1. (a) A homomorphism from the vector space $\mathbb{R}^3$ to the set of traceless Hermitian 2×2 matrices is defined by $\vec{\boldsymbol{x}}\to\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{\sigma}}$. First, show that $\det(\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{\sigma}})=-|\vec{\boldsymbol{x}}|^2$. Second, prove the identity:

$$x_i=\tfrac{1}{2}\mathrm{Tr}\,(\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{\sigma}}\,\sigma_i)\,.$$

This identity provides the inverse transformation from the set of traceless 2×2 Hermitian matrices to the vector space $\mathbb{R}^3$.

Explicitly, we have

$$\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{\sigma}}=\begin{pmatrix}x_3 & x_1-ix_2\\ x_1+ix_2 & -x_3\end{pmatrix}\,.$$

It follows that

$$\det\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{\sigma}}=-x_1^2-x_2^2-x_3^2=-|\vec{\boldsymbol{x}}|^2\,. \tag{1}$$

To invert the transformation $\vec{\boldsymbol{x}}\longrightarrow\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{\sigma}}$, note that:

$$\mathrm{Tr}\,\big[(\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{\sigma}})\sigma_i\big]=x_j\mathrm{Tr}(\sigma_j\sigma_i)=2x_i\,,$$

after using $\mathrm{Tr}(\sigma_i\sigma_j)=2\delta_{ij}$. Hence, it follows that

$$x_i=\tfrac{1}{2}\mathrm{Tr}\big[(\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{\sigma}})\sigma_i\big]\,.$$

(b) Let $U\in\mathrm{SU}(2)$. Show that $U\,\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{\sigma}}\,U^{-1}=\vec{\boldsymbol{y}}\cdot\vec{\boldsymbol{\sigma}}$ for some vector $\vec{\boldsymbol{y}}$. Using the results of part (a), prove that an element of the rotation group exists such that $\vec{\boldsymbol{y}}=R\vec{\boldsymbol{x}}$ and determine an explicit form for $R\in\mathrm{SO}(3)$. Display a homomorphism from SU(2) onto SO(3) and prove that $\mathrm{SO}(3)\cong\mathrm{SU}(2)/\mathbb{Z}_2$.

We can evaluate $U\,\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{\sigma}}\,U^{-1}$ by writing $U=\exp(-i\theta\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{\sigma}}/2)$ and using the results of part (c) of problem 7 of Solution Set 2. It follows that:

$$U\sigma_jU^{-1}=e^{-i\theta\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{\sigma}}/2}\,\sigma_j\,e^{i\theta\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{\sigma}}/2}=R_{ij}(\hat{\boldsymbol{n}},\theta)\,\sigma_i\,,$$

where R_{ij} is given by

$$R_{ij}(\hat{\boldsymbol{n}},\theta)=n_in_j+(\delta_{ij}-n_in_j)\cos\theta-\epsilon_{ijk}n_k\sin\theta\,. \tag{2}$$

Thus,

$$U\,\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{\sigma}}\,U^{-1}=x_jR_{ij}\sigma_i=\vec{\boldsymbol{y}}\cdot\vec{\boldsymbol{\sigma}}\,,$$

where $y_i\equiv R_{ij}x_j$. Thus, we have explicitly identified the matrix R which is an orthogonal 3×3 matrix of unit determinant, i.e. $R\in\mathrm{SO}(3)$.

Alternative derivation. To prove that a real vector $\vec{\boldsymbol{y}}$ exists such that

$$\vec{\boldsymbol{y}}\cdot\vec{\boldsymbol{\sigma}} = U\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{\sigma}}\,U^{-1}\,,$$

for any real vector $\vec{\boldsymbol{x}}$, it is sufficient to prove that $U\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{\sigma}}\,U^{-1}$ is a 2×2 traceless hermitian matrix, since the most general 2×2 traceless hermitian matrix is always a real linear combination of the three Pauli matrices. First, we note that

$$\mathrm{Tr}(U\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{\sigma}}\,U^{-1}) = \mathrm{Tr}\,\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{\sigma}} = 0\,.$$

Next, since $U^\dagger = U^{-1}$ and $\vec{\boldsymbol{\sigma}}^\dagger = \vec{\boldsymbol{\sigma}}$, it follows that

$$(U\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{\sigma}}\,U^{-1})^\dagger = U(\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{\sigma}})^\dagger\,U^{-1} = U\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{\sigma}}\,U^{-1}\,,$$

where we have used the fact that the components of $\vec{\boldsymbol{x}}$ are real. Hence, $U\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{\sigma}}\,U^{-1}$ is a 2×2 traceless hermitian matrix.

We now use the results of part (a) to conclude that:

$$y_j = \tfrac{1}{2}\mathrm{Tr}\big(\vec{\boldsymbol{y}}\cdot\vec{\boldsymbol{\sigma}}\big)\sigma_i = \tfrac{1}{2}\mathrm{Tr}(U\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{\sigma}}U^{-1}\sigma_i) = \tfrac{1}{2}x_j\mathrm{Tr}(U\sigma_jU^{-1}\sigma_i) = R_{ij}x_j\,,$$

which yields an expression for R_{ij} in terms of the the SU(2) matrix U,

$$R_{ij} \equiv \tfrac{1}{2}\mathrm{Tr}(U\sigma_jU^{-1}\sigma_i)\,. \tag{3}$$

In fact, this definition coincides with eq. (49) of Solution Set 2, and we conclude that R_{ij} is given explicitly by eq. (2). Indeed $R\in\mathrm{SO}(3)$ as expected.

However, one does not require an explicit calculation of eq. (3) to conclude that $R\in\mathrm{SO}(3)$. First, we note that

$$\det\vec{\boldsymbol{y}}\cdot\vec{\boldsymbol{\sigma}} = \det(U\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{\sigma}}U^{-1}) = \det\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{\sigma}}\,.$$

Using eq. (1), it follows that $|\vec{\boldsymbol{y}}|^2 = |\vec{\boldsymbol{x}}|^2$. Thus the linear transformation $\vec{\boldsymbol{y}} = R\vec{\boldsymbol{x}}$ preserve the length of the vector. This observation proves that $R\in\mathrm{O}(3)$. To show that $R\in\mathrm{SO}(3)$, we simply note that R is continuously connected to the identity, corresponding to $U=\pm\mathbf{I}$. At this point, $R_{ij} = \frac{1}{2}\mathrm{Tr}(\sigma_j\sigma_i) = \delta_{ij}$, which implies that $\det R = 1$. Since SU(2) is a connected group, $\det R$ cannot change discontinuously as we vary $U\in\mathrm{SU}(2)$. Thus, we conclude that $\det R = 1$ for all $U\in\mathrm{SU}(2)$, which means that $R\in\mathrm{SO}(3)$ as expected.

The homomorphism of SU(2) onto SO(3) is easily obtained. It consists of the mapping of

$$\{U\,,\,-U\}\longmapsto R\,,\qquad \text{where } U\in\mathrm{SU}(2) \text{ and } R\in\mathrm{SO}(3)\,.$$

To prove that this is a homomorphism, we must show that the group multiplication law is preserved. If we define

$$U_2\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{\sigma}}U_2^{-1} = \vec{\boldsymbol{y}}\cdot\vec{\boldsymbol{\sigma}} \quad\Longrightarrow\quad \vec{\boldsymbol{y}} = R(U_2)\vec{\boldsymbol{x}}\,, \tag{4}$$

$$U_1\vec{\boldsymbol{y}}\cdot\vec{\boldsymbol{\sigma}}U_1^{-1} = \vec{\boldsymbol{z}}\cdot\vec{\boldsymbol{\sigma}} \quad\Longrightarrow\quad \vec{\boldsymbol{z}} = R(U_2)\vec{\boldsymbol{y}}\,, \tag{5}$$

then

$$U_1U_2\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{\sigma}}(U_1U_2)^{-1} = \vec{\boldsymbol{z}}\cdot\vec{\boldsymbol{\sigma}}\,,\quad\Longrightarrow\quad \vec{\boldsymbol{z}} = R(U_1U_2)\vec{\boldsymbol{x}}\,.$$

However, eqs. (4) and (5) yield

$$\vec{\boldsymbol{z}} = R(U_1)R(U_2)\vec{\boldsymbol{x}}\,.$$

Since these equations hold for arbitrary vectors, it follows that

$$R(U_1U_2) = R(U_1)R(U_2)\,,$$

which demonstrates that the group multiplication law is preserved.

From the explicit form for R_{ij} given in eq. (3), it is clear that both U and $-U$ correspond to the same rotation R. Equivalently, the kernel of the homomorphism defined by eq. (3) is $\mathbb{Z}_2 \cong \{\mathbf{I}\,,\,-\mathbf{I}\}$, as these are the only two SU(2) matrices that are mapped onto the identity element, $R_{ij} = \delta_{ij}$ of SO(3). Hence, the homomorphism defined by eq. (3) is a two-to-one mapping (also called a double-valued homomorphism). Hence, one can introduce an equivalence relation among SU(2) matrices such that $U_1 \sim U_2$ if either $U_1 = U_2$ or $U_1 = -U_2$. It therefore follows that

$$\mathrm{SO}(3) \cong \mathrm{SU}(2)/\mathbb{Z}_2\,.$$

(c) The Lie group SU(1,1) is defined as the group of 2×2 matrices V that satisfy $V\sigma_3V^\dagger = \sigma_3$ and $\det V = 1$. (Note that V is *not* a unitary matrix.) The Lie group SO(2,1) is the group of transformations on vectors $\vec{\boldsymbol{x}} \in \mathbb{R}^3$ (with determinant equal to one) that preserves $x_1^2 + x_2^2 - x_3^2$. Display the homomorphism from SU(1,1) onto SO(2,1) and compare with part (b).

