, P^\lambda] = -\eta^{\mu\lambda} P^\nu + \eta^{\nu\lambda} P^\mu$$

$$[J^{\mu\nu}, K^\lambda] = -\eta^{\mu\lambda} K^\nu + \eta^{\nu\lambda} K^\mu$$

$$[J^{\mu\nu}, J^{\lambda\rho}] = -\eta^{\mu\lambda} J^{\nu\rho} - \eta^{\nu\rho} J^{\mu\lambda} + \eta^{\nu\lambda} J^{\mu\rho} + \eta^{\mu\rho} J^{\nu\lambda}$$

$$[K^\mu, P^\nu] = 2(J^{\mu\nu} + \eta^{\mu\nu} D)$$

(All the ones not mentioned are zero.)

Counting the number of generator ($d > 2$):

$$\underbrace{d}_{\text{translation}} + \frac{\overbrace{d(d-1)}^{\text{Lorentz}}}{2} + \underbrace{1}_{\text{scale}} + \underbrace{d}_{\text{Special conformal}} = \frac{\overbrace{(d+2)(d+1)}^{\text{\#of gen. of SO}(d,2)}}{2}$$

Gen. of conformal (d-D) || Gen. of SO(d,2) (d+2-D):

Denote generators of SO(2,d) by J^{MN} with

$M, N = 0, 1, \dots, d-1, d, d+1$ and $(\mu, \nu = 0, 1, \dots, d-1)$

Then we have the following commutation relation between $J^{\mu\nu}$ s

with $\overbrace{\eta^{00} = +1}^{\text{time-like}}, \overbrace{\eta^{ij} = -\delta^{ij}, \eta^{dd} = -1}^{\text{space-like}}, \overbrace{\eta^{d+1,d+1} = +1}^{\text{time-like}}$:

$$[J^{MN}, J^{PQ}] = -\eta^{MP} J^{NQ} - \eta^{NQ} J^{MP} + \eta^{NP} J^{MQ} + \eta^{MQ} J^{NP}$$

$$D = J^{d,d+1}$$

P^μ and $K^\mu \rightsquigarrow$ L.C. of $J^{\mu,d}$ and $J^{\mu,d+1}$

Guess:

$$J^{\mu,d} = \frac{1}{2}(K^\mu + P^\mu)$$

$$J^{\mu,d+1} = \frac{1}{2}(K^\mu - P^\mu)$$

These can be checked using commutation relations of Js.

By symmetry considerations, conformal algebra of d-D Minkowski spacetime with isometry group of $SO(d-1,1)$, is isomorphic to $SO(d, 2)$ algebra.

2D Conformal Algebra

The Reduced conformal killing condition becomes the Cauchy-Riemann equation in 2D:

$$\partial_0 \zeta_0 = +\partial_1 \zeta_1 \quad , \quad \partial_0 \zeta_1 = -\partial_1 \zeta_0$$

with $g_{\mu\nu} = \delta_{\mu\nu}$.

Complex function satisfying above eq. \rightsquigarrow Holomorphic

For a holomorphic $\zeta(z)$, function $f(z) = z + \zeta(z)$ gives rise to an infinitesimal 2D conformal transformation $z \rightarrow f(z)$.

Generators of 2D Conformal Algebra

In search of 2D Conformal generators,
expand $\zeta(z)$ around $z=0$ using a Laurent expansion:

$$z' = z + \zeta(z) = z + \sum_{n \in \mathbb{Z}} \zeta_n (-z^{n+1}),$$

$$\bar{z}' = \bar{z} + \bar{\zeta}(\bar{z}) = \bar{z} + \sum_{n \in \mathbb{Z}} \bar{\zeta}_n (-\bar{z}^{n+1}),$$

with infinitesimal parameters ζ_n and $\bar{\zeta}_n$ constants.