First we work out the Lie algebra of SU(1,1). Expanding V about the identity, we write $V \simeq \mathbf{I} + W$. Then, using the definition of SU(1,1),

$$\sigma_3 \simeq (\mathbf{I} + W)\sigma_3(\mathbf{I} + W^\dagger)\,,$$

To first order, it follows that

$$W\sigma_3 = -\sigma_3 W^\dagger\,. \tag{6}$$

Moreover,

$$1 = \det(\mathbf{I} + W) \simeq 1 + \mathrm{Tr}\,W\,,$$

yields $\mathrm{Tr}W = 0$. The latter implies that W is a complex linear combination of Pauli matrices,

$$W = w_1\sigma_1 + w_2\sigma_2 + w_3\sigma_3\,.$$

If we now impose eq. (6), it follows that

$$(w_1\sigma_1 + w_2\sigma_2 + w_3\sigma_3)\sigma_3 = -\sigma_3(w_1^*\sigma_1 + w_2^*\sigma_2 + w_3^*\sigma_3)\sigma_3)\,.$$

Using $\sigma_i\sigma_j = \mathbf{I}\delta_{ij} + i\epsilon_{ijk}\sigma_k$, this condition simplifies to:

$$-iw_1\sigma_2 + iw_2\sigma_1 + w_3\mathbf{I} = -iw_1^*\sigma_2 + iw_2^*\sigma_1 - w_3^*\mathbf{I}\,.$$

That is, w_1 and w_2 are real and w_3 is pure imaginary. It is convenient to define three matrices,

$$\vec{\boldsymbol{\tau}} = (i\sigma_1\,,\, i\sigma_2\,,\, \sigma_3)\,.$$

Then, we may choose $\{-\frac{1}{2}i\tau_k\}$ as a basis for the real Lie algebra of $\mathrm{SU}(1\,,1)$. That is, the most general element of the real Lie algebra of $\mathrm{SU}(1\,,1)$ is

$$W = \sum_{k=1}^{3} c_k(-\tfrac{1}{2}i\tau_k)\,, \qquad \text{where } c_k \in \mathbb{R}\,,$$

where the factor of $\frac{1}{2}$ is conventional. The Lie group elements are obtained by exponentiation:

$$e^{-i\theta\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{\tau}}/2} = e^{\theta(n_1\sigma_1+n_2\sigma_2-in_3\sigma_3)/2} \in\ \mathrm{SU}(1\,,1)\,,$$

where $\hat{\boldsymbol{n}}$ is a unit vector.

We now repeat the analysis of parts (a) and (b). Note that

$$\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{\tau}} = \begin{pmatrix} x_3 & ix_1+x_2 \\ ix_1-x_2 & -x_3 \end{pmatrix}\,,$$

and

$$\det\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{\tau}} = x_1^2+x_2^2-x_3^2\,.$$

If we define:

$$\varepsilon_k = \begin{cases} -1\,, & \text{for } k=1,2\,, \\ +1\,, & \text{for } k=3\,, \end{cases}$$

then,

$$\mathrm{Tr}\big[(\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{\tau}})\tau_k\big] = x_j\mathrm{Tr}\,\tau_j\tau_k = 2\varepsilon_k x_k\,, \qquad (\text{no sum over } k)\,.$$

hence,

$$x_k = \tfrac{1}{2}\varepsilon_k\mathrm{Tr}\big[(\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{\tau}})\tau_k\big]\,. \tag{7}$$

Next, we write:

$$\vec{\boldsymbol{y}}\cdot\vec{\boldsymbol{\tau}} = V\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{\tau}}\,V^{-1}\,, \qquad \text{where } V\in\mathrm{SU}(1\,,1)\,,$$

from which it follows that

$$\det\vec{\boldsymbol{y}}\cdot\vec{\boldsymbol{\tau}} = \det(V\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{\tau}}\,V^{-1}) = \det\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{\tau}}\,.$$

Hence,

$$x_1^2+x_2^2-x_3^2 = y_1^2+y_2^2-y_3^2\,.$$

This is equivalent to the statement that

$$y_i = \mathcal{R}_{ij}x_j\,, \qquad \text{where } \mathcal{R}\in\mathrm{SO}(2\,,1)\,, \tag{8}$$

since $\mathrm{O}(2\,,1)$ transformations preserve the quadratic form $x_1^2+x_2^2-x_3^2$. Moreover, $\det\mathcal{R}=1$, since $\mathcal{R}$ is continuously connected to the identity. Thus, it follows that $\mathcal{R}\in\mathrm{SO}(2\,,1)$.

Hence, eqs. (7) and (8) yield

$$\mathcal{R}_{jk} = \tfrac{1}{2}\varepsilon_k \mathrm{Tr}(V\tau_k V^{-1}\tau_j)\,, \tag{9}$$

which is a homomorphism of $\mathrm{SU}(1\,,1)$ onto $\mathrm{SO}(2\,,1)$ that maps

$$\{V\,,\,-V\} \longmapsto \mathcal{R}\,, \qquad \text{where } V \in \mathrm{SU}(1\,,1) \text{ and } \mathcal{R} \in \mathrm{SO}(2\,,1)\,.$$

As in part (b), it is easy to check that the multiplication law is preserved (the proof is almost identical to the one previously given and will not be repeated here). From the explicit form for $\mathcal{R}_{jk}$ given in eq. (9), it is clear that both V and $-V$ correspond to the same $\mathrm{SO}(2\,,1)$ element $\mathcal{R}$. Equivalently, the kernel of the homomorphism defined by eq. (9) is $\mathbb{Z}_2 \cong \{\mathbf{I}\,,\,-\mathbf{I}\}$, as these are the only two $\mathrm{SU}(1\,,1)$ matrices that are mapped onto the identity element, $\mathcal{R}_{ij} = \delta_{ij}$ of $\mathrm{SO}(2\,,1)$. Hence, the homomorphism defined by eq. (9) is a two-to-one mapping. Hence, one can introduce an equivalence relation among $\mathrm{SU}(1\,,1)$ matrices such that $V_1 \sim V_2$ if either $V_1 = V_2$ or $V_1 = -V_2$. It therefore follows that

$$\mathrm{SO}(2\,,1) \cong \mathrm{SU}(1\,,1)/\mathbb{Z}_2\,.$$

2. The Möbius group is defined as the set of linear fractional transformations:

$$M = \left\{ m(z) = \frac{az+b}{cz+d}\,, \quad ad-bc = 1 \right\}\,,$$

where a, b, c, d and z are complex numbers.

(a) Show that the mapping $f : \mathrm{SL}(2,\mathbb{C}) \to M$ defined by:

$$f : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto m(z) \tag{10}$$

is a group homomorphism.

To demonstrate the homomorphism, we first consider the group multiplication of $\mathrm{SL}(2,\mathbb{C})$,

$$\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = \begin{pmatrix} a_2a_1 + b_2c_1 & a_2b_1 + b_2d_1 \\ c_2a_1 + d_2c_1 & c_2b_1 + d_2d_1 \end{pmatrix}\,.$$

Compare this with the product of two linear fractional transformations. If we define

$$m_1(z) = \frac{a_1z+b_1}{c_1z+d_1}\,,$$

then the composition of two successive linear fractional transformations is given by:

$$m_2(m_1(z)) = \frac{a_2\left(\dfrac{a_1z+b_1}{c_1z+d_1}\right) + b_2}{c_2\left(\dfrac{a_1z+b_1}{c_1z+d_1}\right) + d_2} = \frac{a_2(a_1z+b_1)+b_2(c_1z+d_1)}{c_2(a_1z+b_1)+d_2(c_1z+d_1)}$$

$$= \frac{(a_2a_1+b_2c_1)z + a_2b_1 + b_2d_1}{(c_2a_1+d_2c_1)z + c_2b_1 + d_2d_1}\,.$$

Thus, the multiplication law of $\mathrm{SL}(2,\mathbb{C})$ is preserved. Moreover, the identity of $\mathrm{SL}(2,\mathbb{C})$, denoted by $\mathbf{I}$ (the 2×2 identity matrix), corresponding to $a=d=1$ and $b=c=0$ is mapped to the identity linear fractional transformation $m(z)=z$. Finally, one can easily verify that the inverse $\mathrm{SL}(2,\mathbb{C})$ matrix,[1]

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tag{11}$$

is mapped to the inverse linear fractional transformation,

$$m^{-1}(z) = \frac{dz-b}{-cz+a}\,,$$

since $mm^{-1}(z)=z$ as shown by the following computation (which uses $ad-bc=1$),

$$m(m^{-1}(z)) = \frac{a\left(\dfrac{dz-b}{-cz+a}\right)+b}{c\left(\dfrac{dz-b}{-cz+a}\right)+d} = \frac{a(dz-b)+b(-cz+a)}{c(dz-b)+d(-cz+a)} = z\,.$$

Hence, the mapping specified by eq. (10) is a homomorphism. It is not an isomorphism since for any $T\in\mathrm{SL}(2,\mathbb{C})$, the two elements T and $-T$ are mapped into the same element of M. The kernel of the mapping f is $\mathbb{Z}_2=\{\mathbf{I}\,,-\mathbf{I}\}$. We conclude that

$$M\cong \mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2\,.$$

(b) Prove that M is not simply connected and identify its universal covering group.

M is *not* simply connected. Consider a path in the $\mathrm{SL}(2,\mathbb{C})$ manifold that starts at the identity of the group manifold, $\mathbf{I}\in\mathrm{SL}(2,\mathbb{C})$, and ends at $-\mathbf{I}\in\mathrm{SL}(2,\mathbb{C})$. If we map this path onto M it corresponds to a closed curve since $\mathbf{I}$ and $-\mathbf{I}$ are both mapped to the identity of M. Hence, this is a closed path in M that cannot be contracted to a point. Hence, we conclude that M is not simply connected. In fact, M is doubly connected just like $\mathrm{SO}(3)$. The universal covering group of M is $\mathrm{SL}(2,\mathbb{C})$, which is a simply connected group.

To prove that $\mathrm{SL}(2,\mathbb{C})$ is simply connected, we shall employ the polar decomposition of an invertible complex matrix S. Namely, S can be uniquely written as $S=HU$ where H is a positive definite hermitian matrix and U is unitary.[2] Note that $|\det U|=1$ and $\det H>0$, since the eigenvalues of H are all positive. Hence, it follows that any matrix $T\in\mathrm{SL}(2,\mathbb{C})$,

[1] In computing the inverse of the matrix in eq. (11), we have employed the relation $ad-bc=1$.