Generators corresponding to a transformation for a particular n are:

$$l_n = -z^{n+1} \partial_z \quad , \quad \bar{l}_n = -\bar{z}^{n+1} \partial_{\bar{z}},$$

with $n \in \mathbb{Z}$. Commutators are as follows:

$$[l_m, l_n] = (m - n)l_{m+n}$$

$$[\bar{l}_m, \bar{l}_n] = (m - n)\bar{l}_{m+n}$$

$$[l_m, \bar{l}_n] = 0$$

This is known as the Witt algebra.

Number of independent infinitesimal conformal transformations in 2D is infinite.

Global Conformal Algebra

Examining the ambiguity at $x=0$ and $x \rightarrow \infty$ and work on Riemann sphere $S^2 \simeq \mathbb{C} \cup \infty$:

- $l_n = -z^{n+1}\partial_z$ non-singular at $z=0$ only for $n \geq -1$.
- use conformal transformation $z = -\frac{1}{w}$ $l_n = -(-\frac{1}{w})^{n-1}\partial_w$
non-singular at $w = 0$ only for $n \leq +1$.

Globally defined conformal transformations on $S^2 \simeq \mathbb{C} \cup \infty$ are generated by l_{-1}, l_0, l_{+1} .

Determining the global conformal group generated by these generators:

- $l_{-1} = -\partial_z$ ($z \rightarrow z + b$): Translation.
- $l_0 = -z\partial_z$ ($z \rightarrow az$) (remember $z \in \mathbb{C}$)

After Change of variable $z = re^{i\phi}$, we find:

$$l_0 = -\frac{1}{2}r\partial_r + \frac{i}{2}\partial_\phi, \quad \bar{l}_0 = -\frac{1}{2}r\partial_r - \frac{i}{2}\partial_\phi$$

The linear combinations $l_0 + \bar{l}_0 = -r\partial_r$ and $i(l_0 - \bar{l}_0) = -\partial_\phi$ are the generators of **dilation** and **rotation**, respectively.

- $l_{+1} = -z^2 \partial_z$ Special conformal transformation and translation for $w = -\frac{1}{z}$.

$c l_1 z = -cz^2$ is the infinitesimal version of $z \rightarrow \frac{z}{cz+1}$.

Operators l_{-1}, l_0, l_{+1} generate transformations of the form:

$$z \rightarrow \frac{az+b}{cz+d} \quad \text{with } a, b, c, d \in \mathbb{C} \quad \text{requiring } ad - bc \neq 0$$

The conformal group of $S^2 \simeq \mathbb{C} \cup \infty$ is the Mobius group $\frac{SL(2, \mathbb{C})}{\mathbb{Z}_2}$.

Conformal Field Theory

A field theory that is invariant under conformal transformations.
 \Rightarrow The physics of the theory looks the same at all length scales (theory can't have any preferred length scale. There can not be anything like a mass, Compton wavelength etc.)

Consider the fields ϕ of our theory on \mathbb{C}^2 : $\phi(z, \bar{z})$.

Define:

Chiral fields as $\phi(z)$ and anti-Chiral fields as $\phi(\bar{z})$.

If a field transforms under conformal transformation $z \rightarrow f(z)$ according to:

$$\phi(z, \bar{z}) \rightarrow \phi'(z, \bar{z}) = \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \phi(f(z), \bar{f}(\bar{z}))$$

It is called **primary** a field of conformal dimensions (h, \bar{h}) .

If this holds for $f \in SL(2, \mathbb{C})/\mathbb{Z}_2$, i.e. only for global conformal transformations, then ϕ is called a **quasi-primary field**.

**Note: Not all fields in CFT are primary (or quasi-primary).

The energy-momentum tensor:

Consider scale invariance by transformation $x^\mu \rightarrow \lambda x^\mu$ which is part of global conformal group;

The associated current is : $J_\mu = x^\nu T_{\mu\nu}$.

Applying Noether's theorem:

$$T^\mu_{\mu} = 0$$

This is the key feature of a conformal field theory in any dimension.