[2] To prove this result, we first note that $SS^\dagger$ is a positive definite hermitian matrix. Thus, we define $H\equiv(SS^\dagger)^{1/2}$, which is the unique positive definite hermitian square root of $SS^\dagger$. One can now show that U is unitary since $U=H^{-1}S$, in which case $U^\dagger U=S^\dagger H^{-1}H^{-1}S=S^\dagger(SS^\dagger)^{-1}S=\mathbf{I}$. For further details (including a proof that the polar decomposition is unique), see e.g. Chapter 10 of Denis Serre, *Matrices: Theory and Applications*, Second Edition (Springer Science, New York, 2010).

which by definition has unit determinant, can be uniquely expressed as the product $T = PU$, where $U \in \mathrm{SU}(2)$ and P is a 2×2 positive definite hermitian matrix with unit determinant. Moreover, one can uniquely write $P = e^X = \exp\{\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{\sigma}}\}$ where $\vec{\boldsymbol{x}} \in \mathbb{R}^3$ and X is a 2×2 traceless hermitian matrix.[3] That is, the set of all 2×2 positive definite hermitian matrices with unit determinant is topologically equivalent to $\mathbb{R}^3$.

Thus, there is a continuous invertible map, $T \mapsto (U\,,\,\vec{\boldsymbol{x}})$, from $\mathrm{SL}(2,\mathbb{C})$ to $\mathrm{SU}(2) \times \mathbb{R}^3$. Using the fact that $\mathrm{SU}(2) \simeq S^3$ (as shown in class), it follows that $\mathrm{SL}(2,\mathbb{C})$ is topologically equivalent to the direct product space $S^3 \times \mathbb{R}^3$. Since both S^3 and $\mathbb{R}^3$ are simply connected, so is their direct product. Hence, $\mathrm{SL}(2,\mathbb{C})$ is simply connected.[4]

3. SO(3) can be represented by a ball of radius π with antipodal points identified. A point in the SO(3) group manifold is specified by a vector $\vec{\boldsymbol{\xi}}$ with $|\vec{\boldsymbol{\xi}}| \leq \pi$. Thus, the SO(3) manifold is parameterized by $\vec{\boldsymbol{\xi}} = (\xi, \theta, \phi)$, where (θ, ϕ) are the spherical angles and ξ is the magnitude of the vector $\vec{\boldsymbol{\xi}}$.

[*NOTE:* This is equivalent to the angle-and-axis parameterization where the rotation angle is ξ and the rotation axis, $\hat{\boldsymbol{\xi}}$, is specified by a polar angle θ and an azimuthal angle ϕ.]

(a) Show that the invariant integration measure of SO(3) is given by

$$d\mu(\vec{\boldsymbol{\xi}}) = \det c(\vec{\boldsymbol{\xi}}) \prod_i d\xi_i\,,$$

where the matrix elements of $c(\vec{\boldsymbol{\xi}})$ are

$$c(\vec{\boldsymbol{\xi}})_{nk} = \tfrac{1}{2}\epsilon_{\ell nj} R^{-1}_{\ell i} \frac{dR_{ij}}{d\xi_k}\,,$$

and $R_{ij} \equiv R_{ij}(\vec{\boldsymbol{\xi}})$ is the SO(3) matrix given in problem 7(b) of problem set 2.

In class, we showed that the invariant measure is given by

$$d\mu(g) = \det c(\vec{\boldsymbol{\xi}})\, d\xi_1 d\xi_2 \cdots d\xi_n\,, \tag{12}$$

where the the matrix elements $c_{jk}(\vec{\boldsymbol{\xi}})$ are the coefficients of the Lie algebra element $A^{-1}\partial A/\partial\xi_k$ with respect to some basis, and $A(\vec{\boldsymbol{\xi}})$ are elements of the corresponding Lie group that is

[3]The most general 2×2 traceless hermitian matrix can be written as $\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{\sigma}}$ for some real three-vector, $\vec{\boldsymbol{x}} \in \mathbb{R}^3$. One can easily compute $\exp\{\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{\sigma}}\} = \cosh x + (\vec{x}\cdot\vec{\sigma}/x)\sinh x$, where $x \equiv |\vec{\boldsymbol{x}}|$, and show that this matrix is a positive definite hermitian matrix with unit determinant. In particular, the eigenvalues of $\exp\{\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{\sigma}}\} = \exp\{x\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{\sigma}}\}$ are easily seen to be e^x and e^{-x} (since the eigenvalues of $\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{\sigma}}$ are ± 1), which are manifestly positive. Moreover, allowing $\vec{\boldsymbol{x}}$ to range over all possible vectors in $\mathbb{R}^3$ yields all possible positive definite hermitian matrices with unit determinant, $\exp\{\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{\sigma}}\}$. Hence there exists a map from P to $\vec{\boldsymbol{x}}$ that is one-to-one and onto.

[4]For further details, see e.g. Brian C. Hall, *Lie Groups, Lie Algebras. and Representations: An Elementary Introduction* (Springer Science, New York, 2003).

parameterized by the coordinates $\vec{\boldsymbol{\xi}}$. That is, given an n-dimensional Lie group G and its real Lie algebra $\mathfrak{g}$, an arbitrary Lie algebra element is given by real linear combination of basis vectors $\mathcal{A}_i \in \mathfrak{g}$. Then, we can write

$$A^{-1}\frac{\partial A}{\partial x_k} = \sum_{i=1}^{n} c_{jk}(\vec{\boldsymbol{\xi}})\mathcal{A}_j\,, \tag{13}$$

which defines the coefficients $c_{jk}(\vec{\boldsymbol{\xi}})$ needed in the determination of the invariant measure.

We shall apply this procedure to determine the invariant measure of SO(3). The most general element of the Lie group SO(3) is given in problem 7(b) of problem set 2 [cf. eq. (2)],

$$R_{ij}(\vec{\boldsymbol{\xi}}) = \delta_{ij}\cos\xi + \xi_i\xi_j\left(\frac{1-\cos\xi}{\xi^2}\right) - \epsilon_{ijm}\xi_m\frac{\sin\xi}{\xi}\,, \tag{14}$$

where $\vec{\boldsymbol{\xi}} \equiv \xi\hat{\boldsymbol{n}}$ and $\xi \equiv |\vec{\boldsymbol{\xi}}|$. The matrix generators of the SO(3) Lie algebra are defined by:

$$\mathcal{A}_m = \left.\frac{\partial R}{\partial \xi_m}\right|_{\vec{\boldsymbol{\xi}}=0}\,.$$

It suffices to consider infinitesimal ξ, in which case,

$$R_{ij}(\vec{\boldsymbol{\xi}}) = \delta_{ij} - \epsilon_{ijm}\xi_m + \mathcal{O}(\xi^2)\,.$$

Hence,

$$(\mathcal{A}_m)_{ij} = -\epsilon_{ijm}\,.$$

Using the above results, eq. (13) takes the following form:

$$R^{-1}_{\ell i}\frac{dR_{ij}}{d\xi_k} = -\epsilon_{\ell jm}c_{mk}(\vec{\boldsymbol{x}})\,.$$

Multiplying both sides of this equation by $\epsilon_{\ell nj}$ and using the identity, $\epsilon_{\ell nj}\epsilon_{\ell jm} = -2\delta_{mn}$, one obtains:

$$c(\vec{\boldsymbol{\xi}})_{nk} = \tfrac{1}{2}\epsilon_{\ell nj}R^{-1}_{\ell i}\frac{dR_{ij}}{d\xi_k}\,. \tag{15}$$

(b) Evaluate the expression for $d\mu(\vec{\boldsymbol{\xi}})$ obtained in part (a) and show that

$$d\mu(\vec{\boldsymbol{\xi}}) = 2(1-\cos\xi)\sin\theta d\theta d\phi d\xi\,.$$

We now compute $\det c(\vec{\boldsymbol{\xi}})$ given in eq. (15). Recall that $[R(\hat{\boldsymbol{n}},\xi)]^{-1} = R(-\hat{\boldsymbol{n}},\xi)$. Using $\vec{\boldsymbol{\xi}} = \xi\hat{\boldsymbol{n}}$, it follows that $R^{-1}(\vec{\boldsymbol{\xi}}) = R(-\vec{\boldsymbol{\xi}})$. Hence,

$$R^{-1}_{ij}(\vec{\boldsymbol{\xi}}) = \delta_{ij}\cos\xi + \xi_i\xi_j\left(\frac{1-\cos\xi}{\xi^2}\right) + \epsilon_{ijm}\xi_m\frac{\sin\xi}{\xi}\,. \tag{16}$$

It follows that:

$$\epsilon_{\ell nj}R^{-1}_{\ell i} = \epsilon_{inj}\cos\xi + \xi_\ell\xi_i\epsilon_{\ell nj}\left(\frac{1-\cos\xi}{\xi^2}\right) + (\delta_{in}\xi_j - \delta_{ij}\xi_n)\frac{\sin\xi}{\xi}\,, \tag{17}$$

after employing the identity, $\epsilon_{\ell nj}\epsilon_{\ell im} = \delta_{in}\delta_{jm} - \delta_{ij}\delta_{nm}$ and summing over the repeated index m. Next, we make use of the chain rule,

$$\frac{dR_{ij}}{d\xi_k} = \frac{\partial R_{ij}}{\partial\xi_k} + \frac{\partial R_{ij}}{\partial\xi}\frac{\partial\xi}{\partial\xi_k} = \frac{\partial R_{ij}}{\partial\xi_k} + \frac{\xi_k}{\xi}\frac{\partial R_{ij}}{\partial\xi}\,,$$

where in the the last step, we used $\xi \equiv (\xi_1^2+\xi_2^2+\xi_3^2)^{1/2}$ to compute

$$\frac{\partial\xi}{\partial\xi_k} = \frac{\xi_k}{(\xi_1^2+\xi_2^2+\xi_3^2)^{1/2}} = \frac{\xi_k}{\xi}\,.$$

Hence, using eq. (14),

$$\begin{aligned}\frac{dR_{ij}}{d\xi_k} &= \big(\xi_i\delta_{jk} + \xi_j\delta_{ik}\big)\left(\frac{1-\cos\xi}{\xi^2}\right) - \epsilon_{ijk}\frac{\sin\xi}{\xi} \\ &\quad + \frac{\xi_k}{\xi}\left[-\delta_{ij}\sin\xi + \xi_i\xi_j\left(\frac{\sin\xi}{\xi^2} - \frac{2}{\xi^2}\big(1-\cos\xi\big)\right) - \epsilon_{ijm}\xi_m\left(\frac{\cos\xi}{\xi} - \frac{\sin\xi}{\xi^2}\right)\right]. \end{aligned}\tag{18}$$