2D Conformal Field Theory

2D CFTs have ∞ -D symmetry \rightarrow severe restrictions on CFTs.

(In some cases) can be solved exactly, using the conformal bootstrap method. Ex of 2D CFTs: massless free bosonic theories, and certain sigma models.

Consider 2D Euclidean:

$$T_{\mu\nu} \text{ transforms as } T_{\mu\nu} = \frac{\partial x^\alpha}{\partial x^\mu} \frac{\partial x^\beta}{\partial x^\nu} T_{\alpha\beta}.$$

By a quick change of variable: $x^0 = \frac{1}{2}(z + \bar{z})$ and $x^1 = \frac{1}{2i}(z - \bar{z})$ we find:

$$\partial_{\bar{z}} T_{zz} = 0 \quad \text{and} \quad \partial_z T_{\bar{z}\bar{z}} = 0$$

The two non-vanishing components of the energy-momentum tensors are a **chiral** and **anti-chiral** field:

$$T_{zz}(z, \bar{z}) =: T(z) \quad , \quad T_{\bar{z}\bar{z}}(z, \bar{z}) =: \bar{T}(\bar{z}).$$

Conformal transformations of the energy–momentum tensor:
Under conformal transformation $z \rightarrow f(z)$, T changes as follows:

$$T'(z) = \left(\frac{\partial f}{\partial z}\right)^2 T(f(z)) + \frac{c}{12} S(f(z), z)$$

c is called the central charge and $S(w, z)$ denotes the Schwarzian derivative.

$$S(w, z) = \frac{1}{(\partial_z w)^2} ((\partial_z^3 w)(\partial_z w) - \frac{3}{2}(\partial_z^2 w)^2).$$

- For non-vanishing central charges, $T(z)$ is not a primary field.
- Schwarzian derivative $S(w, z)$ vanishes for $SL(2, \mathbb{C})$ transformations $w = f(z) \Rightarrow T(z)$ is a quasi-primary field.

Central Charge

Introducing Virasoro Algebra:

Remember Witt Algebra: $[l_m, l_n] = (m - n)l_{m+n}$

with $\overbrace{l_1}^{\text{translation}}$, $\overbrace{l_0}^{\text{dilation \& rotation}}$, $\overbrace{l_{-1}}^{\text{special conformal}}$

This algebra has a **central extension** i.e.

Group A is abelian and its image is in the center of E , for an extension of G by A , given by exact sequence of group homomorphism: $1 \rightarrow A \rightarrow E \rightarrow G \rightarrow 1$.

If Lie algebra of G is \mathfrak{g} , that of A is \mathfrak{a} , and that of E is \mathfrak{e} , then \mathfrak{e} is a **central Lie algebra extension** of \mathfrak{g} by \mathfrak{a} .

Denote elements of the central extension of the Witt algebra by L_n with $n \in \mathbb{Z}$. Their Virasoro Algebra, with central charge c has following commutation relations:

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}$$

The central charge or conformal anomaly c does not occur in the classical theory and is thus a purely quantum effect.

translation dilation&rotation special conformal
 ($\overbrace{L_1}$, $\overbrace{L_0}$, $\overbrace{L_{+1}}$)

are still generators of $SL(2, \mathbb{C})/\mathbb{Z}_2$ transformations.

Application in Statistical Mechanics

Let $\phi(x)$ be a generic order parameter at second order phase transitions and $\delta\phi \equiv \phi(x) - \langle \phi(x) \rangle$ be the fluctuation of the order parameter about its mean value.

Away from critical point:

$$\langle \delta\phi(x)\delta\phi(0) \rangle \approx e^{-\frac{x}{\zeta}},$$

where ζ is the correlation length. The above is no longer true at critical point. :

$\zeta \approx L$ where L is the length of the system.

The divergence of ζ at the critical point implies that the system is scale invariant.