Multiplying eqs. (17) and (18) and employing the relevant identities for the product of Levi-Civita tensors, we obtain:

$$\begin{aligned}\tfrac{1}{2}\epsilon_{\ell nj}R^{-1}_{\ell i}\frac{dR_{ij}}{d\xi_k} &= \tfrac{1}{2}(\epsilon_{ink}\xi_i + \epsilon_{knj}\xi_j)\cos\xi\left(\frac{1-\cos\xi}{\xi^2}\right) + \delta_{n\ell}\frac{\cos\xi\sin\xi}{\xi} + \xi_n\xi_k\frac{\cos\xi}{\xi}\left(\frac{\cos\xi}{\xi} - \frac{\sin\xi}{\xi^2}\right) \\ &\quad + \tfrac{1}{2}\epsilon_{\ell nk}\xi_\ell(1-\cos\xi)\left(\frac{1-\cos\xi}{\xi^2}\right) + \tfrac{1}{2}(\delta_{i\ell}\delta_{kn} - \delta_{in}\delta_{k\ell})\xi_i\xi_\ell\frac{\sin\xi}{\xi}\left(\frac{1-\cos\xi}{\xi^2}\right) \\ &\quad + \tfrac{1}{2}(\xi_i\delta_{jk} + \xi_j\delta_{ik})(\xi_j\delta_{in} - \xi_n\delta_{ij})\frac{\sin\xi}{\xi}\left(\frac{1-\cos\xi}{\xi^2}\right) - \tfrac{1}{2}\epsilon_{njk}\xi_j\left(\frac{\sin^2\xi}{\xi^2}\right) \\ &\quad - \tfrac{1}{2}\delta_{ij}(\xi_j\delta_{in} - \xi_n\delta_{ij})\xi_k\left(\frac{\sin^2\xi}{\xi^2}\right) + \tfrac{1}{2}\xi_i\xi_j\xi_k(\xi_j\delta_{in} - \xi_n\delta_{ij})\frac{\sin\xi}{\xi^2}\left[\frac{\sin\xi}{\xi^2} - \frac{2}{\xi^2}\big(1-\cos\xi\big)\right]. \end{aligned}$$

The first and last terms above vanish. Collecting the remaining terms, we find:

$$\begin{aligned}\tfrac{1}{2}\epsilon_{\ell nj}R^{-1}_{\ell i}\frac{dR_{ij}}{d\xi_k} &= \delta_{nk}\left[\frac{\cos\xi\sin\xi}{\xi} + \frac{\sin\xi}{\xi}\big(1-\cos\xi\big)\right] + \tfrac{1}{2}\epsilon_{nk\ell}\xi_\ell\left[\frac{(1-\cos\xi)^2}{\xi^2} + \frac{\sin^2\xi}{\xi^2}\right] \\ &\quad + \xi_n\xi_k\left[\frac{\cos\xi}{\xi}\left(\frac{\cos\xi}{\xi} - \frac{\sin\xi}{\xi^2}\right) - \frac{\sin\xi}{\xi}\left(\frac{1-\cos\xi}{\xi^2}\right) + \frac{\sin^2\xi}{\xi^2}\right]. \end{aligned}$$

Simplifying the trigonometric factors, we end up with

$$c(\vec{\boldsymbol{\xi}})_{nk} = \tfrac{1}{2}\epsilon_{\ell nj}R^{-1}_{\ell i}\frac{dR_{ij}}{d\xi_k} = \left(\delta_{nk} - \frac{\xi_n\xi_k}{\xi^2}\right)\frac{\sin\xi}{\xi} + \frac{\xi_n\xi_k}{\xi^2} + \epsilon_{nk\ell}\xi_\ell\left(\frac{1-\cos\xi}{\xi^2}\right). \tag{19}$$

The invariant measure is given by eq. (12). A direct evaluation of $\det c(\vec{\boldsymbol{\xi}})$ seems extremely tedious without the help of algebraic computing software. Nevertheless, with a little ingenuity, one can carry out this computation explicitly as shown in Appendix A. However, one can simplify the computation of the determinant significantly by making use of the fact that the measure is invariant under rotations of the vector $\vec{\boldsymbol{\xi}}$. Hence, it follows that $\det c(\vec{\boldsymbol{\xi}})$ only depends on the length ξ. To see this explicitly, consider a rotation $\xi_k \longrightarrow R_{kj}\xi_j$. Then the elements of c_{nk} also transform,[5]

$$c_{nk} \longrightarrow R_{kj}R_{n\ell}c_{j\ell} = (RcR^{\mathsf{T}})_{nk}\,. \tag{20}$$

But, note that

$$\det(RcR^{\mathsf{T}}) = \det(R^{\mathsf{T}}Rc) = \det c\,,$$

since $R^{\mathsf{T}}R = \mathbf{I}$. Hence, $\det c$ is independent of the direction of $\vec{\boldsymbol{\xi}}$.

This means that we are free to choose the direction of $\vec{\boldsymbol{\xi}}$ in order to make the computation of $\det c(\vec{\boldsymbol{\xi}})$ as simple as possible. Choosing $\vec{\boldsymbol{\xi}} = \xi\hat{\boldsymbol{z}}$, eq. (19) reduces to:

$$c(\xi\hat{\boldsymbol{z}}) = \begin{pmatrix} \dfrac{\sin\xi}{\xi} & \dfrac{1-\cos\xi}{\xi} & 0 \\ -\left(\dfrac{1-\cos\xi}{\xi}\right) & \dfrac{\sin\xi}{\xi} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The determinant of this matrix is now easily computed,

$$\det c(\xi\hat{\boldsymbol{z}}) = \frac{1}{\xi^2}\left[\sin^2\xi + (1-\cos\xi)^2\right] = \frac{2(1-\cos\xi)}{\xi^2}\,. \tag{21}$$

Hence, the invariant measure of SO(3) is given by:[6]

$$d\mu(\vec{\boldsymbol{\xi}}) = \frac{2(1-\cos\xi)}{\xi^2}\,d^3\xi = 2(1-\cos\xi)\sin\theta d\theta d\phi d\xi\,.$$

(c) Compute the total volume of SO(3). Compare this with the total volume of SU(2).

To compute the volume of SO(3), we note that the parameters of the SO(3) manifold satisfies $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$ and $0 \leq \xi \leq \pi$. Hence,

$$V(\mathrm{SO}(3)) = 2\int_0^\pi d\xi(1-\cos\xi)\int d\Omega = 8\pi^2\,.$$

[5]In obtaining eq. (20), we use the results of Appendix C of the class handout entitled, *Properties of Proper and Improper Rotation Matrices*.

[6]In spherical coordinates, $\vec{\boldsymbol{\xi}} = \xi(\sin\theta\cos\phi\,,\,\sin\theta\sin\phi\,,\,\cos\theta)$ and $d^3\xi = \xi^2 d\xi d\Omega = \xi^2 d\xi\sin\theta d\theta d\phi$.

SU(2) and SO(3) are locally isomorphic groups and hence have the same invariant measure. Globally, the two groups differ since the angle ξ of the SU(2) manifold runs from 0 to 2π. That is, in the angle and axis parameterization, the SU(2) manifold satisfies satisfies $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$ and $0 \leq \xi \leq 2\pi$. Hence,

$$V(\mathrm{SU}(2)) = 2\int_0^{2\pi} d\xi(1-\cos\xi)\int d\Omega = 16\pi^2\,.$$

Thus, the volume of SU(2) is twice that of SO(3), which is not surprising given that $\mathrm{SO}(3) \cong \mathrm{SU}(2)/\mathbb{Z}_2$.

4. Consider a Lie group of transformations G acting on a manifold M. That is, for every $g \in G$, we have $gx = y$ for some $x, y \in M$.

(a) Let H be the set of all transformations in G that map a given point $x \in M$ into itself. Show that H is a subgroup. H has at least three names in the mathematical literature: the little group, the isotropy group, or the stability group of the point x.

Suppose that $h \in H$ such that $hx = x$. First, we note that $ex = x$, so that $e \in H$. If $h_1\,,\, h_2 \in H$, then $h_1h_2 \in H$, since $h_1h_2x = h_1x = x$. Moreover, $hx = x$ implies that $x = h^{-1}x$ (by multiplication by h^{-1} on both sides of the equation), so that $h^{-1} \in H$. Hence, H is a group. Since H is a subset of G, we conclude that H is a subgroup of G.

(b) Consider the submanifold of M defined by $\{gx \,|\, g \in G\}$, for fixed $x \in M$. This is called the *orbit* through x with respect to G. Show that there is a one-to-one correspondence between the points of the orbit and the set of left cosets of H. Explain why we may conclude that $\{gx \,|\, g \in G\} = G/H$. Show that the coset space G/H is homogeneous.

The subgroup H is defined as the set of all $h \in H$ such that $hx = x$. This condition can be rewritten as $Hx = x$. To show that there is a one-to-one correspondence between the points in the orbit through x and the left cosets of H, consider the following result:

$$g_iHx = g_ix\,, \qquad \text{where } g_i \in G\,.$$

As g_i varies over the group, g_iH ranges over all left cosets of H and g_ix ranges over all points in the orbit through x. Thus, all be have to show is

$$g_1H = g_2H \quad \Longleftrightarrow \quad g_1x = g_2x\,,$$

Note that $g_1H = g_2H \implies g_1 = g_2h$ for some $h \in H$. Hence, $g_1x = g_2hx = g_2x$. Likewise, $g_1x = g_2x \implies g_2^{-1}g_1x = x \implies g_2^{-1}g_1 \in H$. Since $HH = H$ (this is simply the statement that the subgroup H is closed under group multiplication), it follows that $g_2^{-1}g_1H \in H$. Since two cosets that are not disjoint must coincide, $g_2^{-1}g_1H = eH = H$. Multiplying both sides by g_2 then yields $g_1H = g_2H$.

We conclude that the points of the orbit are in one-to-one correspondence with the distinct left cosets of H. The set of left cosets is denoted by G/H. Hence,

$$G/H = \{gx \,|\, g \in G\}\,, \qquad \text{for a fixed } x \in M\,.$$

To prove that G/H is homogeneous, we must exhibit a homeomorphism ϕ that maps any point of G/H to any other point. Given the points $y, z \in G/H$ such that $y = g_1 x$ and $z = g_2 x$, consider the homeomorphism,

$$\phi(gx) = g_2 g_1^{-1} gx\,.$$

For $g = g_1$, we have $\phi(y) = z$. That is, the homogeneous property of G/H derives from the homogeneous property of the Lie group G.

(c) Prove that $S^{n-1} = \mathrm{SO}(n)/\mathrm{SO}(n-1)$ by considering the action of the rotation group on the point $(1, 0, 0, \ldots, 0) \in \mathbb{R}^n$.

Consider the action of the rotation group $\mathrm{SO}(n)$ on vectors in $\mathbb{R}^n$. Let us focus on the point $\vec{\boldsymbol{x}} \in \mathbb{R}^n$. The orbit through x under the group action $\mathrm{SO}(n)$ consists of all vectors $\vec{\boldsymbol{y}} \in \mathbb{R}^n$ such that $|\vec{\boldsymbol{y}}| = 1$. One can always find a rotation matrix that rotates the point $\vec{\boldsymbol{x}}$ to the point $\vec{\boldsymbol{y}}$. Thus, the orbit is S^{n-1}, i.e. the set of all unit vectors in $\mathbb{R}^n$.

To determine the little group, we must find the subgroup of $\mathrm{SO}(n)$ that leaves $\vec{\boldsymbol{x}}$ invariant. In the case of $\vec{\boldsymbol{x}} = (1, 0, 0, \ldots, 0)$, the little group is the subgroup of $\mathrm{SO}(n)$ that leaves the x_1-axis fixed, namely $\mathrm{SO}(n-1)$. This can be seen explicitly as follows:

$$\left(\begin{array}{c|ccc} 1 & 0 & 0 & \cdots \quad 0 \\ \hline 0 & & & \\ 0 & & \mathrm{SO}(n-1) & \\ \vdots & & & \\ 0 & & & \end{array}\right)\begin{pmatrix}1\\0\\0\\\vdots\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\\\vdots\\0\end{pmatrix}\,,$$

where the matrix block obtained by eliminating the first row and column is an $\mathrm{SO}(n-1)$ matrix. Thus, using the results of part (b), we conclude that:

$$S^{n-1} \cong \frac{\mathrm{SO}(n)}{\mathrm{SO}(n-1)}\,. \tag{22}$$

Remark: In general, the little group H is *not* an invariant subgroup of G, in which case the manifold G/H is not a Lie group. In light of eq. (22), this last remark is consistent with the theorem mentioned in class that S^{n-1} is not a group manifold for all positive integers n (excluding the special cases of $n = 1, 2$ and 4).

(d) Prove that $S^{2n-1} = \mathrm{U}(n)/\mathrm{U}(n-1)$ by considering the action of the $\mathrm{U}(n)$ matrices on the point $(1, 0, 0, \ldots, 0) \in \mathbb{C}^n$.

Consider the action of $\mathrm{U}(n)$ on vectors in $\mathbb{C}^n$. We repeat the analysis give in part (c) almost verbatim. Focus on the point $\vec{\boldsymbol{w}} = (1\,,\,0\,,\,0\,,\,\ldots\,,\,0) \in \mathbb{C}^n$. The orbit through $\vec{\boldsymbol{w}}$ under the action of $\mathrm{U}(n)$ consists of all complex vectors $\vec{\boldsymbol{z}} \in \mathbb{C}^n$ such that $|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2 = 1$. Writing $z_i = x_i + iy_i$ where x_i and y_i are real numbers, it follows that

$$x_1^2 + x_2^2 + \cdots + x_n^2 + y_1^2 + y_2^2 + \cdots + y_n^2 = 1\,.$$

This is the equation for the $(2n-1)$-dimensional sphere, S^{2n-1}. To determine the little group, we must find the subgroup of $\mathrm{U}(n)$ that leaves $\vec{\boldsymbol{w}}$ invariant. In the case of $\vec{\boldsymbol{w}} = (1,0,0,\ldots,0)$, the little group is the subgroup of $\mathrm{U}(n)$ that leaves the complex w_1-axis fixed, namely $\mathrm{U}(n-1)$. This can be seen explicitly as follows:

$$\left(\begin{array}{c|ccc} 1 & 0 & 0 & \cdots \quad 0 \\ \hline 0 & & & \\ 0 & & \mathrm{U}(n-1) & \\ \vdots & & & \\ 0 & & & \end{array}\right)\begin{pmatrix}1\\0\\0\\\vdots\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\\\vdots\\0\end{pmatrix}\,,$$

where the matrix block obtained by eliminating the first row and column is an $\mathrm{U}(n-1)$ matrix. Thus, using the results of part (b), we conclude that:

$$S^{2n-1} \cong \frac{\mathrm{U}(n)}{\mathrm{U}(n-1)}\,.$$

(e) Complex projective space $\mathbb{CP}^n$ is defined as the space of complex lines in $\mathbb{C}^{n+1}$ through the origin. That is, $\mathbb{CP}^n$ consists of the set of vectors in $\mathbb{C}^{n+1}$ (omitting the zero vector) where we identify $(z_0, z_1, \ldots, z_n) \sim \lambda(z_0, z_1, \ldots, z_n)$, for any nonzero complex number λ. Without loss of generality, we can restrict our considerations to vectors in $\mathbb{C}^{n+1}$ of modulus 1. Show that $\mathrm{U}(1) \otimes \mathrm{U}(n)$ is the little group of the point $z = (1,0,0,\ldots,0) \in \mathbb{CP}^n$, and that $\mathbb{CP}^n$ is the orbit through z with respect to $\mathrm{U}(n+1)$. Conclude that $\mathbb{CP}^n = \mathrm{U}(n+1)/\mathrm{U}(1) \otimes \mathrm{U}(n)$.

A complex line through the origin is an equivalence relation of points in $\mathbb{C}^{n+1}$ given by:

$$(z_0\,,\,z_1\,,\,\ldots\,,\,z_n) \;\sim\; \lambda(z_0\,,\,z_1\,,\,\ldots\,,\,z_n)\,, \qquad \text{where } \lambda \in \mathbb{C}^*\,, \tag{23}$$

where $\mathbb{C}^* \equiv \mathbb{C} - \{0\}$ is the set of nonzero complex numbers and $(z_0\,,\,z_1\,,\,\ldots\,,\,z_n) \in \mathbb{C}^{n+1}$. Without loss of generality, we can restrict our considerations to vectors in $\mathbb{C}^{n+1}$ of modulus 1. This means that we impose the condition

$$\sum_{i=0}^{n} |z_i|^2 = \sum_{i=0}^{n} (x^2 + y^2) = 1\,, \tag{24}$$

where we have written $z_i = x_i + iy_i$ where x_i and y_i are real numbers. Eq. (24) is an equation for the $(2n+1)$-dimensional sphere, S^{2n+1}. When eq. (24) is imposed, the complex number

λ in eq. (23) must satisfy $|\lambda| = 1$. Hence, $\lambda = e^{i\theta}$, where $0 \leq \theta < 2\pi$, which is in one-to-one correspondence with a one-dimensional circle, S^1. Thus, complex projective space $\mathbb{CP}^n$ can also be described as:

$$\mathrm{CP}^n = \{\text{points on } S^{2n+1} \text{ with a circle of phases } S^1 \text{ identified as equivalent}\}\,.$$

This definition of CP^n is mathematically written as:

$$\mathrm{CP}^n = S^{2n+1}/S^1\,.$$

Consider the action of $\mathrm{U}(n+1)$ on $\mathrm{CP}^n \subset \mathbb{C}^{n+1}$. Pick the point in CP^n corresponding to the $(n+1)$ dimensional vector, $z = (1\,,0\,,0\,,\ldots\,,0)$. Note that as a point in CP^n, z is equivalent to all $(n+1)$-dimensional vectors in $\mathbb{C}^{n+1}$ of the form $(e^{i\theta}\,,0\,,0\,,\ldots\,,0)$. It is easy to prove that any point in CP^n can be obtained by the action of some element of $\mathrm{U}(n+1)$ on z, since one can always find an element of $\mathrm{U}(n+1)$ to map z to an arbitrary $(n+1)$-dimensional vector of $\mathbb{C}^{n+1}$ of unit modulus.

To determine the little group, consider the following mapping,

$$\left(\begin{array}{c|ccc} e^{i\theta} & 0 & \cdots & 0 \\ \hline 0 & & & \\ 0 & & \mathrm{U}(n) & \\ \vdots & & & \\ 0 & & & \end{array}\right)\begin{pmatrix}1\\0\\0\\\vdots\\0\end{pmatrix} = \begin{pmatrix}e^{i\theta}\\0\\0\\\vdots\\0\end{pmatrix}\,. \tag{25}$$

Since the following $(n+1)$-dimensional vectors are equivalent,

$$(e^{i\theta}\,,0\,,0\,,\ldots\,,0) \sim (1\,,0\,,0\,,\ldots\,,0)\,,$$

eq. (25) implies that the little group consists of all $(n+1)\times(n+1)$ matrices of the form

$$\left(\begin{array}{c|ccc} e^{i\theta} & 0 & \cdots & 0 \\ \hline 0 & & & \\ 0 & & \mathrm{U}(n) & \\ \vdots & & & \\ 0 & & & \end{array}\right) \in \mathrm{U}(1)\otimes\mathrm{U}(n)\,, \tag{26}$$

where the matrix block obtained by eliminating the first row and column is a $\mathrm{U}(n)$ matrix. The matrices specified in eq. (26) constitute the group $\mathrm{U}(1)\otimes\mathrm{U}(n)$. Hence, using the results of part (b), it follows that:

$$\mathrm{CP}^n \cong \frac{\mathrm{U}(n+1)}{\mathrm{U}(1)\otimes\mathrm{U}(n)}\,.$$

(f) Real projective space $\mathbb{RP}^n$ can be defined analogously to $\mathbb{CP}^n$ of part (e) by replacing the field of complex numbers with the field of real numbers. What coset space can be identified with $\mathbb{RP}^n$?