Physical quantities are given by: Scaling fields.

Looking at scaling transformation $x^\lambda \rightarrow \lambda x$, the scaling fields transform as $\phi(x) \rightarrow \lambda^\Delta \phi(x)$

where $\Delta = h + \bar{h}$: scaling dimension of field ϕ .

Systems exhibiting scale invariance in 2D, are also invariant under conformal transformation.

2D Ising Model at its critical point is an ex. of a CFT.

It has 3 primary operators:

Identity with $h = \bar{h} = 0$,

"spin field σ " with $h = \bar{h} = \frac{1}{16}$,

"energy density ϵ " with $h = \bar{h} = \frac{1}{2}$.

It also has central charge $c = \frac{1}{2}$.

2D CFT: Massless Free Boson Example

Alternative definition of a primary field:

A field $\phi(z, \bar{z})$ is primary with conformal dimensions (h, \bar{h}) , if the operator product expansion (OPE) between the $T_{\mu\nu}$ and $\phi(z, \bar{z})$ is:

$$T(z)\phi(w, \bar{w}) = \frac{h}{(z-w)^2}\phi(w, \bar{w}) + \frac{1}{(z-w)}\partial_w\phi(w, \bar{w}) + \dots$$

$$\bar{T}(\bar{z})\phi(w, \bar{w}) = \frac{\bar{h}}{(\bar{z}-\bar{w})^2}\phi(w, \bar{w}) + \frac{1}{(\bar{z}-\bar{w})}\partial_{\bar{w}}\phi(w, \bar{w}) + \dots$$

OPE of chiral $T_{\mu\nu}$ with itself reads:

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + \dots$$

with $(|z| > |w|)$.

The action of a single free massless scalar boson reads:

$$S_\phi = \frac{1}{4\pi} \int dz^2 (\partial\phi\bar{\partial}\phi) \quad \text{defined on 2D complex plane.}$$

Want to show that even though $\phi(z)$ itself does not have a conformal dimension, the fields $\partial_z\phi(z)$ have definite dimensions, i.e. they transform as primary fields:

We need:

$$\partial\phi(z)\partial\phi(w) = -\frac{1}{(z-w)^2} + \dots$$

and from the action, we get:

$$T_\phi(z) = -\frac{1}{2} : \partial\phi(z)\partial\phi(z) := -\frac{1}{2} \lim_{w \rightarrow z} [\partial\phi(z)\partial\phi(w) + \frac{1}{(z-w)^2}]$$

$$T_\phi(z)\partial\phi(w) \approx \frac{\partial\phi(w)}{(z-w)^2} + \frac{1}{(z-w)}\partial^2\phi(w).$$

$\partial\phi$ is a primary field of dimension $h=1$.

We can also calculate the value of central charge c , by calculating OPE of $T_\phi = -\frac{1}{2} : \partial\phi\partial\phi :$ with itself:

$$T_\phi(z) T_\phi(w) \approx \frac{1/2}{(z-w)^4} + \frac{2}{(z-w)^2} \left[-\frac{1}{2} (\partial\phi(w))^2 \right] + \dots$$

From this we conclude that a single free chiral boson has $c=1$.

CFTs in higher dimensions

Higher-dimensional CFTs are prominent in the AdS/CFT correspondence

Gravitational theory in anti-de Sitter space (AdS)

||

Conformal field theory on the AdS boundary.

Well studied example is the $d=4$, $N = 4$ supersymmetric Yang–Mills theory, that is dual to Type *IIB* string theory on $AdS_5 \times S^5$.

References

- Group Theory in a Nutshell for Physicists, by Anthony Zee.
- Conformal Field Theory, by Sergei V. Ketov.
- Introduction to Conformal Field Theory: With Applications to String Theory, by Ralph Blumenhagen, Erik Plauschinn.
- Govindarajan, Suresh. Introduction to Conformal Field Theory, March 1993, The Institute of Mathematical Sciences.