A real line through the origin is an equivalence relation of points in $\mathbb{R}^{n+1}$ given by:

$$(x_0\,,\,x_1\,,\,\ldots\,,\,x_n)\;\sim\;\lambda(x_0\,,\,x_1\,,\,\ldots\,,\,x_n)\,,\qquad\text{where }\lambda\in\mathbb{R}^*\,,\tag{27}$$

where $\mathbb{R}^*\equiv\mathbb{R}-\{0\}$ is the set of nonzero real numbers and $(x_0\,,\,x_1\,,\,\ldots\,,\,x_n)\in\mathbb{R}^{n+1}$. Without loss of generality, we can restrict our considerations to vectors in $\mathbb{R}^{n+1}$ of unit length. This means that we impose the condition

$$\sum_{i=0}^{n}x_i^2=1\,,\tag{28}$$

which is an equation for the n-dimensional sphere, S^n. When eq. (28) is imposed, the real number λ in eq. (27) must satisfy $\lambda^2=1$. Hence, $\lambda=\pm1$, which is in one-to-one correspondence with the discrete group $\mathbb{Z}_2$ Thus, real projective space $\mathbb{RP}^n$ can also be described as:

$$\mathrm{RP}^n=\{\text{points on }S^n\text{ with antipodal points identified as equivalent}\}\,.$$

This definition of RP^n is mathematically written as:

$$\mathrm{RP}^n\cong S^n/\mathbb{Z}_2\,.\tag{29}$$

Consider the action of $\mathrm{O}(n+1)$ on $\mathrm{RP}^n\subset\mathbb{R}^{n+1}$. Pick the point in RP^n corresponding to the $(n+1)$-dimensional vector, $\vec{\boldsymbol{x}}=(1\,,\,0\,,\,0\,,\,\ldots\,,\,0)$. Note that as a point in RP^n, $\vec{\boldsymbol{x}}$ is equivalent $(-1\,,\,0\,,\,0\,,\,\ldots\,,\,0)$. It is easy to prove that any point in RP^n can be obtained by the action of some element of $\mathrm{O}(n+1)$ on $\vec{\boldsymbol{x}}$, since one can always find an element of $\mathrm{O}(n+1)$ to map $\vec{\boldsymbol{x}}$ to an arbitrary $(n+1)$-dimensional vector of $\mathbb{R}^{n+1}$ of unit length.

To determine the little group, consider the following mapping,

$$\left(\begin{array}{c|ccc}\pm1&0&0&\cdots\;\;0\\\hline0&&&\\0&&\mathrm{O}(n)&\\\vdots&&&\\0&&&\end{array}\right)\begin{pmatrix}1\\0\\0\\\vdots\\0\end{pmatrix}=\begin{pmatrix}\pm1\\0\\0\\\vdots\\0\end{pmatrix}\,.\tag{30}$$

Since $(1\,,\,0\,,\,0\,,\,\ldots\,,\,0)\;\sim\;(-1\,,\,0\,,\,0\,,\,\ldots\,,\,0)$, the little group consists of all $(n+1)\times(n+1)$ matrices of the form

$$\left(\begin{array}{c|ccc}\pm1&0&0&\cdots\;\;0\\\hline0&&&\\0&&\mathrm{O}(n)&\\\vdots&&&\\0&&&\end{array}\right)\in\mathrm{O}(1)\otimes\mathrm{O}(n)\,,\tag{31}$$

where the matrix block obtained by eliminating the first row and column is an $\mathrm{O}(n)$ matrix. Note that the orthogonal group, $\mathrm{O}(1)=\{+1,-1\}$, which implies that the matrices of eq. (31) constitute the group $\mathrm{O}(1)\otimes\mathrm{O}(n)$. Hence, using the results of part (b), it follows that:

$$\mathrm{RP}^n\cong\frac{\mathrm{O}(n+1)}{\mathrm{O}(1)\otimes\mathrm{O}(n)}\,.\tag{32}$$

Remark: By a computation that is nearly identical to the one given in part (c), one can prove that $S^n \cong \mathrm{O}(n+1)/\mathrm{O}(n)$. Hence, it follows from eq. (29) that:

$$\mathrm{RP}^n \cong \frac{\mathrm{O}(n+1)/\mathrm{O}(n)}{\mathbb{Z}_2}\,.$$

It is tempting to claim that this result immediately implies eq. (32) in light of the fact that $\mathrm{O}(1) \cong \mathbb{Z}_2$, but this last step is not well-defined until one specifies the nature of the equivalence relation implicit in eq. (32) as well as the precise embedding of the groups. Thus, one cannot use this shortcut to derive eq. (32).

(g) In parts (c)–(f), check that $\dim(G/H) = \dim G - \dim H$.

In part (c), $\dim S^{n-1} = n-1$, $\dim \mathrm{SO}(n) = \frac{1}{2}n(n-1)$ and $\dim \mathrm{SO}(n-1) = (n-1)^2$. Hence, we check that

$$\tfrac{1}{2}n(n-1) - \tfrac{1}{2}(n-1)n - 2) = \tfrac{1}{2}(n-1)[n-(n-2)] = n-1\,. \quad \checkmark$$

In part (d), $\dim S^{2n-1} = 2n-1$, $\dim \mathrm{U}(n) = n^2$ and $\dim \mathrm{U}(n-1) = \frac{1}{2}(n-1)(n-2)$. Hence, we check that

$$n^2 - (n-1)^2 = 2n-1\,. \quad \checkmark$$

In part (e), $\dim \mathbb{CP}^n = 2n$, $\dim \mathrm{U}(n+1) = (n+1)^2$ and $\dim \mathrm{U}(n)\otimes\mathrm{U}(n) = 1+n^2$. Hence, we check that

$$(n+1)^2 - 1 - n^2 = 2n\,. \quad \checkmark$$

In part (f), $\dim \mathbb{RP}^n = n$, $\dim \mathrm{O}(n+1) = \frac{1}{2}n(n+1)$ and $\dim \mathrm{O}(1)\otimes\mathrm{O}(n) = \frac{1}{2}n(n-1)$, since the discrete group $\mathrm{O}(1) \cong \mathbb{Z}_2$ does not contribute the dimension. Hence, we check that

$$\tfrac{1}{2}n(n+1) - \tfrac{1}{2}n(n-1) = n\,. \quad \checkmark$$

In all cases above, dim refers to *real* dimensions (sometimes denoted by $\dim_{\mathbb{R}}$). In particular, since $\mathrm{U}(n+1)$ is *not* a complex manifold, it does not make sense to employ complex dimensions in the case of part (e). Note that $\mathbb{CP}^n$ is a complex manifold of n complex dimensions, so that $\dim_{\mathbb{R}} \mathbb{CP}^n = 2n$ as stated above.

(h) $\mathbb{CP}^n$ is a manifold of n complex (or $2n$ real) dimensions. $\mathbb{CP}^1$ is homeomorphic to which well-known two-dimensional real manifold?

$\mathbb{CP}^1$ consists of all complex 2-vectors such that $(z_0\,,\,z_1) \sim \lambda(z_0\,,\,z_1)$ for all $\lambda \in \mathbb{C}^*$. Thus, we can express $\mathbb{CP}^1$ as the union of two sets:

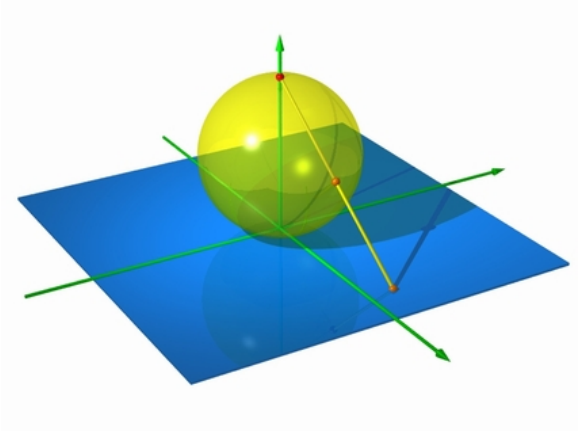
$$\{\text{all vectors equivalent to } (z_0\,,\,z_1)\,,\ z_0 \neq 0\} \cup \{\text{all vectors equivalent to } (0\,,\,z_1)\,, z_1 \neq 0\}\,.$$

If $z_0 \neq 0$, then $(z_0\,,\,z_1) \sim (1\,,\,z_1/z_0) \equiv (1\,,\,z)$, where z ranges over all numbers in the finite complex plane (including $z=0$). Likewise, $(0\,,\,z_1) \sim (0,1)$ for all $z_1 \in \mathbb{C}^*$. Hence, we conclude that:

$$\mathbb{CP}^1 \cong \mathbb{C} \cup \{(0,1)\}\,.$$

In words, $\mathbb{CP}^1$ is isomorphic to the union of the finite complex plane and a single point.

This should remind you of the representation of the extended complex plane as the union of the finite complex plane and the point of infinity. This representation is topologically equivalent to the stereographic projection of the two-dimensional sphere (called the Riemann sphere). In particular, we represent $\mathbb{C} \cong \mathbb{R}^2 \cong S^2 - \{\text{north pole}\}$ by the mapping illustrated in the figure below.



The south pole of the Riemann sphere is tangent to the complex plane. A straight line is drawn from the north pole (which lies above the complex plane) to the complex plane. This line intersects the sphere (i.e. the surface of the ball) once and intersects the complex plane once.

Thus, this establishes a one-to-one correspondence between the finite complex plane and $S^2 - \{\text{north pole}\}$. The north pole itself maps onto the point of infinity. Hence,

$$S^2 \cong \mathbb{C} \cup \{\text{point of infinity}\}\,,$$

as a topological homeomorphism. Thus, using the stereographic projection, we have a one-to-one onto map between $\mathbb{CP}^1$ and the Riemann sphere. It follows that:

$$\mathbb{CP}^1 \cong S^2\,.$$

That is, one can introduce a complex structure on the two-dimensional sphere and realize it as a complex manifold of one complex dimension. Indeed, this manifold is $\mathbb{CP}^1$.

5. Let A be an even-dimensional complex antisymmetric $2n \times 2n$ matrix, where n is a positive integer. We define the *pfaffian* of A, denoted by $\mathrm{pf}\,A$, by:

$$\mathrm{pf}\,A = \frac{1}{2^n n!} \sum_{p \in S_{2n}} (-1)^p\, A_{i_1 i_2} A_{i_3 i_4} \cdots A_{i_{2n-1} i_{2n}}\,, \tag{33}$$

where the sum is taken over all permutations

$$p = \begin{pmatrix} 1 & 2 & \cdots & 2n \\ i_1 & i_2 & \cdots & i_{2n} \end{pmatrix}$$

and $(-1)^p$ is the sign of the permutation $p \in S_{2n}$. If A is an odd-dimensional complex antisymmetric matrix, the corresponding pfaffian is defined to be zero.

(a) By explicit calculation, show that[7]

$$\det A = (\mathrm{pf}\, A)^2\,, \tag{34}$$

for any 2×2 and 4×4 complex antisymmetric matrix A.

The most general 2×2 antisymmetric matrix is:

$$A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}\,.$$

By definition of the pfaffian given in eq. (33),

$$\mathrm{pf}\, A = \tfrac{1}{2}[a-(-a)] = a\,,$$

and $\det A = a^2$. Hence, $\det A = (\mathrm{pf}\, A)^2$. ✓

The most general 4×4 antisymmetric matrix is:

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix}\,.$$

Consider the 24 elements of the permutation group S_4. Using the definition of the pfaffian given in eq. (33), three of the terms that appear in the sum are:

$$\mathrm{pf}\, A = \tfrac{1}{8}\left[a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} + \cdots\right]\,, \tag{35}$$

where we have noted that $\left(\begin{smallmatrix}1&2&3&4\\1&3&2&4\end{smallmatrix}\right)$ is an odd permutation and $\left(\begin{smallmatrix}1&2&3&4\\1&4&2&3\end{smallmatrix}\right)$ is an even permutation. It is clear that the other 21 terms that appear in the sum just reproduce the same three terms displayed in eq. (35) with the same signs. Hence, we conclude that

$$\mathrm{pf}\, A = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}\,.$$

The computation of the determinant of A is straightforward.

$$\det A = -a_{12}\det\begin{pmatrix} -a_{12} & a_{23} & a_{24} \\ -a_{13} & 0 & a_{34} \\ -a_{14} & -a_{34} & 0 \end{pmatrix} + a_{13}\det\begin{pmatrix} -a_{12} & 0 & a_{24} \\ -a_{13} & -a_{23} & a_{34} \\ -a_{14} & -a_{24} & 0 \end{pmatrix} - a_{14}\det\begin{pmatrix} -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \\ -a_{14} & -a_{24} & -a_{34} \end{pmatrix}$$

$$= -a_{12}a_{24}a_{13}a_{34} + a_{12}a_{34}(a_{12}a_{34} + a_{23}a_{14}) - a_{13}a_{12}a_{24}a_{34} + a_{13}a_{24}(a_{13}a_{24} - a_{14}a_{23})$$

$$+a_{14}a_{12}a_{23}a_{34} - a_{14}a_{23}(a_{13}a_{24} - a_{14}a_{23})$$

$$= a_{14}^2a_{23}^2 + a_{13}^2a_{24}^2 + a_{12}^2a_{34}^2 - 2a_{12}a_{13}a_{24}a_{34} + 2a_{12}a_{14}a_{23}a_{34} - 2a_{13}a_{14}a_{23}a_{24}$$

$$= (a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})^2\,.$$

Once again, we see that $\det A = (\mathrm{pf}\, A)^2$. ✓

[7]Eq. (34) holds for all complex antisymmetric $2n\times 2n$ matrices, where n is any positive number. A general proof is provided in a class handout entitled, *Antisymmetric Matrices and the Pfaffian*.

(b) Prove that the determinant of any odd-dimensional complex antisymmetric matrix vanishes. As a result, the definition of the pfaffian in the odd-dimensional case is consistent with the result of eq. (34).

Starting with $A = -A^{\mathsf{T}}$ and taking the determinant of both sides of this equation,

$$\det A = \det(-A^{\mathsf{T}}) = (-1)^n \det A^{\mathsf{T}} = (-1)^n \det A\,,$$

where n is the dimension of the matrix A. If n is odd, then $(-1)^n = -1$, in which case,

$$\det A = -\det A\,,$$

from which we conclude that $\det A = 0$.

(c) Given an arbitrary $2n \times 2n$ complex matrix B and complex antisymmetric $2n \times 2n$ matrix A, use eq. (34) to prove the following identity:

$$\mathrm{pf}\,(BAB^{\mathsf{T}}) = \mathrm{pf}\,A\,\det B\,. \tag{36}$$

Using eq. (33),

$$\begin{aligned}\mathrm{pf}\,(BAB^{\mathsf{T}}) &= \frac{1}{2^n n!}\epsilon_{i_1 j_1 i_2 j_2 \cdots i_n j_n}(B_{i_1 k_1}A_{k_1 \ell_1}B_{j_1 \ell_1})(B_{i_2 k_2}A_{k_2 \ell_2}B_{j_2 \ell_2})\cdots(B_{i_n k_n}A_{k_n \ell_n}B_{j_n \ell_n}) \\ &= \frac{1}{2^n n!}\epsilon_{i_1 j_1 i_2 j_2 \cdots i_n j_n}B_{i_1 k_1}B_{j_1 \ell_1}B_{i_2 k_2}B_{j_2 \ell_2}\cdots B_{i_n k_n}B_{j_n \ell_n}A_{k_1 \ell_1}A_{k_2 \ell_2}\cdots A_{k_n \ell_n}\,,\end{aligned}$$

after rearranging the order of the matrix elements of A and B. We recognize the definition of the determinant of a $2n \times 2n$-dimensional matrix,

$$\det B\,\epsilon_{k_1 \ell_1 k_2 \ell_2 \cdots k_n \ell_n} = \epsilon_{i_1 j_1 i_2 j_2 \cdots i_n j_n}B_{i_1 k_1}B_{j_1 \ell_1}B_{i_2 k_2}B_{j_2 \ell_2}\cdots B_{i_n k_n}B_{j_n \ell_n}\,. \tag{37}$$

Inserting eq. (37) into the expression for $\mathrm{pf}\,(BAB^{\mathsf{T}})$ yields

$$\mathrm{pf}\,(BAB^{\mathsf{T}}) = \frac{1}{2^n n!}\,\det B\,\epsilon_{k_1 \ell_1 k_2 \ell_2 \cdots k_n \ell_n}A_{k_1 \ell_1}A_{k_2 \ell_2}\cdots A_{k_n \ell_n} = \mathrm{pf}\,A\,\det B\,.$$

(d) A complex $2n \times 2n$ matrix S is called *symplectic* if $S^{\mathsf{T}}JS = J$, where S^{T} is the transpose of S and

$$J \equiv \begin{pmatrix} \mathbb{O}_n & \mathbb{1}_n \\ -\mathbb{1}_n & \mathbb{O}_n \end{pmatrix}\,, \tag{38}$$

where $\mathbb{1}_n$ is the $n \times n$ identity matrix and $\mathbb{O}_n$ is the $n \times n$ zero matrix. Prove that the set of $2n \times 2n$ complex symplectic matrices, denoted by $\mathrm{Sp}(n, \mathbb{C})$, is a matrix Lie group[8] [*i.e.*, it is a topologically closed subgroup of $\mathrm{GL}(2n, \mathbb{C})$].

[8]Warning: many authors denote the group of $2n \times 2n$ complex symplectic matrices by $\mathrm{Sp}(2n, \mathbb{C})$.

Consider the set,

$$\mathrm{Sp}(n,\mathbb{C}) = \{S \in \mathrm{GL}(2n,\mathbb{C}) \,|\, S^{\mathsf{T}} J S = J\}\,,$$

where J is defined in eq. (38). To prove that this is a subgroup of $\mathrm{GL}(2n,\mathbb{C})$, we first verify the closure property. If $S_1^{\mathsf{T}} J S_1 = J$ and $S_2^{\mathsf{T}} J S_2 = J$, then

$$(S_1 S_2)^{\mathsf{T}} J (S_1 S_2) = S_2^{\mathsf{T}} S_1^{\mathsf{T}} J S_1 S_2 = S_2^{\mathsf{T}} J S_2 = J\,. \quad \checkmark$$

Next, we note that the identity matrix $\mathbb{1}_{2n} \in \mathrm{GL}(2n,\mathbb{C})$ is an element of $\mathrm{Sp}(n,\mathbb{C})$, since

$$\mathbb{1}_{2n}^{\mathsf{T}} J \mathbb{1}_{2n} = J\,. \quad \checkmark$$

Finally, we check that if $S \in \mathrm{Sp}(n,\mathbb{C})$ then $S^{-1} \in \mathrm{Sp}(n,\mathbb{C})$, by verifying that

$$(S^{-1})^{\mathsf{T}} J S^{-1} = (S^{-1})^{\mathsf{T}} (S^{\mathsf{T}} J S) S^{-1} = J\,,$$

where we have used $S^{\mathsf{T}} J S = J$. $\checkmark$

Note that $\mathrm{Sp}(n,\mathbb{C})$ is a closed subgroup of $\mathrm{GL}(2n,\mathbb{C})$ by virtue of the theorem given in class that states if f and g are two functions that map Hausdorff spaces $X \longrightarrow Y$, then the set $\{x \in X \,|\, f(x) = g(x)\}$ is closed. We apply this theorem to the current problem by choosing $f(S) = S^{\mathsf{T}} J S$ and $g(S) = J$.

(e) Prove that if S is a symplectic matrix, then $\det S = 1$.

Since S is symplectic, $J = S^{\mathsf{T}} J S$. Taking the determinant of this equation and noting that $\det(S^{\mathsf{T}}) = \det S$, it follows that,

$$\det J = \det(S^{\mathsf{T}} J S) = [\det S]^2 \det J\,. \tag{39}$$

A simple computation gives $\det J = 1$. Hence, eq. (39) implies that $[\det S]^2 = 1$, and it follows that $\det S = \pm 1$.

To prove that $\det S = 1$ for all $S \in \mathrm{Sp}(n,\mathbb{C})$, we shall make use of part (c). In particular, using $\det(S^{\mathsf{T}}) = \det S$ and eq. (36),

$$\mathrm{pf}(S^{\mathsf{T}} J S) = \mathrm{pf}\, J \det S\,. \tag{40}$$

Since S is symplectic, $S^{\mathsf{T}} J S = J$, which when inserted into eq. (40) gives

$$\mathrm{pf}\, J = \mathrm{pf}\, J \det S\,. \tag{41}$$

Using the definition of the pfaffian given in eq. (33), it follows that $\mathrm{pf}\, J = 1$. Thus, it is permissible to divide eq. (41) by $\mathrm{pf}\, J$, which immediately yields

$$\det S = 1\,,$$

for all complex symplectic matrices S.

First alternative derivation. Another way to show that there are no symplectic matrices with $\det S = -1$ is to prove that $\mathrm{Sp}(n,\mathbb{C})$ is a connected group.[9] Thus, all elements of $\mathrm{Sp}(n,\mathbb{C})$ are continuously connected to the identity element. Since $\mathbb{1}_{2n} \in \mathrm{Sp}(n,\mathbb{C})$ is a matrix with determinant equal to 1, it follows by continuity that all elements of $\mathrm{Sp}(n,\mathbb{C})$ must have unit determinant.

Second alternative derivation. In Anthony Zee, *Group Theory in a Nutshell for Physicists*, another proof that the determinant of a symplectic matrix is equal to one is provided. However, this proof provided is not complete. Here, I will provide Zee's argument and then show how to complete the proof.

Zee examines the characteristic polynomial, $P(z)$, of the symplectic matrix S, which can be manipulated as follows.

$$P(z) = \det(S - z\mathbb{1}_{2n}) = z^{2n}\det(z^{-1}S - \mathbb{1}_{2n}) = z^{2n}\det S\det(z^{-1}\mathbb{1}_{2n} - S^{-1})\,. \tag{42}$$

Since S is symplectic, we have $J = S^{\mathsf{T}}JS$, which implies that $S^{-1} = -JS^{\mathsf{T}}J$, after making use of $J^{-1} = -J$. Hence, it follows that

$$P(z) = z^{2n}\det S\det(z^{-1}\mathbb{1}_{2n} + JS^{T}J) = z^{2n}(\det J)^2\det S\det(S^{\mathsf{T}} - z^{-1}\mathbb{1}_{2n})\,.$$

As previously noted, $\det J = 1$. Hence,

$$P(z) = z^{2n}\det S\det\big[(S - z^{-1}\mathbb{1}_{2n})^{\mathsf{T}}\big] = z^{2n}\det S\det(S - z^{-1}\mathbb{1}_{2n}) = z^{2n}(\det S)P(z^{-1})\,. \tag{43}$$

That is, we have proved that the characteristic equation for a symplectic matrix S satisfies,

$$P(z) = z^{2n}(\det S)P(z^{-1})\,. \tag{44}$$

The eigenvalues of S are obtained by solving the equation $P(z) = 0$. Since S is invertible, S does not possess any zero eigenvalues. Hence, $P(z) = 0$ implies that $P(z^{-1}) = 0$. We conclude that if λ is an eigenvalue of S then λ^{-1} is also an eigenvalue of S. That is, the eigenvalues of S come in pairs, $\{\lambda\,,\,\lambda^{-1}\}$. Moreover, if $\lambda \neq \pm 1$, then it follows that the multiplicity of the eigenvalues λ and λ^{-1} are the same. This latter result follows after taking r derivatives of eq. (44) with respect to z, where r is the multiplicity of the eigenvalue λ, and then setting $z = \lambda$.

What happens when $\lambda = \lambda^{-1}$ or equivalently when $\lambda = \pm 1$? Zee does not consider this possibility, but it is critical for the argument, because the determinant of S is a product of its eigenvalues. It then follows that

$$\det S = (-1)^d\,, \tag{45}$$

where d is the multiplicity of the eigenvalue -1. We shall now argue that d must be even. Suppose that -1 is an eigenvalue of S, in which case $\mathbb{1}_{2n} + S$ is not invertible. However,

[9]The proof is a generalization of the proof that $\mathrm{SL}(2,\mathbb{C})$ is simply connected given in part (b) of problem 2. In this proof, one shows that $\mathrm{Sp}(n,\mathbb{C})$ is topologically equivalent to $\mathrm{U}(n) \times \mathbb{R}^{n(n+1)}$, which is a connected space. We do not present this proof here; for further details, see e.g. Chapter 10 of Denis Serre, *Matrices: Theory and Applications*, Second Edition (Springer Science, New York, 2010).

it is possible to perturb the matrix $S \to S_\epsilon$ in such a way that it remains symplectic such that $\mathbb{1}_{2n} + S_\epsilon$ is invertible, implying that -1 is not an eigenvalue of S_ϵ.[10] If the perturbation is small, then S_ϵ will possess an eigenvalue λ with multiplicity r that is very close to -1. But, using the results obtained above, S_ϵ will possess a second eigenvalue λ^{-1} with the same multiplicity r that is also very close to -1. In the limit of $\epsilon \to 0$ where the perturbation is removed, the eigenvalues λ and λ^{-1} coalesce, resulting in a degenerate eigenvalue -1 with multiplicity $2r$.[11] We conclude that $d = 2r$ in eq. (45), and it follows that $\det S = 1$.

(f) Using the results of parts (d) and (e), prove that the matrix Lie groups $\mathrm{Sp}(1,\mathbb{C})$ and $\mathrm{SL}(2,\mathbb{C})$ are isomorphic.

Consider an arbitrary complex 2×2 symplectic matrix,

$$S \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}(1,\mathbb{C})\,,$$

where $a, b, c, d \in \mathbb{C}$. Imposing the condition $S^{\mathsf{T}} J S = J$, we require

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,.$$

Multiplying the matrices on the left hand side of the above equation yields

$$\begin{pmatrix} 0 & ad - bc \\ bc - ad & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,.$$

Thus, we conclude that $ad - bc = 1$, which implies that $\det S = 1$. Hence, S is a complex matrix of unit determinant, which means that $S \in \mathrm{SL}(2,\mathbb{C})$. The analysis above also implies that any $\mathrm{SL}(2,\mathbb{C})$ matrix automatically satisfies $S^{\mathsf{T}} J S = J$. Hence, we have proven that

$$\mathrm{Sp}(1,\mathbb{C}) \cong \mathrm{SL}(2,\mathbb{C})\,.$$

Of course, this result is consistent with the general result that all symplectic matrices have unit determinant. For 2×2 symplectic matrices, the determinental constraint is the *only* constraint imposed on a complex 2×2 matrix. For values of $n > 1$, the condition $S^{\mathsf{T}} J S = J$ imposes additional constraints beyond that of $\det S = 1$.

Remark: For $n = 2$, one can prove the isomorphism $\mathrm{Sp}(2,\mathbb{C}) \cong \mathrm{SO}(5,\mathbb{C})$. For values of $n \geq 3$, $\mathrm{Sp}(n,\mathbb{C})$ is not isomorphic to any of the other classical groups, and thus must be considered as an independent family of simple Lie groups.

[10] Details are given in Israel Gohberg, Peter Lancaster and Leiba Rodman, *Indefinite Linear Algebra and Applications* (Birkhäuser,Basel, Switzerland, 2005) pp. 64–66. In particular, see Lemma 4.8.6 of this reference, which was inspired by an appendix on symplectic matrices that appears in Peter D. Lax, *Linear Algebra and its Applications*, 2nd edition (John Wiley & Sons, Inc., Hoboken, NJ, 2007) pp. 308–312.

[11] By a similar argument, the multiplicity of the eigenvalue $+1$ must also be even.

APPENDIX: A simple method for evaluating the determinant of eq. (19)

Consider the following 3×3 hermitian matrices:

$$A_{nk} = \delta_{nk} - \frac{\xi_n \xi_k}{\xi^2}\,, \qquad B_{nk} = \frac{\xi_n \xi_k}{\xi^2}\,, \qquad C_{nk} = i\epsilon_{nk\ell}\frac{\xi_\ell}{\xi}\,,$$

where the indices n and k run over 1,2,3 and ξ is the magnitude of the vector $\vec{\boldsymbol{\xi}}$. It is easy to verify that these matrices satisfy the following relations:

$$A^2 = A\,, \quad B^2 = B\,, \quad C^2 = C\,, \quad AC = CA = C\,, \quad AB = BA = BC = CB = 0\,.$$

We now define three projection matrices:

$$E_1 = \tfrac{1}{2}(A + C)\,, \qquad E_2 \equiv \tfrac{1}{2}(A - C)\,, \qquad E_3 = B\,, \tag{46}$$

which satisfy:

$$E_i^\dagger = E_i\,, \qquad E_i E_j = \delta_{ij} E_j \ \text{(no sum over } i)\,, \quad \text{and} \quad \sum_{i=1}^{3} E_i = \mathbf{I}\,,$$

where $\mathbf{I}$ is the 3×3 identity matrix. Using the spectral decomposition of linear algebra, any diagonalizable complex 3×3 matrix can be written in the form

$$M = \sum_i \lambda_i E_i\,, \tag{47}$$

where the λ_i are the eigenvalues of M and the E_i are a projection operators that project vectors of $\mathbb{R}^3$ to the one-dimensional linear subspace spanned by the corresponding eigenvector $\vec{\boldsymbol{v}}_i$, respectively. In particular, $E_i \vec{\boldsymbol{v}}_j = \vec{\boldsymbol{v}}_i \delta_{ij}$ (no sum over i), which implies [after using eq. (47)] that $M\vec{\boldsymbol{v}}_i = \lambda_j \vec{\boldsymbol{v}}_i$, as expected. The determinant of M is then given by:

$$\det M = \lambda_1 \lambda_2 \lambda_3\,.$$

We now use the above results to evaluate the determinant of eq. (19), which can be rewritten in the following form,

$$c(\vec{\boldsymbol{\xi}}) = \frac{\sin\xi}{\xi} A + B - i\left(\frac{1 - \cos\xi}{\xi}\right) C\,.$$

One can re-express $c(\vec{\boldsymbol{\xi}})$ in terms of the projection operators defined in eq. (46),

$$c(\vec{\boldsymbol{\xi}}) = \left[\frac{\sin\xi}{\xi} - i\left(\frac{1 - \cos\xi}{\xi}\right)\right] E_1 + \left[\frac{\sin\xi}{\xi} + i\left(\frac{1 - \cos\xi}{\xi}\right)\right] E_2 + E_3\,.$$

Comparing with eq. (47), we can read off the eigenvalues of $c(\vec{\boldsymbol{\xi}})$,

$$\lambda_{1,2} = \frac{\sin\xi}{\xi} \mp i\left(\frac{1 - \cos\xi}{\xi}\right)\,, \qquad \lambda_3 = 1\,.$$

Hence,

$$\det c(\vec{\boldsymbol{\xi}}) = \lambda_1 \lambda_2 \lambda_3 = \frac{1}{\xi^2}\left[\sin^2\xi + (1 - \cos\xi)^2\right] = \frac{2(1 - \cos\xi)}{\xi^2}\,,$$

which reproduces the result obtained in eq. (21).